I. INTRODUCTION

The swirling flow driven in an enclosed cylinder by the constant rotation of one of its end walls gives rise to a rich array of dynamical behavior, including vortex breakdown and the existence of multiple solutions. Recently, a series of experiments with a variation on the above setup have been conducted. These examined the swirling flow in a cylinder driven by the constant rotation of its bottom end wall while the top was a free surface rather than a rigid wall. Hyun has studied this flow numerically; however, the study was restricted to the time-independent equations and covered only a very small part of parameter space. It did not reveal the rich dynamical behavior observed experimentally. 

Daube presented some calculations that match many aspects of the aforementioned experiments. However, the bifurcation structure of the flow was not examined and the onset of time dependence was assumed to be via a supercritical Hopf bifurcation, which the results here show not to be the case. 

Spohn et al. remark that for the parameter range considered in their experiments, the Froude number was negligibly small. The Froude number gives a relative measure of the extent to which the free surface is deformed by inertial forces such as the centrifugal force due to the swirling motion compared to the restoring gravitational force. Given that the Froude number is essentially zero under the conditions considered, the free surface can be treated as a flat stress-free surface. This flow then corresponds to the situation where the cylinder is twice as long and the flow is driven by the constant corotation of both a top and bottom rigid end wall rotating at the same rate in the same direction and the mid-plane is a plane of reflectional symmetry ($Z_2$ symmetry). In the corotating end walls situation, $Z_2$ symmetry is introduced into the flow, in contrast with the original flows considered by Escudier, Spohn, and Spohn et al. which had no symmetries in the meridional plane. Enforcing the $Z_2$ symmetry allows one to follow solution branches beyond the point where symmetry breaking would occur. The experiments of Spohn et al. essentially do this up to the point where free surface deformations become important. Valentine and Jahnke have also studied this flow numerically for the corotating end walls case, but have concentrated on the steady solutions that retain the $Z_2$ symmetry; for the most part, they imposed this symmetry and hence did not explore the nature of any symmetry-breaking bifurcations nor the nature of the onset of time dependence.

In the concluding remarks of Spohn et al. it is noted that the presence of the free surface leads to very different flow structures compared to when the top is a rigid end wall. It will be demonstrated that the formation of the recirculation zones, referred to as vortex breakdown bubbles, attached to the free surface are a result of the flow responding to axial gradients in the vortex lines, leading to the turning of meridional vorticity into azimuthal vorticity and inducing the reversed meridional circulations, just as in the case when the top is a rigid end wall. The vortex breakdown phenomenon in both the stationary top case and the case with the imposed $Z_2$ symmetry is qualitatively the same, as is their bifurcation structure. When the $Z_2$-symmetry condition is relaxed, the bifurcation structure of the corotating case may be richer due to the breaking of $Z_2$ symmetry. We find that the $Z_2$ symmetry is broken only in the time-dependent solutions, and then, only on "coarse" grids. All steady solutions without $Z_2$ symmetry imposed were found to be $Z_2$ symmetric.

II. GOVERNING EQUATIONS AND THEIR NUMERICAL SOLUTION

The equations governing the flow are the axisymmetric Navier–Stokes equations, together with the continuity equation and appropriate boundary and initial conditions. It is convenient to write these using a cylindrical polar coordinate system $(r, \theta, z)$, with the origin at the center of the bottom rotating end wall and the positive $z$ axial direction being toward the top. Since the flow is axisymmetric, there exists a Stokes streamfunction $\psi$ and the velocity vector in cylindrical polars is

$$\mathbf{u} = \left( -\frac{1}{r} \psi_z, \psi_r, \frac{1}{r} \psi_r \right). \quad (1)$$

Subscripts denote partial differentiation with respect to the subscript variable. This form of the velocity automatically satisfies the continuity equation. It is also convenient to in-
roduce a new variable, the angular momentum \( \Gamma = rv \). Here \( \Gamma \) is proportional to the circulation. The vorticity field corresponding to (1) is

\[
\omega = (\xi, \eta, \zeta) = \left( -\frac{1}{r} \Gamma_z, -\frac{1}{r} \nabla^2 \psi, \frac{1}{r} \Gamma_r \right),
\]

where

\[
\nabla^2 = (\ )_{zz} + (\ )_{rr} - \frac{1}{r} (\ )_r.
\]

The velocity and vorticity fields can be decomposed into azimuthal (\( \theta \)) and meridional (\( m \)) fields, where

\[
\begin{align*}
\mathbf{u}^\theta &= \left( 0, -\frac{1}{r} \Gamma, 0 \right), \\
\mathbf{\omega}^\theta &= \left( 0, -\frac{1}{r} \nabla^2 \psi, 0 \right), \\
\mathbf{u}^m &= \left( -\frac{1}{r} \psi_z, 0, \frac{1}{r} \psi_r \right), \\
\mathbf{\omega}^m &= \left( -\frac{1}{r} \Gamma_z, 0, -\frac{1}{r} \Gamma_r \right);
\end{align*}
\]

so \( \omega^\theta = \nabla \times \mathbf{u}^\theta \) and \( \omega^m = \nabla \times \mathbf{u}^m \). Further, \( \Gamma \) plays the role of a streamfunction for the meridional vorticity field.\( ^{10} \) In other words, contours of \( \Gamma \) in a meridional plane are cross sections of vortex surfaces (vortex lines), just as contours of \( \psi \) are cross sections of streamsurfaces (streamlines). These give the local direction of the vorticity and velocity vectors in the plane, respectively, and the azimuthal components of the vectors give the degree to which the vectors are directed out of the plane. If the flow were inviscid and steady, \( \Gamma \) would be a function of \( \psi \) alone.

