CELL SIZES AT THE ONSET OF OSCILLATORY CONVECTIVE INSTABILITY IN A LAYER OF LOW-PRANDTL-NUMBER FLUID SUBJECT TO ROTATION AND A VERTICAL MAGNETIC FIELD

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[Received 21 April 1986. Revise 18 August 1986]

SUMMARY

The computed theoretical predictions of cell sizes at the onset of direct and oscillatory instabilities when a rotating layer of mercury is heated from below and subjected to the influence of a vertical magnetic field are compared with the experimentally determined values of Nakagawa (1). The point of transition from oscillatory instability to direct instability which is manifested in a discontinuous change in the cell sizes is also investigated.

1. Introduction

ALTHOUGH many physically significant astrophysical and geophysical situations are encountered where both rotation and a magnetic field are associated with a convective region, little experimental investigation, apart from the pioneering work of Nakagawa (1, 2) published some twenty-five years ago, appears to have been undertaken into their combined effects, which are known to generally inhibit both the onset of convection and the total heat energy transported by established cellular convective motions. Further experimental results establishing the critical parameter values which determine the onset of convective instability in electrically conducting low-Prandtl-number fluids would be of particular interest to astrophysics where, for example, the prevailing conditions in the solar photospheric convective zone are characterized by Rayleigh numbers of very high order and very small Prandtl numbers. Nakagawa, in a series of experiments, recorded the occurrence of convective instability in a layer of mercury heated from below and subject to separate application of rotation about a vertical axis or a vertical magnetic field of varying strength or both in
The use of mercury is significant as it has a relatively low Prandtl number, \( \sigma = 0.025 \), compared with \( \sigma = 0.7 \) for air and water with \( \sigma = 7 \). Liquid sodium with \( \sigma = 0.005 \) as an alternative to the use of mercury in the experiments would undoubtedly further clarify the general features of convection at low Prandtl numbers.

In two experimental investigations Nakagawa \((3, 4)\) established the dependence of the critical Rayleigh number for the onset of stationary convection on the vertical magnetic-field strength, and the effect of the impressed field on the heat transported by an established convective regime. In both these cases only stationary cellular convection prevailed. However, when rotation was substituted for the magnetic field as the external inhibition force in the experimental arrangement \((5 \text{ to } 7)\) it was observed that initially, at the critical Rayleigh number, the layer of mercury exhibited oscillatory motions designating oscillatory convection whereas for considerably larger values of the Rayleigh number the mode of instability corresponded to stationary convection. When small sand particles were placed on top of the mercury layer together with a small amount of distilled water to prevent oxidation at the top surface, Nakagawa \((5)\) was able to demonstrate in an explicit way the special characteristics of oscillatory convection by recording the paths of individual tracers. The period of these oscillations was estimated by plotting the movement of each tracer in the \(x\)-and \(y\)-directions and measuring the time interval between two consecutive maxima and minima.

In another series of experiments Nakagawa used the same apparatus, incorporating some essential modifications, to investigate the kind of instability prevailing and to determine the associated critical Rayleigh numbers governing convective instability in a layer of mercury now subject to the simultaneous interaction of a magnetic field and rotation. As the temperature difference across the fluid layer is increased (this in effect corresponds to an increase in the Rayleigh number and hence in the degree of instability) from that determining the conductive state for specified rate of rotation and magnetic-field strength, oscillatory convection was found to set initially. The existence of this oscillatory instability was detected indirectly from the sinusoidal nature of the oscillations in the record of the adverse temperature gradients; the presence of a rigid oxidation film which covered the top surface of the mercury layer prevented direct observations being made \((2)\). The periods of these oscillations were also determined from these results. As before, the addition of a thin layer of water and sand particle tracers allowed Nakagawa \((1)\) to observe the cellular nature of convection at marginal stability, classify it as oscillatory or steady, measure the cell sizes and hence define the horizontal wave number of the hexagonal convective planform. At various constant rotation rates he also established the dependence of the critical Rayleigh number on the Chandrasekhar number, which is proportional to the square of the applied magnetic-field.
strength and is explicitly defined later in this section, and observed a
discontinuous change in the cell size from the streak photographs, at the
value of the Chandrasekhar number where the mode of instability changes
from oscillatory to steady convection. It was also observed that the type of
motion present at marginal stability persists long after the critical condition
has been exceeded, by virtue of increasing the rate of heating at the bottom
plate; this appeared also to be stable. Moreover, in extending these
experiments a decrease in the critical Rayleigh number with increasing
magnetic-field strength over part of the convective branch was observed for
some constant rotation rate.

