Rigorous proof of two conjectures for the distribution of money

Nicolas Lanchier
Joint work with Stephanie Reed
– Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. 
– Finite connected graph \( \mathcal{G} = (V, E) \).
– Each vertex is an agent with some coins.
– Finite connected graph $G = (V, E)$.
– Each vertex is an agent with some coins.
– Number of agents $= \text{card}(V) = N$. 
– Finite connected graph $G = (V, E)$.
– Each vertex is an agent with some coins.
– Number of agents $= \text{card} (V) = N$.
– Number of coins $= M$. 
– Finite connected graph $G = (V, E)$.
– Each vertex is an agent with some coins.
– Number of agents $= \text{card}(V) = N$.
– Number of coins $= M$.
– Money temperature $= M/N = T$. 
Distribution of money

Uniform reshuffling

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of agents $= \text{card}(\mathcal{V}) = N$.
- Number of coins $= M$.
- Money temperature $= M/N = T$.
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of agents $= \text{card} (\mathcal{V}) = N$.
- Number of coins $= M$.
- Money temperature $= M/N = T$. 

Distribution of money

Uniform reshuffling
Distribution of money

Uniform reshuffling

Discrete-time Markov chain $\xi_t: \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

where $U = \text{Uniform}\{0, 1, \ldots, 7\}$.

Finite connected graph $G = (\mathcal{V}, \mathcal{E})$.
Each vertex is an agent with some coins.
Number of agents $= \text{card}(\mathcal{V}) = N$.
Number of coins $= M$.
Money temperature $= M/N = T$.
Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

where $U = \text{Uniform} \{0, 1, \ldots, a + b\}$.

Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
Each vertex is an agent with some coins.
Number of agents $= \text{card}(\mathcal{V}) = N$.
Number of coins $= M$.
Money temperature $= M/N = T$. 

Uniform reshuffling (2000)
Distribution of money

Uniform reshuffling

Discrete-time Markov chain $\xi_t : V \to \mathbb{N}$

where $\xi_t(x) = \text{number of coins agent } x \text{ has}.$

Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
Each vertex is an agent with some coins.
Number of agents $= \text{card } (\mathcal{V}) = N$.
Number of coins $= M$.
Money temperature $= M/N = T$.

Uniform reshuffling (2000)

Conjecture – When $N$ and $T$ are large and
the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Distribution of money

Exponential distribution

uniform reshuffling

1000000 updates

200 agents
Distribution of money

Exponential distribution

uniform reshuffling
1000000 updates
1000 agents
Distribution of money

Exponential distribution

uniform reshuffling
1,000,000 updates
10,000 agents
Distribution of money

Exponential distribution

uniform reshuffling
1000000 updates
100000 agents
distribution of money

exponential distribution

Discrete-time Markov chain $\xi_t : \mathcal{Y} \rightarrow \mathbb{N}$

where $\xi_t(x) =$ number of coins agent $x$ has.

Uniform reshuffling (2000)

Finite connected graph $\mathcal{G} = (\mathcal{Y}, \mathcal{E})$.

Each vertex is an agent with some coins.

Number of agents = $\text{card} (\mathcal{Y}) = N$.

Number of coins = $M$.

Money temperature = $M/N = T$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Distribution of money

Immediate exchange

Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of agents $=$ card ($\mathcal{V}$) $= N$.
- Number of coins $= M$.
- Money temperature $= M/N = T$.

Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

Conjecture – When $N$ and $T$ are large and
the graph is complete

\[
P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.
\]
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Each vertex is an agent with some coins.

Number of agents $= \text{card} (\mathcal{V}) = N$.

Number of coins $= M$.

Money temperature $= M/N = T$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Uniform reshuffling (2000)
Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

where $U = \text{Uniform}\{0,1,\ldots,a+b\}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Distribution of money

Immediate exchange

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$

where $\xi_t(x) =$ number of coins agent $x$ has.

Uniform reshuffling (2000)

- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of agents $= \text{card}(\mathcal{V}) = N$.
- Number of coins $= M$.
- Money temperature $= M/N = T$.