The axisymmetric Navier–Stokes equations, in terms of \( \psi_t, \Gamma, \) and \( \eta \), are

\[
\begin{align*}
D \Gamma &= \nabla^2 \Gamma / \text{Re}, \\
D \left( \frac{\eta}{r} \right) &= \left[ \nabla^2 \left( \frac{\eta}{r} \right) + \frac{2}{r} \left( \frac{\eta}{r} \right)_r \right] / \text{Re} + \left( \frac{\Gamma^2}{r^2} \right)_z, \\
\end{align*}
\]

where

\[
\begin{align*}
\nabla^2 \psi &= -r \eta, \\
D &= \left( \right)_z - \frac{1}{r} \psi_z \left( \right) + \frac{1}{r} \psi_r \left( \right)_z, \\
\nabla^2 &= \left( \right)_{zz} + \left( \right)_{rr} - \frac{1}{r} \left( \right)_r.
\end{align*}
\]

and \( \text{Re} = \Omega R^2 / \nu \). The length scale is the radius of the cylinder \( R \), the time scale is \( 1/\Omega \), \( \Omega \) is the constant angular speed of the rotating end wall, \( \nu \) is the kinematic viscosity, and the other governing nondimensional parameter is the cylinder aspect ratio \( H/R \); \( H \) being the cylinder height in the case of a stationary rigid end wall, or, in the corotating end walls case, it is the distance from an end wall to the midplane of the cylinder.

Equation (3) shows that inertial change in the azimuthal vorticity, and hence a source of the overturning meridional flow, is driven by axial gradients in the angular momentum. The physical origin\( ^{11} \) of the source term on the right-hand side of (3) is the azimuthal component of \( \nabla \times (u^\theta \times \omega^m) \). It corresponds to the turning of meridional vorticity into the azimuthal direction by the azimuthal velocity. Brown and Lopez\( ^2 \) found the necessary condition, in the limit of steady, inviscid flow, such that the turning of the meridional vorticity produces azimuthal vorticity of the correct sign so as to locally induce a reversal in the direction of the meridional flow. The condition for this flow reversal to be possible is that the ratio of the tangents of the helix angles of the velocity and vorticity vectors be greater than unity on streamsurfaces upstream of the reversal in the meridional flow.

A. Computational technique

Details of the computational technique, together with accuracy and resolution tests, are given in Lopez\( ^1 \) for the stationary top end wall case. In summary, the governing equations are discretized on a uniform finite-difference grid using second-order central differences to approximate all spatial derivatives except those in the advection terms, which are approximated using the second-order conservative scheme of Arakawa.\( ^{12} \) No artificial viscosity is used; instead, a sufficiently fine grid is used to ensure proper resolution of spatial scales and grid-independent solutions. For the steady solution branch with \( \text{Re} \) up to approximately 3000 and \( H/R = 1.5 \), this is achieved on a uniform grid consisting of \( n_r = 61 \) nodes in the radial and \( n_z = 91 \) nodes in the axial directions. The steady computations of Valentine and Jahnke\( ^6 \) were performed on uniform grids using second-order central differences for \( \text{Re} \) up to 3000, and they report that there were no significant differences between the results using grid sizes of 1/60 and 1/120. Further grid refinement tests and the use of a stretched grid are detailed in the following section for higher \( \text{Re} \) flows than those considered in Lopez,\( ^1 \) including time-dependent solutions, and for the corotating end walls case.

Time integration uses the explicit alternating time-step scheme of Miller and Pearce.\( ^{13} \) For \( 1000 < \text{Re} < 4000 \), stability of the scheme is primarily governed by the Courant–Friedrichs–Levy condition, and this is amply satisfied by a time step \( \delta t = 0.05 \) on the 61X91 grid. For \( \text{Re} < 1000 \), the diffusion requirement,\( ^{14} \)

\[
\delta t < \delta r^2 \text{Re} / \Omega.
\]

dominates the stability of the scheme, where \( \delta r \) is the spatial resolution (all results presented have \( \delta r = \delta z \)).

A number of different types of initial conditions have been employed in this study. The first consists of an impulsive start from rest, where initially all cylinder walls and the fluid are at rest. At \( t = 0 \), one end wall or both end walls (depending on the particular case being studied) are set to rotate at a constant angular speed \( \Omega \). A second type of initial condition consists of taking a steady solution at some \( \text{Re} \) as the initial condition for a calculation with a different value of \( \text{Re} \). This second initiation was used to continue the steady branch to large \( \text{Re} \), using small increments in \( \text{Re} \) so as not to start the calculations outside the basin of attraction of the
steady solution branch. A steady state is determined to have been reached when the relative change between $\psi$, $\eta$, and $\Gamma$ at time step $k$ and at time step $k+1$, at all grid points, is less than $10^{-6}$. Another initial condition, albeit nonphysical, consists of a fully developed flow in the bottom half (the flow with $Z_2$ symmetry imposed at the same or nearby parameter values) and setting everything to zero in the top half of the cylinder. This initial condition is used to determine the robustness of the $Z_2$ symmetry.

The boundary conditions on the free surface are $\eta=0$, $\Gamma=0$, and $\psi=0$; on the axis of symmetry, $\eta=0$, $\Gamma=0$, and $\psi=0$; on stationary rigid walls, $\eta=-\psi_n/r$, $\Gamma=0$, and $\psi=0$; and on rotating rigid end walls, $\eta=-\psi_n/r$, $\Gamma=r^2$, and $\psi=0$. The subscript $n$ denotes differentiation normal to the wall. One-sided differences from a Taylor-series expansion to second-order centered about one grid point in from the boundary are used to discretize derivative boundary conditions. Use is also made of the fact that normal derivatives of $\psi$ on rigid walls are zero. The discrete version of the vorticity boundary conditions are $\eta(i,j)=0$; $\eta(i,n+1,j)=2(\psi(i,j)-\psi(i,j-1))/\delta z^2$; $\eta(i,1,1)=2(\eta(i+1,1,1)-\eta(i,1,1))/\delta r^2$ (stationary rigid top); $\eta(i,n,1)=0$ (corotating end walls with imposed $Z_2$ symmetry); and $\eta(i,2n-1,1)=-2(\psi(i,2n-2,1)-\psi(i,2n-1,1))/\delta r^2$ (corotating end walls without imposed $Z_2$ symmetry).