Theoretical considerations show that a Hopf bifurcation is not possible in
the classical Rayleigh–Benard problem which characterizes convective
motions in a horizontal layer of viscous fluid, of depth $d$, heated from
below. In the absence of external constraints instability can only manifest
itself via a simple bifurcation at the marginal state provided the Rayleigh
number is equal to the critical value $R_c$ for a particular horizontal wave
number denoted by $a_c$. However, the linear theory also confirms that, when
rotation and a magnetic field act simultaneously, the conditions for the
occurrence of oscillatory motions depend upon the non-dimensional para-
meters incorporating the angular speed of rotation about a vertical axis $\Omega_0$,
strength of the applied vertical magnetic field $H_0$, and also on the relative
values of the thermal $\kappa$, viscous $\nu$, and magnetic $\eta$, diffusivities in a
complex manner. Specifically, Chandrasekhar (8) investigated this situation
in the case when a mercury layer is bounded by two free or slippery type
boundaries, although his analysis ignored the ratio of the Prandtl number
$\sigma=\nu/\kappa$ to magnetic Prandtl number $\tau=\eta/\kappa$, where

$$\frac{\sigma}{\tau} = \frac{\nu}{\eta} = 1.5 \times 10^{-7}$$

is small in comparison to $\sigma=0.025$. These results were displayed in two
composite diagrams from which the Rayleigh number, and corresponding
wave number, determining the onset of time dependent and stationary
convection could be ascertained for specific speeds of rotation as a function
of the magnetic field strength.

Eltayeb (9) studied four models with different orientations of the rotation
axis and direction of the magnetic field when the Chandrasekhar number

$$Q = \frac{\mu^* H_0^2 d^2}{4 \pi \mu \eta}$$

(where $\mu^*$ is the magnetic permeability and $\mu$ is the viscosity) and the
Taylor number

$$T = \frac{4 \Omega_0^2 d^4}{\nu^2}$$
were both large and with a variety of boundary conditions applying. He found that the nature of the motions at the onset of instability could be categorized according to the relative dominance or equality of the Lorentz and Coriolis forces. Specifically, he established approximations for the critical Rayleigh number, wave number and frequency of the oscillatory instabilities, for applicable ranges of the Taylor number, when the magnetic field, rotation axis and gravitational force were parallel and \( \sigma \ll \tau \ll 1 \).

We now proceed to formulate a variational principle that will allow the computation of the theoretical cell sizes for the onset of oscillatory motions when the rigid–free, conducting–non-conducting boundary conditions apply—these being appropriate for the experimental configuration of Nakagawa (1), where the mercury layer was confined between a stainless steel base and a layer of distilled water. This combination of boundary conditions presents a more complicated problem, compared to the free–free boundaries case, and does not allow solutions which can be expressed analytically in terms of the relevant parameters. The necessity to adopt numerical procedures at least provides an approach which is valid for all parameter values and is not constrained to the rather impractical asymptotic case of large \( Q \) and \( T \). As the principal objective of this study is to establish the correspondence between the theoretical and experimental cell sizes for oscillatory convection we have only presented results over a range of magnetic field strengths at a constant rotation speed as determined by the experimental parameter values adopted by Nakagawa (1).

2. Basic equations and method of solution

A theoretical analogue of the experimental work undertaken by Nakagawa (1, 2) is considered where an infinite horizontal layer of fluid is subjected to uniform rotation, with an angular velocity \( \Omega \), and a uniform magnetic field \( \mathbf{H} \) when \( \mathbf{G}, \mathbf{H} \) and \( \Omega \) all act in the same direction. The fluid layer is contained by two isothermal boundaries and heated from below, with a temperature difference of \( \Delta T \) maintained across them.