Immediate exchange (2014)

- Number of agents $= \text{card}(\mathcal{V}) = N$.
- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of coins $= M$.
- Money temperature $= M/N = T$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

![Graph with nodes labeled 0 to 7 and edges showing a possible transition]

where $U = \text{Uniform}\{0, 1, \ldots, a + b\}.$

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$ 

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

![Graph with nodes labeled 0 to 7 and edges showing a possible transition]

where $U = \text{Uniform}\{(0, 1, \ldots, a) \times (0, 1, \ldots, b)\}.$

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{4}{T^2} c e^{-2c/T}.$$
Distribution of money

Gamma distribution

immediate exchange
Distribution of money

Gamma distribution

immediate exchange
1000000 updates
200 agents
Distribution of money

Gamma distribution

immediate exchange
1000000 updates
1000 agents
Distribution of money

Gamma distribution

immediate exchange
1000000 updates
10000 agents
Distribution of money

Gamma distribution

immediate exchange
1,000,000 updates
100,000 agents
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{array}{c}
a \\
\hline
\end{array} \quad \begin{array}{c}
b \\
\hline
\end{array} \quad \begin{array}{c}
U \\
\hline
\end{array} \quad \begin{array}{c}
a + b - U \\
\hline
\end{array}
\]

where $U =$ Uniform $\{0, 1, \ldots, a + b\}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[P(\xi_\infty(x) = c) \approx \frac{1}{T} \ e^{-c/T}.
\]

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{array}{c}
a \\
\hline
\end{array} \quad \begin{array}{c}
b \\
\hline
\end{array} \quad \begin{array}{c}
a - U_1 + U_2 \\
\hline
\end{array} \quad \begin{array}{c}
b + U_1 - U_2 \\
\hline
\end{array}
\]

where $U =$ Uniform $(\{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\})$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[P(\xi_\infty(x) = c) \approx \frac{4}{T^2} \ c e^{-2c/T}.
\]
Distribution of money

Main results

Discrete-time Markov chain \( \xi_t : \mathcal{V} \to \mathbb{N} \)
where \( \xi_t(x) = \) number of coins agent \( x \) has.

- Finite connected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \).
- Each vertex is an agent with some coins.
- Number of agents = \( \text{card}(\mathcal{V}) = N \).
- Number of coins = \( M \).
- Money temperature = \( M/N = T \).

Uniform reshuffling (2000)

Immediate exchange (2014)

Discrete-time Markov chain \( \xi_t : \mathcal{V} \to \mathbb{N} \)
where \( \xi_t(x) = \) number of coins agent \( x \) has.

- Finite connected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \).
- Each vertex is an agent with some coins.
- Number of agents = \( \text{card}(\mathcal{V}) = N \).
- Number of coins = \( M \).
- Money temperature = \( M/N = T \).

Conjecture – When \( N \) and \( T \) are large and the graph is complete

\[ P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T} . \]

Theorem – For all \( N \) and \( M \) and all finite connected graphs

\[ P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M-c+N-2}{N-2} . \]

where \( \Lambda_{N,M} = \) Uniform \( \{0,1,\ldots,a+b\} \).

where \( U = \) Uniform \( \{0,1,\ldots,a\} \times \{0,1,\ldots,b\} \).

Conjecture – When \( N \) and \( T \) are large and the graph is complete

\[ P(\xi_\infty(x) = c) \approx \frac{4}{T^2} \ c e^{-2c/T} . \]

where \( U = \) Uniform \( \{0,1,\ldots,a\} \times \{0,1,\ldots,b\} \).
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) = \text{number of coins agent } x \text{ has}$.

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) = \text{number of coins agent } x \text{ has}$.

**Main results**

- Finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Each vertex is an agent with some coins.
- Number of agents $= \text{card } (\mathcal{V}) = N$.
- Number of coins $= M$.
- Money temperature $= M/N = T$.


Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \left( M - c + N - 2 \right).$$

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$

**Immediate exchange (2014)**

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c+1) \binom{M-c+2N-3}{2N-3}.$$
The process is
- irreducible because $G$ is connected,


Discrete-time Markov chain $\xi_t : \mathcal{Y} \rightarrow \mathbb{N}$ where $\xi_t(x) = \text{number of coins agent } x \text{ has.}$

\[
\begin{array}{ccc}
\text{ } & \text{a} & \text{b} \\
\text{ } & \text{ } & \text{ } & \text{U} \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{a+b-U} \\
\end{array}
\]

where $U = \text{Uniform } \{0, 1, \ldots, a+b\}.$

**Conjecture** – When $N$ and $T$ are large and the graph is complete

\[P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.\]

**Theorem** – For all $N$ and $M$ and all finite connected graphs

\[P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.\]
Distribution of money

Proof (uniform reshuffling)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

The process is
− irreducible because $\mathcal{G}$ is connected,
− finite because $N$ and $M$ are,

Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

The process is

Conjecture – When $N$ and $T$ are large and
the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$  

Theorem – For all $N$ and $M$ and all finite
connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.$$  

where $U = \text{Uniform} \{0, 1, \ldots, a + b\}$.  

3 5 2 7 0 4 8
Distribution of money

Proof (uniform reshuffling)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) = \text{number of coins agent } x \text{ has.}$

The process is

- irreducible because $G$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) = \text{number of coins agent } x \text{ has.}$

where $U = \text{Uniform } \{0, 1, \ldots, a + b\}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$  

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.$$
The process is
- irreducible because $G$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x)$ = number of coins agent $x$ has.

$$a \quad b \quad U \quad a + b - U$$

where $U = \text{Uniform} \{0, 1, \ldots, a + b\}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$ 

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.$$
Distribution of money

Proof (uniform reshuffling)

The process is
- irreducible because $G$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi,\xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Key: Find a reversible distribution

\[ \pi(\xi) p(\xi,\xi') = \pi(\xi') p(\xi',\xi) \text{ for all } \xi,\xi'. \]

Uniform reshuffling (2000)

At any time $t$, $\xi_t : V \to \mathbb{N}$ is a discrete-time Markov chain where $\xi_t(x) =$ number of coins agent $x$ has.

The graph is complete

Disconnected graphs

Theorem – For all $N$ and $M$ and all finite connected graphs

\[ P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \begin{pmatrix} M - c + N - 2 \end{pmatrix}. \]

Conjecture – When $N$ and $T$ are large and the graph is complete

\[ P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}. \]
Distribution of money

Proof (uniform reshuffling)

Discrete-time Markov chain $\xi_t: \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) = $ number of coins agent $x$ has.

The process is
- irreducible because $\mathcal{G}$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Key: Find a reversible distribution

\[ \pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi) \text{ for all } \xi, \xi'. \]

Assume $p(\xi, \xi') > 0$. Then,

\[ \xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y) \]

for some $e = \{x, y\} \in \mathcal{E}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[ P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}. \]

Theorem – For all $N$ and $M$ and all finite connected graphs

\[ P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}. \]
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

![Graph of the distribution of coins](image)

where $U =$ Uniform $\{0, 1, \ldots, a + b\}$.

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$  

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.$$  

The process is

– irreducible because $\mathcal{G}$ is connected,
– finite because $N$ and $M$ are,
– aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Key: Find a reversible distribution

$$\pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi) \text{ for all } \xi, \xi'.$$

Assume $p(\xi, \xi') > 0$. Then,

$$\xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y)$$

for some $e = \{x, y\} \in \mathcal{E}$. 
Uniform reshuffling (2000)

Discrete-time Markov chain $\xi_t: \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

The process is
- irreducible because $\mathcal{G}$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Key: Find a reversible distribution

$\pi(\xi)p(\xi, \xi') = \pi(\xi')p(\xi', \xi)$ for all $\xi, \xi'$.

Assume $p(\xi, \xi') > 0$. Then,

$\xi = \xi'$ on $\mathcal{V} - e$ and $\xi(x) + \xi(y) = \xi'(x) + \xi'(y)$

for some $e = \{x, y\} \in \mathcal{E}$.

$$p(\xi, \xi') = \frac{1}{\text{card}(\mathcal{E})} \frac{1}{a + b + 1} = p(\xi', \xi).$$

$\xi_t$:

1. $\xi_{t+1}(x) = \xi_t(x) + U$ for $x \in \mathcal{V}$
2. $\xi_{t+1}(y) = \xi_t(y) - U$ for $y \in \mathcal{V}$

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.$$
Uniform reshuffling (2000)

Discrete-time Markov chain \( \xi_t : \mathcal{V} \rightarrow \mathbb{N} \)
where \( \xi_t(x) = \) number of coins agent \( x \) has.

\[
\begin{array}{ccc}
\text{a} & \text{b} & U \\
\text{a} + b & - & U
\end{array}
\]

where \( U = \) Uniform \( \{0, 1, \ldots, a + b\} \).