### B. Grid resolution study

Of great concern to this study is the question of sufficient grid resolution. This is particularly important, as the discretized set of governing equations constitute a finite-dimensional dynamical system whose bifurcation structure (singular points of the dynamical system) can differ significantly from those of the continuous equations (the infinite-dimensional dynamical system). There are numerous examples in the literature demonstrating this. Particular examples are that of solutal convection and flow past a sphere in a pipe, where the existence of spurious Hopf bifurcations as a result of insufficient grid resolution have been revealed.

In regions of parameter space where only a unique steady solution exists in the continuous system, beyond a certain level of resolution in the discrete system, the corresponding solution (termed the “low” resolution solution) does not change qualitatively as the level of grid resolution is increased, and the solution changes quantitatively in an asymptotic manner, converging to the solution of the continuous system. There are formal theoretical results for the Galerkin discretization of the Navier–Stokes equations, which suggest generalizations to other discretizations. Constantin, Foias, and Temam have shown that if the computed approximations of the time-dependent equations “seem to converge” to some limit as $t \to \infty$, then the same is true for the exact problem and the two limits are related.

However, in regions of parameter space where multiple solutions exist, formal results concerning convergence of the discrete system appear to be lacking. This situation is far more complicated. In the previous situation, the phase space of the system had only one attractor (the unique steady solution). Now there is more than one attractor and hence one needs to consider not only whether the attractors are converging to those of the exact problem, but also if their basins of attraction are converging. Even if the level of resolution is such that the attractors of the discrete system are asymptotically similar to those of the exact problem, their basins of attraction may not have converged at that level of resolution, and hence conclusions about the stability of the attractors of the continuous system may not be correct. Also, for similar initial conditions, evolution on systems of different resolution may not end up on the “same” attractor, because the structure of the basins of attraction may be such that the initial condition at one level of resolution is in the basin of attraction of a different attractor to that at another level of resolution. Determining the structure of the basin of attraction for the Navier–Stokes equations is a nontrivial exercise. Consider how complicated it can be for relatively simple equations (the Julia set and the Mandelbrot set).

Here, we determine the level of grid resolution required, within a region of parameter space, for the discretized equations to give consistent (i.e., not changing with further grid resolution) qualitative behavior (the solutions may continue to change quantitatively in an asymptotic manner). We do this in a number of ways. First, in a region of parameter space where only a single steady state exists, we determine the level of grid resolution required for the solution to be quantitatively in the asymptotic region. We monitor the maximum and minimum values of $\psi$ and $\eta$ on the grid points, both on a uniform grid and a stretched grid. Next, using a time-dependent code, we take the steady solutions with different levels of resolution and use continuation to follow the steady branch to higher $Re$. By increasing the grid resolution and reducing the increments in the continuation parameter (typically $Re$), the level of grid resolution for consistent qualitative behavior is determined.

The stretched grid is given by

$$r=x-a \sin(2\pi x)$$

and

$$z=[y-b \sin(2\pi y)]H/R,$$

where $x=il(nr-1)$, for $i=0\ldots(nr-1)$; $y-j(nz-1)$, for $j=0\ldots(nz-1)$; and in all the results presented here, the stretching factors $a=b=0.1$ have been used. This stretching places the grid points more densely near the boundaries, the axis, and the midplane when $Z_2$ symmetry is imposed. When employing the stretched grid, second-order central differences are used to discretize the equations spatially. We begin by determining a suitable level of resolution at a large $Re=2600$, where the only solution is steady (see Sec. IV for this determination). Table I gives the extreme values of $\psi$ and $\eta$ for the case $Re=2600$, $H/R=1.5$, both on the uniform and the stretched grids of various sizes, together with the $\delta t$ used. The $\delta t$'s are approximately the smallest on a given grid for which the system was stable. This information, together with the visual information provided by the plots of the solutions in Fig. 1, indicates that the 61x91 uniform grid solution is in the asymptotic range of the continuous solution. Given that $\delta t$ on a uniform grid with about the same number of grid
TABLE I. Minimum and maximum values of $\phi$ and $\eta$ on the grid points of the various grids, as indicated, for $Re=2600$ and $H/R=1.5$ at steady state.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\delta t$</th>
<th>$\min(\phi)$</th>
<th>$\max(\phi)$</th>
<th>$\min(\eta)$</th>
<th>$\max(\eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31\times 46 uniform</td>
<td>$1.0\times 10^{-1}$</td>
<td>$-7.741 \times 10^{-3}$</td>
<td>$2.860 \times 10^{-4}$</td>
<td>$-6.322$</td>
<td>$11.08$</td>
</tr>
<tr>
<td>61\times 91 uniform</td>
<td>$5.0\times 10^{-2}$</td>
<td>$-7.709 \times 10^{-3}$</td>
<td>$3.777 \times 10^{-4}$</td>
<td>$-4.496$</td>
<td>$17.58$</td>
</tr>
<tr>
<td>121\times 181 uniform</td>
<td>$2.3\times 10^{-3}$</td>
<td>$-7.608 \times 10^{-3}$</td>
<td>$3.747 \times 10^{-4}$</td>
<td>$-4.322$</td>
<td>$20.62$</td>
</tr>
<tr>
<td>241\times 361 uniform</td>
<td>$7.5\times 10^{-4}$</td>
<td>$-7.555 \times 10^{-3}$</td>
<td>$3.714 \times 10^{-4}$</td>
<td>$-4.217$</td>
<td>$21.42$</td>
</tr>
<tr>
<td>51\times 76 stretched</td>
<td>$1.0\times 10^{-2}$</td>
<td>$-7.539 \times 10^{-3}$</td>
<td>$4.234 \times 10^{-4}$</td>
<td>$-4.196$</td>
<td>$20.94$</td>
</tr>
<tr>
<td>101\times 151 stretched</td>
<td>$5.0\times 10^{-3}$</td>
<td>$-7.537 \times 10^{-3}$</td>
<td>$3.832 \times 10^{-4}$</td>
<td>$-4.213$</td>
<td>$21.49$</td>
</tr>
<tr>
<td>151\times 226 stretched</td>
<td>$2.5\times 10^{-3}$</td>
<td>$-7.532 \times 10^{-3}$</td>
<td>$3.756 \times 10^{-4}$</td>
<td>$-4.215$</td>
<td>$21.60$</td>
</tr>
</tbody>
</table>