This physical situation is governed by the basic partial differential equations from Chandrasekhar (8). A normal-mode analysis of the corresponding linear perturbation equations is made in terms of two-dimensional periodic waves with assigned horizontal wave numbers, where typically the vertical component of the velocity \( w(x, y, z, t) \) has the form

\[
w = W(z) \exp \left[ i(k_x + k_y)z + pt \right].
\]

The other perturbation quantities are assumed to have a similar dependence on \( x, y \) and \( t \); \( k = (k_x^2 + k_y^2)^{1/2} \) is the wave number of the disturbance and \( p \), the frequency of the perturbation, is a constant and may be complex.

The following equations now establish the vertical dependence of the
variables using Chandrasekhar's (8) notation:

\[(D^2 - a^2 - P_1 n)F = -Ra^2 W, \tag{1}\]
\[(D^2 - a^2 - P_2 n)H = -(H_0 d/\eta)DW, \tag{2}\]
\[(D^2 - a^2 - P_2 n)\chi = -(H_0 d/\eta)DZ, \tag{3}\]
\[(D^2 - a^2 - n)Z = -(2\Omega_0 d/\nu)DW - (\mu H_0 d/4\pi\rho \nu)D\chi \tag{4}\]

and

\[(D^2 - a^2)(D^2 - a^2 - n)W + (\mu H_0 d/4\pi\rho \nu)D(D^2 - a^2)H
- (2\Omega_0 d^3/\nu)DZ = F, \tag{5}\]

where length is expressed in units of \(d\) and time in units of \(d^2/\nu\), with \(\nu = \mu/\rho_0\). As functions of \(z\), \(0 \leq z \leq 1\), the vertical distance across the fluid layer, \(W(z)\), \(F(z)\), \(Z(z)\), \(H(z)\) and \(\chi(z)\) give the vertical velocity, temperature fluctuation, vertical vorticity, perturbed magnetic field and current density, respectively. Also, \(D = d/dz\), \(n = pd^2/\nu\) is a non-dimensional time constant, \(a = kd\) is the horizontal wave number which determines the aspect ratio of a convection cell, \(P_1 = \sigma\) and \(P_2 = P_1/\tau\). The Rayleigh number \(R\) is a non-dimensional measure of the temperature difference across the layer and is defined by

\[R = \frac{g\alpha d^3 \Delta T}{\kappa \nu}.\]

The critical value of the Rayleigh number at the onset of convective instability is denoted by \(R_c\) with the corresponding horizontal scale given by \(a_c\). The Chandrasekhar and Taylor numbers, as previously defined, can now be introduced into the above system to parametrize the effects of the magnetic field and rotation.

The boundary conditions necessary to solve the eigenvalue system (1) to (5) must now be specified. Those appropriate for the experimental arrangement used by Nakagawa (1) on the basis of a rigid conducting lower boundary and a free non-conducting upper boundary are

\[W = DW = F = Z = D\chi = H = 0 \quad \text{at } z = 0\]

and

\[W = D^2 W = F = DZ = \chi = DH = 0 \quad \text{at } z = 1.\]

Due to the symmetry of the differential system, the problem can be simplified mathematically by considering the fluid layer to be contained between two rigid boundaries at \(z = -\frac{1}{2}\) and \(z = \frac{1}{2}\). Then taking odd solutions for \(W(z)\) and by considering a fluid layer of depth \(d/2\), \(-\frac{1}{2} < z < 0\), results in having a rigid boundary at \(z = -\frac{1}{2}\) and a free
boundary at $z = 0$. This is coupled with even solutions for $Z(z)$ giving $Z(-\frac{1}{2}) = 0$ and $DZ(0) = 0$. For this layer of $d/2$ rather than the usual depth $d$, the parameters must be appropriately scaled; when $d \to d/2$ then $a \to a/2$, $Q \to Q/4$, $R \to R/16$, $T \to T/16$, $n \to n/4$ and $\Delta T \to \Delta T/2$.