Conjecture – When \( N \) and \( T \) are large and the graph is complete
\[
P(\xi_\infty(x) = c) \approx \frac{1}{T} e^{-c/T}.
\]

Theorem – For all \( N \) and \( M \) and all finite connected graphs
\[
P(\xi_\infty(x) = c) = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}.
\]

The process is
- irreducible because \( \mathcal{G} \) is connected,
- finite because \( N \) and \( M \) are,
- aperiodic because \( p(\xi, \xi) > 0 \) for all \( \xi \).

For any initial configuration, convergence to a unique stationary distribution \( \pi \).

Key: Find a reversible distribution
\[
\pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi) \text{ for all } \xi, \xi'.
\]

Assume \( p(\xi, \xi') > 0 \). Then,
\[
\xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y)
\]
for some \( e = \{x, y\} \in \mathcal{E} \).

\[
p(\xi, \xi') = \frac{1}{\text{card}(\mathcal{E})} \frac{1}{a + b + 1} = p(\xi', \xi).
\]

\( \Rightarrow \) \( \pi = \) Uniform (set of configurations).
Distribution of money

Proof (uniform reshuffling)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

The process is

- irreducible because $\mathcal{G}$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi) > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

**Key:** Find a reversible distribution

$$\pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi) \text{ for all } \xi, \xi'.$$

Assume $p(\xi, \xi') > 0$. Then,

$$\xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y)$$

for some $e = \{x, y\} \in \mathcal{E}$.

$$p(\xi) = \frac{1}{\text{card } (\mathcal{E})} \frac{1}{a + b + 1} = p(\xi', \xi).$$

$\Rightarrow \pi = \text{Uniform (set of configurations)}.$

$$\Lambda_{N,M} = \# \text{ solutions of } c_1 + c_2 + \cdots + c_N = M$$

$$= \binom{M + N - 1}{N - 1}.$$
Distribution of money

Proof (uniform reshuffling)

Discrete-time Markov chain \( \xi_t : \mathcal{V} \rightarrow \mathbb{N} \) where \( \xi_t(x) = \) number of coins agent \( x \) has.

The process is
- irreducible because \( \mathcal{G} \) is connected,
- finite because \( N \) and \( M \) are,
- aperiodic because \( p(\xi, \xi') > 0 \) for all \( \xi \).

For any initial configuration, convergence to a unique stationary distribution \( \pi \).

Key: Find a reversible distribution
\[ \pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi) \text{ for all } \xi, \xi'. \]

Assume \( p(\xi, \xi') > 0 \). Then,
\[ \xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y) \]
for some \( e = \{x, y\} \in \mathcal{E} \).

\[ p(\xi, \xi') = \frac{1}{\text{card}(\mathcal{E})} \frac{1}{a + b + 1} = p(\xi', \xi). \]

\( \Rightarrow \pi = \text{Uniform (set of configurations)}. \)

\( \Lambda_{N,M} = \# \text{ solutions of } c_1 + c_2 + \cdots + c_N = M \)
\[ = \binom{M + N - 1}{N - 1}. \]

\[ P(\xi_\infty(x) = c) = \frac{\Lambda_{N-1,M-c}}{\Lambda_{N,M}} = \frac{1}{\Lambda_{N,M}} \binom{M - c + N - 2}{N - 2}. \]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

The process is
- irreducible because $\mathcal{G}$ is connected,
- finite because $N$ and $M$ are,
- aperiodic because $p(\xi, \xi') > 0$ for all $\xi$.

For any initial configuration, convergence to a unique stationary distribution $\pi$.

Key: Find a reversible distribution

$$\pi(\xi) p(\xi, \xi') = \pi(\xi') p(\xi', \xi)$$ for all $\xi, \xi'$.