As on a stretched grid is larger, the 61\times 91 uniform grid is preferred because it is sufficiently accurate, essentially grid independent, and efficient, at least for the steady solution branch up to $Re\approx 3\times 10^3$.

Testing the grid independence of the solutions in regions of parameter space where multiple solutions exist is more complicated. For $Re>2600$ (and $H/R=1.5$), we have found that using grids with less resolution than the uniform 61\times 91 grid, the steady solution branch loses stability at $Re\approx 2650$ (the exact value of $Re$ depends on the grid resolution). However, for grids with at least the resolution of the uniform 61\times 91 grid, the steady solution branch remains stable up to $Re=3200$. These results are detailed in Sec. III. We have confirmed this by recalculation of a part of the steady branch on the 101\times 151 stretched grid using $\delta t=0.005$ (the uniform 61\times 91 grid used $\delta t=0.05$). The steady solutions were continued from the unique steady solution at $Re=2600$. Using $Re=50$, the steady $Re=2650$ was reached after approximately 1000 time units. Figure 2 gives the time series of $\psi(nr=51, nz=76)$ for the continuation runs. The transient oscillations in the $Re=2650$ calculation, initiated with the $Re=2600$ steady solution, begin to grow, but are soon highly damped as the flow adjusts to the sudden increase in $Re$. Physically, this corresponds to an impulsive increase of $\approx 2\%$ in the rotation rate of the disk, $\Omega$. If a larger $\delta Re$ is used, the initial transients are not damped and the flow evolves to the periodic solution branch. If the same $\delta Re$ is used, starting from the $Re=2650$ steady solution, the flow evolves to the periodic branch. However, a smaller $\delta Re=25$ results in a steady $Re=2675$ solution, and from that solution to the $Re=2700$ solution. To continue beyond $Re=2700$, a $\delta Re=10$ had to be employed. All the transients, up to $Re=2790$, using this $\delta Re$, are damped; as illustrated in Fig. 2. Using $\delta Re=10$ at $Re=2790$ to continue the branch to $Re=2800$ was not successful and the flow evolved to the periodic branch. A detail of the corresponding time series is presented in Fig. 3.
Continuing this calculation for a further 2000 time units results in the periodic solution detailed in Fig. 4 (discussed in Sec. IV). A calculation such as this is suggestive of a supercritical Hopf bifurcation. However, if instead of using \( \delta \Re \) = 10 we use \( \delta \Re \) = 5 from the steady \( \Re \) = 2790 solution, the steady \( \Re \) = 2800 solution is reached, as illustrated in Fig. 2. There is no supercritical Hopf bifurcation here, merely the existence of two stable solution branches, one of them steady and the other time periodic, and which one is realized depends on the initial conditions. Figure 5 gives the steady solutions at \( \Re \) = 2790, 2795, and 2800. Note that the three solutions are virtually indistinguishable, yet the \( \Re \) = 2790 solution is to "far away" (i.e., outside the basin of attraction) from the \( \Re \) = 2800 solution, yet the \( \Re \) = 2795 solution is not, illustrating how sensitive the stable solutions on the steady branch are. Compare the solution for \( \Re \) = 2800 in Fig. 5, on the stretched 101 \( \times \) 151 grid with that in Fig. 6(i), on the uniform 61 \( \times \) 91 grid; they are in very close agreement. The qualitative behavior of the steady branch is the same on both the 61 \( \times \) 91 uniform grid and the higher resolution 101 \( \times \) 151 stretched grid.

The grid resolution study provides confidence in the accuracy of the determined stability of the steady branch, and also that the time-dependent branch is disjoint from the steady branch, or at least it does not originate via a supercritical Hopf bifurcation from it (see Sec. IV).

**III. PRODUCTION OF SECONDARY FLOW BY VORTEX LINE BENDING**

The flows under consideration are a part of a larger class of confined flows driven by angular momentum gradients. The primary motion is due to externally imposed rotation imparting a vertical (i.e., parallel to the rotation axis) component of vorticity to the flow. The particular details of the enclosing geometry imposes kinematic constraints on the flow resulting in the bending of vortex lines. The bending of vortex lines produces secondary motions, primarily associated with the azimuthal component of vorticity. In a large class of these flows, the secondary motions can be comparable to the primary motion, leading to significant nonlinear interactions. One such interaction manifests itself as recirculation bubbles in the interior of the flow, and is often referred to as vortex breakdown. The particular details of the topology of these recirculation zones depends on the details of the geometry of the container and the strength of the driving force, yet they all result from the bending of vortex lines. A great deal is made of the differing details of the streamsurfaces of these recirculation zones, with suggestions that these differences may reflect different processes at work. However, the topology of the streamsurfaces is not a particularly useful diagnostic for uncovering the dynamical processes at play. More suited is a study of the vortex lines (or vortex surfaces) and how the geometry of the container bends these, producing the rich array of secondary motions.