A variational method of solution is described in the Appendix. In the notation defined there, the first approximation for $R$ is given by

$$R = (4\pi^2 + a^2 + P_1 n) / a^2 \left\{ 8\pi \sum_{j=1}^{5} \frac{B_j^{(1)} \sinh (q_j/2)}{(q_j^2 + 4\pi^2)} + C_2 \gamma_2 \right\}$$  \hspace{1cm} (6)$$

and a second approximation is given by the lower root of the determinantal equation:

$$\begin{vmatrix}
\frac{1}{2} \left[ \frac{4\pi^2 + a^2 + P_1 n}{Ra^2} - C_2 \gamma_2 \right] - \left( \frac{1}{1} \right)
- \left( \frac{1}{2} \right) \\
1 \left[ \frac{16\pi^2 + a^2 + P_1 n}{Ra^2} - C_4 \gamma_4 \right] - \left( \frac{2}{2} \right)
\end{vmatrix} = 0.$$  \hspace{1cm} (7)

The application of the variational method gives the characteristic equation (A14), which represents a double eigenvalue problem in $R$ and $n$ involving complex terms and hyperbolic functions with complex arguments arising from the nature of the roots $q_j$. In general both $R$ and $n$ may be complex; however, the physical limitation on the Rayleigh number is that it must always be real. The direct bifurcation curve for cellular convection is defined by $n = 0$, and under these circumstances the formulation given reduces to a single eigenvalue equation (10). For the problem under consideration, at the onset of oscillatory instability $n$ has no real part and the numerical algorithm employed is also based on the observation that it is a purely imaginary quantity. As a computational expedient, $R$ is considered to be complex and following the introduction of an initial estimate for $n$ into (6), further iteration on $n$ is undertaken employing complex computer arithmetic until the imaginary part of $R$ vanishes. These values of $R$ and $n$, representing a first approximation, can be improved by evaluating successive higher-order approximations in the secular determinant (A14), the previous values computed being used at any stage as initial estimates for the iterative procedure in the next higher approximation. It was generally found, however, that the second-order approximation given by (7) provided adequate accuracy and it was not necessary to proceed to higher-order approximations.

For the purposes of comparison, we have adopted in our computations the scaled equivalents of the corresponding experimental values of the physical parameters used by Nakagawa (1); they are given by fixed $T = 7.3 \times 10^5 \pi^4$, $P_1 = 0.025$, $P_2 = 1.5 \times 10^{-7}$ and particular values of $Q$ in the range $\pi^2$ to $\sim 10^{3.5} \pi^2$. For any appropriate value of the horizontal wave
number \( a \) the corresponding Rayleigh number at which the mercury layer first becomes unstable via a Hopf bifurcation can now be computed. As we specifically require the critical value \( R_0 \), which is the minimum \( R \) for varying \( a \) at a particular value of \( Q \) resulting in oscillatory motions, a half-interval search procedure was employed to determine this value. The corresponding value of \( a \) is \( a_0 \) and this value can now be used to determine the critical cell size \( b \), measured in centimetres, from the expression

\[
b = \frac{4\pi}{(3a_0)^{1/2}},
\]

when the mercury layer has depth 3 cm equivalent to \( \frac{1}{3}d \) in our derivation.

To achieve these results some subsidiary numerical procedures to determine \( q_i \) and \( B_j^{(m)} \), \( j = 1, 2, 3, 4, 5, \ m = 1, 2, 3, \ldots, J \) have to be implemented. The \( q_i^2 \) are evaluated from the roots of \((A6)\). As \( n \), the frequency of the oscillatory motion, is imaginary, this quintic polynomial in \( q^2 \) has complex coefficients and the roots were established by utilizing a numerical routine of Jenkins and Traub (11). In turn the \( B_j^{(m)} \) which are determined by the choice of boundary conditions on \( W(z), DW(z), Z(z), D\chi(z), H(z) \) at \( z = \pm \frac{1}{2} \), are governed by the linear non-homogeneous equations \((A8)\) to \((A12)\). In the numerical solution each equation is split into real and imaginary parts, giving a system of ten linear equations which are solved using a Gaussian elimination procedure incorporating pivoting.