Assume $p(\xi, \xi') > 0$. Then,

$$\xi = \xi' \text{ on } \mathcal{V} - e \text{ and } \xi(x) + \xi(y) = \xi'(x) + \xi'(y)$$

for some $e = \{x, y\} \in \mathcal{E}$.

$$p(\xi, \xi') = \frac{1}{\text{card}(\mathcal{E})} \frac{1}{a + b + 1} = p(\xi', \xi).$$

$\Rightarrow \pi = \text{Uniform (set of configurations)}$.

$$\Lambda_{N,M} = \# \text{ solutions of } c_1 + c_2 + \cdots + c_N = M = \binom{M + N - 1}{N - 1}.$$
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x)$ = number of coins agent $x$ has.

$$a \quad b \quad a - U_1 + U_2 \quad b + U_1 - U_2$$

where $U = \text{Uniform} \left( \{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\} \right)$.

Conjecture – When $N$ and $T$ are large and
the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{4}{T^2} c e^{-2c/T}.$$ 

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c + 1) \left( \frac{M - c + 2N - 3}{2N - 3} \right).$$

Key: reversibility. Assume $p(\xi, \xi') > 0$,

$$\xi(x) = a, \; \xi(y) = b, \; \xi'(x) = a', \; \xi'(y) = b'.$$
Distribution of money

Proof (immediate exchange)

Discrete-time Markov chain $\xi_t : \mathcal{V} \to \mathbb{N}$
where $\xi_t(x) = \text{number of coins agent } x \text{ has.}$

Key: reversibility. Assume $p(\xi, \xi') > 0,$

$$\xi(x) = a, \quad \xi(y) = b, \quad \xi'(x) = a', \quad \xi'(y) = b'.$$

Immediate exchange (2014)

Conjecture – When $N$ and $T$ are large and the graph is complete

$$P(\xi_\infty(x) = c) \approx \frac{4}{T^2} \frac{c e^{-2c/T}}{c + 1}.$$

Theorem – For all $N$ and $M$ and all finite connected graphs

$$P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c + 1) \left( \frac{M - c + 2N - 3}{2N - 3} \right).$$

where $U = \text{Uniform}(\{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\}).$
Key: reversibility. Assume $p(\xi, \xi') > 0$,

$\xi(x) = a, \; \xi(y) = b, \; \xi'(x) = a', \; \xi'(y) = b'.$

Number of outcomes such that $\xi'(x) = a'$ is

- $a' + 1$ for $a' \leq \min(a, b) \leq b'$
- $\min(a, b) + 1$ for $\min(a, b) < a', b' < \max(a, b)$
- $b' + 1$ for $a' \geq \max(a, b) \geq b'$.

Immediate exchange (2014)

Discrete-time Markov chain $\xi_t: \mathcal{V} \rightarrow \mathbb{N}$

where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{align*}
  a & \quad b \\
  \quad & \quad \\
  a - U_1 + U_2 & \quad b + U_1 - U_2
\end{align*}
\]

where $U = \text{Uniform}\left(\{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\}\right)$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[
P(\xi_\infty(x) = c) \approx \frac{4}{T^2} ce^{-2c/T}.
\]

Theorem – For all $N$ and $M$ and all finite connected graphs

\[
P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c + 1) \left(\frac{M - c + 2N - 3}{2N - 3}\right).
\]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$
where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{align*}
\begin{array}{c c c c c}
 & a & b & a - U_1 + U_2 & b + U_1 - U_2 \\
\end{array}
\end{align*}
\]

where $U = \text{Uniform}(\{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\})$.