**A. Vortex line structure of the flows**

For the confined flow driven by the bottom rotating end wall with a flat stress-free top surface, the primary flow is due to the rotating end wall. The vortex lines all emanate from this end wall, as the fluid in contact must move with it. The sidewall of the cylinder, being stationary, is a vortex surface corresponding to \( \Gamma = 0 \) and the axis of symmetry, \( r = 0 \), is also a vortex line corresponding to \( \Gamma = 0 \). The vortex lines emanating from the rotating end wall correspond to \( \Gamma(r, 0) = r^2 \) distribution. Hence, the corner where the rotating end wall and the stationary sidewall meet is singular, with \( \Gamma \) varying from 1 on the end wall to 0 on the sidewall. Vortex lines may terminate at this corner. For the case where the top is rigid and stationary, all the vortex lines emanating from the rotating end wall must terminate at this corner. However, when the top is a flat stress-free surface, or equivalently, at the midplane when \( z \) is symmetry imposed in the corotating end walls case, the vortex lines have the option of meeting the surface (midplane) orthogonally. This option is responsible for the secondary motions being different to those of the stationary rigid top case. Also, the vortex lines meeting the midplane divide the fluid into an inner and an outer region. The inner region is \( r < r_1(z) \), where \( r_1(z) \) is the location of the vortex line emanating from the rotating end walls (but not from the corners) with largest radius, which meets the midplane. The two regions are particularly distinct at larger \( \Re \) and their interface supports waves (see Sec. IV).

As in the case of a rigid top, the bending of the vortex lines in the limit of creeping flow is due solely to the kinematic constraints of the container. The vortex line bending

**FIG. 2.** Time series of \( 10^3 \psi (nr=51, nz=76) \) for the continuation run on the 101 \( \times \) 151 stretched grid using \( \delta t = 0.005 \), started from the steady \( \Re = 2600 \) solution, the values of \( \Re \) following each impulsive, incremental increase are noted on the plot.

**FIG. 3.** Details of the time series of \( 10^3 \psi (nr=51, nz=76) \) for the continuation run on the 101 \( \times \) 151 stretched grid using \( \delta t = 0.005 \), started from the steady \( \Re = 2780 \) solution, using \( \delta \Re = 10 \), leading to a periodic solution for \( \Re = 2800 \).
FIG. 4. Eleven equally spaced phases over one period ($T=28.2$) of the periodic $Z_2$-symmetric solution for $Re=2800$ and $H/R=1.5$, and its time average, calculated on the $101 \times 131$ stretched grid with $\delta t=0.005$. The contours are as determined in Fig. 1.

leads to axial gradients in $\Gamma$ and via the term $(\Gamma^2/r^4)_z$ in Eq. (3), azimuthal vorticity is produced, inducing the secondary meridional flow. This secondary meridional flow is responsible for the establishment of the Ekman layer on the rotating end wall and at larger $Re$, advects high $\Gamma$ fluid from the Ekman layer into the interior. In so doing, the secondary meridional flow sweeps the vortex lines near the Ekman layer radially outward. This nonlinear interaction between the primary flow (vertical vorticity) and the secondary flow (azimuthal vorticity) causes further bending of the vortex lines and an enhancement of the meridional flow transporting further fluid with high $\Gamma$ into the interior. This process near the rotating end wall is essentially independent of the form of the top, at least qualitatively, for small $Re$. Hyun\(^5\) also made this observation. However, how the high $\Gamma$ fluid behaves near the top does depend on whether it is a rigid or a flat stress-free surface, as conditions there on $\Gamma$ are different. The conditions on $\eta$ are also different, but $\psi=0$ on the top regardless of whether it is stress-free or rigid. The condition $\psi=0$ at the midplane does not necessarily apply in the corotating end wall case if $Z_2$ symmetry is not imposed.

B. Flow development for increasing $Re$

At low $Re$ ($<1$), the flow is essentially at the limit of Stokes (creeping) flow. At this limit, the flow has all the symmetries of the container, i.e., azimuthal symmetry, and in the corotating end walls case, both the flow and the container have $Z_2$ symmetry. The results of Serrin\(^6\) ensure that in the limit of low $Re$, the flow is unique and steady. The primary

FIG. 5. Steady $Z_2$-symmetric solutions for $H/R=1.5$ and (a) $Re=2790$, (b) $Re=2795$, and (c) $Re=2800$; calculated on the $101 \times 131$ stretched grid with $\delta t=0.005$. The contours are as determined in Fig. 1.

The majority of the vortex lines emanating from the rotating end walls, in the Stokes flow limit, are bent and terminate at the corner \((r=1, z=0)\). Only the vortex lines corresponding to \(\Gamma=1\) meet the symmetry plane. This bending results in a negative (positive) axial gradient in \(1\), turning the meridional vorticity into the negative (positive) azimuthal direction in the bottom (top) half of the cylinder, and hence induces meridional circulations. These secondary circulations consist of radial outflow at the rotating end walls, flow

![FIG. 6. Steady \(Z_2\)-symmetric solutions for \(Re\) as indicated and \(H/R=1.5\), calculated on the 61×91 uniform grid with \(\delta t=0.05\). The contours are as determined in Fig. 1.](image)

![FIG. 7. Contours of \(v/r\) (determined from the steady \(Z_2\)-symmetric solutions calculated on a 61×91 uniform grid with \(\delta t=0.05\)) at the \(Z_2\)-symmetry plane versus \(Re\). The contours are uniformly spaced and the increment between them is 1/40. At \(r=1\), \(v/r=0\); \(v/r\) is a maximum at \(r=0\) and \(Re=650\). From that maximum, there extends a maximum ridge line toward larger \(Re\) with \(r=0.25\). There is also a local minimum in \(v/r\) at \(r=0\) and \(Re=1450\).](image)

![FIG. 8. State diagram using \(\eta_{max}\) as the state variable, for \(H/R=1.5\). This diagram was primarily determined from the 61×91 uniform grid solutions and “spot-checked” using the 101×151 stretched grid solutions.](image)
FIG. 9. Eleven equally spaced phases over one period (T=28.5) of the periodic \( Z_2 \)-symmetric solution for \( Re=2640 \) and \( H/R=1.5 \), and its time average calculated on the 61\( \times \)91 uniform grid with \( \delta t=0.05 \). The contours are as determined in Fig. 1.