3. Results and conclusions

When conducting his initial experiments with rotation and a magnetic field Nakagawa (2) was unable to measure the size of the convective cells associated with the time-dependent motions because of the formation of an opaque, rigid oxidized film on top of the mercury layer. In subsequent experiments a small quantity of distilled water placed on the layer prevented the formation of this film thus allowing the horizontal scale of the cell sizes to be determined. This action changed the nature of the top boundary, in physical terms, to a stress-free non-conducting one, the lower boundary still being rigid and conducting as in previous experiments. The theoretical analogues of these boundary conditions have been incorporated into the linear theory, the results from which are utilized to predict the cell sizes at the onset of oscillatory motions.

An earlier study (10) made a comparison of the cell sizes on the same basis but for stationary convection. However, the discontinuous change in cell size with increasing magnetic field strength in that case was from the state of stationary convection rather than from oscillatory convection as observed experimentally. The critical cell sizes for direct instability have been recalculated here and found to agree precisely with those presented by Murphy (10).

From the relationships between \( R_c \) and \( Q \) determined for both simple and
Hopf bifurcations, the value of \( Q \) where the nature of the onset of instability changes is established. A relative comparison is presented in Fig. 1; overall, this relationship between the simple bifurcation curves for the rigid–free system exhibits the same qualitative features as shown by both the rigid–rigid case (12) and the free–free case (8). It is necessary to note, however, that different values of the Taylor number have been used in these three cases and any further comparison would be only on the basis of Taylor numbers with the same order of magnitude. When the individual oscillatory curves are examined, they show some differences of physical significance arising from the different types of boundary conditions employed. When two rigid non-conducting boundaries were considered, the Hopf bifurcation curve consisted of one branch monotonically increasing with \( Q \). In contrast, the curve for two rigid perfectly conducting boundaries consisted of two branches which intersected at \( Q \approx 10^{2.5} \pi^2 \). For a range of values of \( Q \) which includes this point of intersection, the corresponding \( R-a \) curves have two minima. As a consequence, a discontinuous change in cell size for oscillatory motions could occur with increasing magnetic field strength for this type of boundary. The rigid–free Hopf bifurcation curve in Fig. 1 is also monotonically increasing for the small value of \( \sigma = 0.025 \), whereas the stationary convection curve consists of two branches as a result of the corresponding \( R-a \) curves having two minima in the range of values of \( Q \) indicated in Fig. 1. At \( Q \approx 10^{2.5} \pi^2 \) the Hopf bifurcation curve flattens out just before the transition to steady cellular convection, indicating that the inhibiting effect of the magnetic field on the onset of oscillatory motions is reduced in this range of \( Q \).

![Graph](image_url)

**Fig. 1.** Variation of \( R_c \) with the magnetic field strength \( Q \) when \( T = 7.3 \times 10^5 \pi^4 \), \( P_1 = 0.025 \) and \( P_2 = 1.5 \times 10^{-7} \) in the case of a lower rigid conducting boundary and an upper free non-conducting boundary. The shaded region represents the range of values of \( Q \) where the marginal stability \( R-a \) curve has two minima. Full and dashed curves are for Hopf and direct bifurcations respectively.
Fig. 2. The convective cell size $b$ in centimetres as a function of the Chandrasekhar number $Q$, at the same parameter values and with the boundary conditions as detailed for Fig. 1. Nakagawa's (1) experimental results are: ■, Hopf and ◆, direct bifurcations. The vertical broken line indicates the value of $Q/\pi^2$ at which the two branches of the direct bifurcation curves cross.

Fig. 3. Variation of $n$ with the Chandrasekhar number $Q$ at the same parameter values and with the same boundary conditions as detailed for Fig. 1. The gyration frequency $p/\Omega_0 = 2n/\sqrt{T}$.
The theoretical results presented here confirm the observed (1) discontinuous change in cell size as \( Q \) is increased. Figure 2 clearly shows the experimental measures of the cell width at the onset of instability together with the theoretical predictions for steady and oscillatory convection. The agreement between the two is very good, especially for the Hopf-bifurcation branch. The change-over from narrow to wide cells occurs at the value of \( Q \) where the simple and Hopf bifurcation curves of Fig. 1 intersect. This is preceded by a small increase in the size of the oscillatory cells, which would be difficult to observe as it is within the limits of the error bars.