FIG. 10. Variation of the period of oscillation, \( T \), with \( Re \) on the \( T=28 \) periodic solution branch.
symmetry plane orthogonally. This in effect produces a radial jet of high fluid being injected into the interior at the symmetry plane.

Spohn et al., based on inviscid arguments, suggest that the radial jet of high fluid injects angular momentum in toward small radii only up to the point where the angular velocity of the fluid, \( \omega \), increases to match the angular velocity of the rotating disk, \( \Omega \), which is equal to 1 in non-dimensional units. At about this point (it will vary slightly due to viscous effects) the meridional flow no longer continues to flow radially inward, but instead stagnates on the free surface and turns into the axial direction, thereby forming the observed toroidal recirculation zone attached to the free surface. For low Re flow, the viscous effects near the free surface (even though there is no rigid wall, there are still axial gradients in the velocity contributing to viscous stresses near the free surface) are sufficient for the flow to dissipate enough angular momentum as it flows radially inward so that \( \omega / r \) on the free surface remains sufficiently less than 1 and no separation is required. This experimentally observed behavior is also observed in our calculations of the corotating case with \( Z_2 \) symmetry imposed, where the symmetry plane corresponds to the free surface in the experiments.

Figure 7 shows the development of \( \omega / r \) on the symmetry plane for increasing Re along the steady solution branch. At low Re, \( \omega / r \) is negligibly small on the symmetry plane. With increasing Re, \( \omega / r \) increases monotonically from zero with decreasing r. At Re=680, \( \omega / r \) has a local maximum of \( \approx 0.82 \) at \( r = 0 \). It is at about this Re that \( \eta \) near the symmetry plane changes sign [cf. Figs. 6(d) and 6(e)]. At this Re, the meridional flow has not stagnated on the symmetry plane, but as Re increases, the radius at which \( \omega / r \) attains its maximum value of \( \approx 0.82 \) moves farther out from the axis and the region of positive (negative) \( \eta \) intensifies. By Re=800 [Fig. 6(f)], the locally increased \( |\eta| \) is large enough to induce a reversed meridional flow halting the inward meridional flow provided by the Ekman pumping on the rotating end walls. This results in stagnation on the symmetry plane and the formation of a recirculation zone, termed a vortex breakdown by Spohn et al. Certainly, the reversed meridional flow here is induced by the azimuthal vorticity \( \eta \) resulting from the bending of the vortex lines, just as are the recirculation zones on the axis in both the corotating end walls case and the stationary rigid top case. The radius at which \( \omega / r \) is a maximum corresponds more closely with the radius at which \( \eta \) changes sign near the symmetry plane than with the radius at which the meridional flow stagnates. For Re>700, \( \omega / r \) decreases toward \( r = 0 \), and there is a local minimum in \( \omega / r \) at \( r = 0 \) for Re=1450. From Fig. 6(g), corresponding to Re=1500, \( \omega / r \) has a maximum at \( r = 0.25 \) and a local minimum at \( r = 0.15 \). The maximum corresponds to the change in sign...
of $\eta$ as $r$ decreases and the minimum corresponds to a second change in the sign of $\eta$ as $r$ decreases further. For increasing Re, the inner region of negative (positive) $\eta$ induces radial inflow. By Re=2000 [Fig. 6(h)], this induced radial inflow is large enough to reattach the separated flow onto the symmetry plane, forming the toroidal recirculation zone. Figure 6(h) is in very close agreement with the flow visualization of Spohn et al.\textsuperscript{5} in their Fig. 5(a), corresponding to Re =2095, $H/R = 1.5$.

Figure 8 is a state (bifurcation) diagram for the corotating end walls flow. The bifurcation parameter is Re (a fixed $H = 1.5$ is employed) and the quantitative measure of the flow state is $\eta_{\text{max}}$, the maximum positive value of $\eta$ in the interior of the bottom half of the cylinder. For periodic flows (Sec. IV), $\eta_{\text{max}}$ is the maximum positive value of $\eta$ of the time-averaged flow in the interior of the bottom half of the cylinder. The interior is defined as being the region in from the boundaries delineated by the first zero contour of $\eta$ (note that $\eta=0$ on and near the rigid boundaries in the bottom half and $\approx 0$ in the top half of the cylinder; see Fig. 6). Any quantitative measure of the flow state can be used, and $\eta_{\text{max}}$, a local measure, is not necessarily optimal. However, the qualitative picture of the state diagram is unaffected by the choice and $\eta_{\text{max}}$ is convenient and illustrates some of the flow physics.

IV. ONSET OF UNSTEADY $Z_2$-SYMMETRIC FLOW

The steady solution branch (Fig. 8) described in the preceding section remains stable to time-dependent and $Z_2$-symmetry breaking disturbances (at least for Re =3200). As in the case of the rigid stationary top, a disjoint periodic branch has been found. For $H/R = 1.5$, it originates as a turning point bifurcation at Re =2640 (at Re =2630, only a steady solution is found to exist; whereas, at Re =2640, both a periodic and a steady solution exist). In order for the time-periodic branch to originate as a subcritical Hopf bifurcation from the steady branch, the steady branch would have to be unstable beyond the Re corresponding to the subcritical Hopf bifurcation, but we find the steady solution to be stable at least for Re =3200. The basin of attraction of the steady branch becomes increasingly small for increasing Re and a continuation in Re for Re =3200 requires increments, $\delta$Re, of the order of 0.01% in order to guarantee that the initial conditions remain within the basin of attraction of the steady solution. For Re =2650, increments of the order of 2% are small enough. Some further discussion of the dependence of the basin of attraction on the grid resolution is provided in Sec. II B. Such a small basin of attraction gives the impression that the onset of time dependence is via a supercritical Hopf bifurcation (Dauhe\textsuperscript{7} assumes this to be the case, but
does not explore other possibilities). This impression is caused if the initial conditions are placed outside the basin of attraction of the steady branch, or if an imposed disturbance (physically, this could be due to a slight wobble in the rotating end walls, a nonconstant rotation rate $\Omega$, or temperature fluctuations) perturbs the flow out of the basin. The resulting evolution to a time-periodic state only shows that the steady solution is unstable to *finite-amplitude* disturbances, but does not allow conclusions concerning its *linear* stability. Valentine and Jahynke conclude that their steady solution at $Re = 3000$, $H/R = 1.5$, is unstable to *finite-amplitude* disturbances, the disturbances being the differences between the central difference used to discretize the steady equations to get their steady solution and the upwind differencing used in their time-dependent code. They did not attempt to do time-dependent calculations with more carefully selected initial conditions. Calculations with carefully selected initial conditions placed within the basin of attraction of the steady solution branch, such as those reported here, clearly show the steady solutions to be stable and to coexist with the periodic solutions. Hence, a supercritical Hopf bifurcation from the steady branch cannot be the origin of the periodic solution branch. Otherwise, the steady branch would have to be *linearly* unstable beyond the supercritical Hopf bifurcation point, and a time-dependent calculation using the steady solution as an initial condition, together with a perturbation of any size (even as small as numerical roundoff), would evolve away from the steady solution. As detailed in Sec. II B, on a sufficiently refined grid, this does not happen.

The attractiveness (i.e., larger, in some sense, basin of attraction) of the periodic solution over the steady solution can be understood as follows. For axisymmetric steady flows in the limit as $Re \to \infty$, the streamlines ($\phi$) and the vortex lines ($\Gamma'$) must coincide. However, at the symmetry plane (or flat stress-free surface) they must be orthogonal. At low $Re$, viscosity acts to adjust the flow, but as $Re$ increases, the flow must either lose its axial symmetry or become unsteady in order that the streamlines and vortex lines need not coincide. The experiments of Spohn et al. suggest that the flow first becomes unsteady rather than nonaxisymmetric as $Re$ is increased.

In the periodic flow, the radial jet of high $\Gamma'$ flow at the midplane overshoots the mark at which $u/r^2=1$, and recoils, overcorrecting, then overshoots again and so on. At $Re = 2640$ (see Fig. 9), this results in a weak periodic wobbling, in the radial direction, of the toroidal recirculation bubble. The oscillation is highly damped in the axial direction. As $Re$ is increased, this behavior is further enhanced. By $Re=2800$, each wobble sends a pulse in the axial direction along the interface between the inner flow (i.e., the region near the axis where the vortex lines originating on the end walls meet the symmetry plane orthogonally) and the outer flow. This pulse

![Figure 13](image-url)
just reaches the rotating end wall and is reflected back along the axis. At this Re, the reflected pulse is highly damped by viscous effects, and is just discernible in Fig. 4.

Unlike the rigid top end wall case, where the frequency of oscillation for a particular aspect ratio is independent of Re over a large range, the frequency of the flat stress-free surface case does show a slight variation with Re. This was also noted by Daube in his calculations. Figure 10 shows the variation of the period of oscillation, $T$ (nondimensionalized by $\Omega$), with Re for the $Z_2$-symmetric flow with $H/R = 1.5$ impulsively started from rest. The periodic flow is first observed at Re=2640 with a period $T \approx 28.5$, and, at Re=2800, the period has decreased to $T \approx 28.2$. Figure 10 is constructed using both the 61x91 uniform grid and the 101x151 stretched grid. The period at Re=2800 was determined on both grids and agreed to three figures. At Re=3030, there is period doubling to a flow with $T \approx 56.0$, when calculated on the 61x91 uniform grid. At Re=3050, also on the 61x91 uniform grid, the flow evolves from rest to the period $T \approx 28$ flow and eventually changes to one with $T \approx 18.75$. As Re is further increased, using the same grid, the period increases in contrast to the trend at lower Re, and at Re=3350 a further

FIG. 13. (Continued.)

FIG. 14. Time series of $10^3\psi (n_x=51, n_z=150)$ for Re=3200, $H/R = 1.5$ calculated on the 101x301 stretched grid using $\Delta t = 0.005$, started at $t=3750$ from the asymmetric initial condition described in Sec. II A.
period doubling is observed. However, if these higher Re cases are computed on the 101×151 stretched grid, the aforementioned higher-frequency modes and the period doublings are not observed. Calculations started from rest on the stretched grid at Re=2800, 3200, and 3500 all evolved to the single-frequency periodic flow on the T=28 branch, the only time-dependent branch found with the 101×151 stretched grid (see Figs. 8 and 10) when $Z_2$ symmetry is imposed and the calculations are impulsively started from rest. This strongly suggests that the higher-frequency modes and the period doublings are numerical artifacts, due to insufficient resolution. Figure 11 shows the Re=3500 flow over one period ($T=27.5$). The overshooting of the high radial jet at the symmetry plane causing a pulse to travel along the interface between the inner and outer regions to the rotating end walls, and then its reflection along the axis, as described earlier for Re=2800 (Fig. 4) is more enhanced at this higher Re.