In addition to \( R \), the corresponding non-dimensional frequency of the oscillatory instabilities is also determined from the double eigenvalue problem. Figure 3 gives the dependence of \( n \) on the magnetic-field strength and indicates that the period of the oscillations increases with the strength of the field and that no oscillatory motions are possible beyond \( Q = 10^{3.5} \pi^2 \).

As Nakagawa (1) did not publish any values of the periods of the oscillatory motions, direct comparison with experimental data is not possible. But he did report that motion pictures taken during the course of these experiments clearly exhibited the oscillatory motion of the sand tracers on the surface of the mercury layer. In an earlier paper, Nakagawa (2) determined the periods \( 2\pi/n \) of the overstable oscillations from the oscillations in the time records of the adverse temperature gradients for a heated layer of mercury, but an average value of \( T = 8.05 \times 10^4 \pi^4 \) in a rigid–rigid perfectly conducting system. These results were given in terms of the gyration frequency of the oscillations at the onset of instability denoted by the ratio

\[
p/\Omega_0 = 2n/\sqrt{T},
\]

according to the definition of \( T \). On comparison, good agreement is found between the two sets of values for \( p/\Omega_0 \), albeit at different Taylor numbers, and boundary conditions.

Further experimental results for fluids with different Prandtl numbers would be of considerable interest especially to ascertain if the Hopf-bifurcation curve exhibited any non-monotonic behaviour which would be manifested as step changes in the observed cell sizes for oscillatory motions with increasing magnetic-field strength. This would be a different phenomenon from the already-observed transition from oscillatory motions to steady cellular motions which is accompanied by a discontinuous change in cell size for mercury, and is readily predictable from a theoretical treatment such as the one presented here.

REFERENCES

APPENDIX

The variational method

First \( \chi(z) \) and \( H(z) \) can be eliminated from equations (2) to (5), and the substitution of an odd series expansion for \( F(z) \) of the form

\[
F(z) = \sum_{m=1}^{J} A_m \sin (2m \pi z)
\]

results in

\[
(D^2 - a^2)[(D^2 - a^2 - P_2 n)(D^2 - a^2 - n) - Q D^2] W(z) + T(D^2 - a^2 - P_2 n)^2 D^2 W
\]

\[
= - \sum_{m=1}^{J} A_m \{(4m^2 \pi^2 + a^2 + P_2 n)(4m^2 \pi^2 + a^2 + P_2 n)(4m^2 \pi^2 + a^2 + n) +
+ 4m^2 \pi^2 Q \} \sin (2m \pi z) = - \sum_{m=1}^{J} A_m C_{2m} \sin (2m \pi z),
\]

where

\[
C_{2m} = - [4m^2 \pi^2 + a^2 + P_2 n] \{(4m^2 \pi^2 + a^2 + P_2 n) \times
\times (4m^2 \pi^2 + a^2 + n) + 4m^2 \pi^2 Q \}
\]

\[ (m = 1, 2, 3, \ldots, J). \]

From the linearity of the system, we have

\[
W(z) = \sum_{m=1}^{J} A_m W_m(z), \quad Z(z) = \sum_{m=1}^{J} A_m Z_m(z), \quad \chi(z) = \sum_{m=1}^{J} A_m \chi_m(z)
\]

and

\[
H(z) = \sum_{m=1}^{J} A_m H_m(z).
\]

Substituting for \( W(z) \) into equation (A2) leads to the general solution for \( W_m(z) \) given by

\[
W_m(z) = \sum_{j=1}^{5} B_j^{(m)} \sinh q_i z + C_{2m} \gamma_{2m} \sin (2m \pi z),
\]

where the \( B_j^{(m)} \) are constants of integration, the \( q_i \) are roots of the equation

\[
(q^2 - a^2)[(q^2 - a^2 - P_2 n)(q^2 - a^2 - n) - Q q^2] + T q^2 (q^2 - a^2 - P_2 n)^2 = 0,
\]

and

\[
\gamma_{2m}^{-1} = -(4m^2 \pi^2 + a^2)[(4m^2 \pi^2 + a^2 + P_2 n)(4m^2 \pi^2 + a^2 + n) +
+ 4m^2 \pi^2 Q] - 4m^2 \pi^2 T [4m^2 \pi^2 + a^2 + P_2 n]^2.
\]