V. $Z_2$-SYMMETRY BREAKING

So far, only the $Z_2$-symmetric flow has been considered. The experimental flow with a free surface is the physical analog of this flow, up to the point where free surface deformations are no longer negligible. In this section, we consider the possible breaking of the $Z_2$ symmetry (note that this is not related to free surface deformations). Azimuthal symmetry continues to be imposed, and the breaking of this symmetry remains an open question.

Impulsively started calculations from an initial state of rest without imposed $Z_2$ symmetry reach the same steady state as when the symmetry is imposed for Re≤2600 on both the 61×91 uniform and the 101×151 stretched grids (when $Z_2$ symmetry is not imposed, the grid has $nr\times[2(nz-1)+1]$ grid points, i.e., 61×181 and 101×301 on the uniform and stretched grids, respectively). For 2640≤Re≤2800, the same $Z_2$-symmetric periodic flow was reached as when symmetry was imposed, using either grid. These $Z_2$-symmetric solutions are reached even when very nonsymmetric initial conditions are used (the third initial condition described in Sec. II A). For Re=2800, these initial conditions evolve to the steady $Z_2$-symmetric solution branch. For 2640≤Re≤2800, these initial conditions evolve to the $Z_2$-symmetric periodic branch. However, on the 61×91 uniform grid, for 2850≤Re≤3000, a $Z_2$-symmetric periodic flow was reached, but its period ($T=28.2$) was different to that of the flow when $Z_2$ symmetry was explicitly imposed ($T=28.2$). The initial condition consisted of the $T=18.6$ flow in the lower half and zero flow in the top half. It is curious that $18.6\times1.5=28.2$, remembering that here $H/R=1.5$.

Here $Z_2$-symmetry breaking on the 61×91 uniform grid

![Figure 15](https://example.com/figure15)

**FIG. 15.** Eleven equally spaced solutions, at times as indicated, and their average over that time period, of the modulated periodic flow with $Z_2$ symmetry not imposed for Re=3200, $H/R=1.5$, calculated on the 101×301 stretched grid using $\delta t=0.005$, started at $t=3750$ from the asymmetric initial condition described in Sec. II A. The contours are as determined in Fig. 12.

has only been observed from the periodic branch and for \( \text{Re} \geq 3020 \). This is consistent with the conclusion of Valentine and Jahnke\(^9\) that for \( \text{Re} \approx 3000 \), the \( Z_2 \) symmetry is robust. However, the \( Z_2 \) symmetry breaking observed on the \( 61 \times 91 \) uniform grid for \( \text{Re} > 3020 \) appears to be due to insufficient grid resolution. If the same asymmetric initial conditions are used in a calculation on the \( 101 \times 151 \) stretched grid, the flow eventually, after about 1000 units, evolves to the same \( Z_2 \)-symmetric periodic flow as that found when \( Z_2 \) symmetry was imposed. This resymmetrization of the flow is illustrated in Fig. 12 for \( \text{Re} = 3500 \). For the first 100 or so time units of the evolution, the flow is very non-\( Z_2 \) symmetric. However, by \( t = 500 \), it has settled down to an almost \( Z_2 \) symmetric flow; the small perturbations away from \( Z_2 \) symmetry are gradually damped, but are still discernible at \( t = 1250 \), particularly in the \( \eta \) contours about the midplane. By \( t = 2025 \), as illustrated in Fig. 13, the flow has settled onto the \( Z_2 \)-symmetric, \( T \approx 28 \) periodic branch found when \( Z_2 \) symmetry was imposed. Figure 13 shows the flow over one period of this oscillation, together with its time average. The figure should be compared with Fig. 11, showing the \( T \approx 28 \) periodic flow for the same \( \text{Re} \) and \( H/R \) when \( Z_2 \) symmetry is imposed; the two flows are the same. This would suggest that the \( Z_2 \) symmetry is very robust. However, the situation is not entirely resolved. A similar calculation at \( \text{Re} = 3200 \) does not resymmetrize on the \( 101 \times 151 \) stretched grid, but instead evolves to a modulated, \( T \approx 18.9 \), non-\( Z_2 \)-symmetry flow (the time series for this flow is shown in Fig. 14 along with contours of the flow over approximately 18.9 time units in Fig. 15, showing the “flip-flopping” of the high \( \Gamma \) radial jet across the midplane), similar to that found on the \( 61 \times 91 \) grid. Whereas a calculation started impulsively from rest at the same parameter values without the \( Z_2 \) symmetry imposed evolves to the \( T \approx 28 \) periodic \( Z_2 \)-symmetric flow found when the symmetry is imposed. This all suggests that the \( Z_2 \)-symmetry flows are stable for \( \text{Re} \leq 3500 \), but that there also exist non-\( Z_2 \)-symmetric flows in that parameter range. However, it is not clear whether the observed non-\( Z_2 \)-symmetric flows are present due to insufficient grid resolution, or if the non-\( Z_2 \)-symmetric flow branch is actually disjoint from the \( Z_2 \)-symmetric branches and we are able to reach it via “nonstandard” initial conditions. In order to determine this, one would need to double the grid points to \( 201 \times 601 \) and then probably to \( 401 \times 1201 \), with a corresponding halving of \( \Delta t \) to 0.0001 and 0.00005, until consistent dynamical behavior is achieved on two consecutive doublings of the resolution, and compute several cases at various \( \text{Re} \) with a variety of initial conditions. Such a study, given the long evolution times required (about 10 000 time units) is simply beyond the computational resources presently available to the author.