Similar expressions result for \( Z_m(z) \), \( \chi_m(z) \) and \( H_m(z) \).
When the rigid, perfectly conducting boundary conditions are taken into account at \( z = -\frac{1}{2} \) and \( z = \frac{1}{2} \), the following equations for the constants \( B_j^{(m)} \) result:

(i) \( W_m(z) = 0 \) at \( z = \pm \frac{1}{2} \),

\[
\sum_{j=1}^{5} B_j^{(m)} \sinh \left( \frac{1}{2} q_j \right) = 0. \quad (A8)
\]

(ii) \( DW_m(z) = 0 \) at \( z = \pm \frac{1}{2} \),

\[
\sum_{j=1}^{5} B_j^{(m)} q_j \cosh \left( \frac{1}{2} q_j \right) = (-1)^{m+1} C_{2m} \gamma_{2m} (2m\pi). \quad (A9)
\]

(iii) \( Z_m(z) = 0 \) at \( z = \pm \frac{1}{2} \),

\[
\sum_{j=1}^{5} \frac{B_j^{(m)} q_j (q_j^2 - a^2 - P_2n) \cosh \left( \frac{1}{2} q_j \right)}{((q_j^2 - a^2 - P_2n)(q_j^2 - a^2 - n) - Qq_j^2)} = \frac{(-1)^m C_{2m} \gamma_{2m} (4m^2\pi^2 + a^2 + P_2n)(2m\pi)}{((4m^2\pi^2 + a^2 + P_2n)(4m^2\pi^2 + a^2 + n) + 4m^2\pi^2 Q)}. \quad (A10)
\]

(iv) \( DX_m(z) = 0 \) at \( z = \pm \frac{1}{2} \),

\[
\sum_{j=1}^{5} \frac{B_j^{(m)} q_j^3 \cosh \left( \frac{1}{2} q_j \right)}{((q_j^2 - a^2 - P_2n)(q_j^2 - a^2 - n) - Qq_j^2)} = \frac{(-1)^m C_{2m} \gamma_{2m} (2m\pi)^3}{((4m^2\pi^2 + a^2 + P_2n)(4m^2\pi^2 + a^2 + n) + 4m^2\pi^2 Q)}. \quad (A11)
\]

(v) \( H_m(z) = 0 \) at \( z = \pm \frac{1}{2} \),

\[
\sum_{j=1}^{5} \frac{B_j^{(m)} q_j \cosh \left( \frac{1}{2} q_j \right)}{(q_j^2 - a^2 - P_2n)} = \frac{(-1)^m C_{2m} \gamma_{2m} (2m\pi)}{(4m^2\pi^2 + a^2 + P_2n)}. \quad (A12)
\]

These equations are also consistent with the free, non-conducting boundary conditions at \( z = 0 \).

Substituting for \( F(z) \) and \( W(z) \) into equation (1) we have

\[
\sum_{m=1}^{J} A_m [4m^2\pi^2 + a^2 + P_1n] \sin (2m\pi z) = Ra^2 \sum_{m=1}^{J} A_m [C_{2m} \gamma_{2m} \sin (2m\pi z)] + \sum_{j=1}^{5} B_j^{(m)} \sinh q_j z. \quad (A13)
\]

Multiplying (A13) by \( \sin (2l\pi z) \) and integrating from \( z = -\frac{1}{2} \) to \( z = \frac{1}{2} \) leads to a linear homogeneous system of equations in \( A_m \), for which unique solutions exist if

\[
\left[ \frac{4l^2\pi^2 + a^2 + P_1n}{Ra^2} - C_{2\ell} \gamma_{2\ell} \right] \delta_{ml} / 2 - \left( \frac{l}{m} \right) = 0, \quad (A14)
\]

where

\[
\left( \frac{l}{m} \right) = \sum_{j=1}^{5} B_j^{(m)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sinh q_j z \sin (2l\pi z) \, dz = (-1)^{l+1} 4l\pi \sum_{j=1}^{5} \frac{B_j^{(m)} \sinh \left( \frac{1}{2} q_j \right)}{(q_j^2 + 4l^2\pi^2)} \quad (l, m = 1, 2, \ldots, J). \quad (A15)
\]