Conjecture – When $N$ and $T$ are large and
the graph is complete

\[
P(\xi_\infty(x) = c) \approx \frac{4}{T^2} c e^{-2c/T}.
\]

Key: reversibility. Assume $p(\xi, \xi') > 0$,

\[
\begin{align*}
\xi(x) &= a, \quad \xi(y) = b, \quad \xi'(x) = a', \quad \xi'(y) = b'.
\end{align*}
\]

Number of outcomes such that $\xi'(x) = a'$ is

\[
\begin{align*}
& a' + 1 \quad \text{for} \quad a' \leq \min(a, b) \leq b' \\
& \min(a, b) + 1 \quad \text{for} \quad \min(a, b) < a', b' < \max(a, b) \\
& b' + 1 \quad \text{for} \quad a' \geq \max(a, b) \geq b'.
\end{align*}
\]

This implies that

\[
\text{card}(\mathcal{E})(a + 1)(b + 1) p(\xi, \xi') = \min(a, b, a', b') + 1
\]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{array}{ccc}
& a & b \\
\rightarrow & a - U_1 + U_2 & b + U_1 - U_2 \\
\end{array}
\]

where $U = \text{Uniform}(\{0,1,\ldots,a\} \times \{0,1,\ldots,b\})$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[P(\xi_\infty(x) = c) \approx \frac{4}{T^2} \ c e^{-2c/T}.
\]

Theorem – For all $N$ and $M$ and all finite connected graphs

\[P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c+1) \binom{M-c+2N-3}{2N-3}.
\]

Key: reversibility. Assume $p(\xi, \xi') > 0$,

$\xi(x) = a, \ \xi(y) = b, \ \xi'(x) = a', \ \xi'(y) = b'$.

Number of outcomes such that $\xi'(x) = a'$ is

- $a' + 1$ for $a' \leq \min(a, b) \leq b'$
- $\min(a, b) + 1$ for $\min(a, b) < a', b' < \max(a, b)$
- $b' + 1$ for $a' \geq \max(a, b) \geq b'$.

This implies that

\[
\text{card}(\mathcal{E})(a+1)(b+1) p(\xi, \xi') = \min(a, b, a', b') + 1
= \text{card}(\mathcal{E})(a'+1)(b'+1) p(\xi', \xi).
\]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{array}{c}
\text{a} & \text{b} \\
\text{a} - U_1 + U_2 & \text{b} + U_1 - U_2
\end{array}
\]

where $U = \text{Uniform} (\{0, 1, \ldots, a\} \times \{0, 1, \ldots, b\})$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[P(\xi_{\infty}(x) = c) \approx \frac{4}{T^2} c e^{-2c/T}.\]

Theorem – For all $N$ and $M$ and all finite connected graphs

\[P(\xi_{\infty}(x) = c) = \frac{1}{\Gamma_{N,M}} (c + 1) \left( \frac{M - c + 2N - 3}{2N - 3} \right).\]

Key: reversibility. Assume $p(\xi, \xi') > 0,$

\[
\xi(x) = a, \ \xi(y) = b, \ \xi'(x) = a', \ \xi'(y) = b'.
\]

Number of outcomes such that $\xi'(x) = a'$ is

\[
a' + 1 \quad \text{for} \quad a' \leq \min(a, b) \leq b' \\
\min(a, b) + 1 \quad \text{for} \quad \min(a, b) < a', b' < \max(a, b) \\
b' + 1 \quad \text{for} \quad a' \geq \max(a, b) \geq b'.
\]

This implies that

\[
\text{card} (\mathcal{E})(a + 1)(b + 1) p(\xi, \xi') = \min(a, b, a', b') + 1 \\
= \text{card} (\mathcal{E})(a' + 1)(b' + 1) p(\xi', \xi).
\]

$\Rightarrow$ reversible distribution

\[
\pi(\xi) = \frac{\mu(\xi)}{\sum_{\eta} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{x \in \mathcal{V}} (\xi(x) + 1).
\]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t : \mathcal{V} \rightarrow \mathbb{N}$ where $\xi_t(x)$ = number of coins agent $x$ has.

Key: reversibility. Assume $p(\xi, \xi') > 0$,

\[\xi(x) = a, \ \xi(y) = b, \ \xi'(x) = a', \ \xi'(y) = b'.\]

Number of outcomes such that $\xi'(x) = a'$ is

- $a' + 1$ for $a' \leq \min(a, b) \leq b'$
- $\min(a, b) + 1$ for $\min(a, b) < a', b' < \max(a, b)$
- $b' + 1$ for $a' \geq \max(a, b) \geq b'$.

This implies that

\[\text{card}(\mathcal{E})(a + 1)(b + 1) p(\xi, \xi') = \min(a, b, a', b') + 1 \]

\[= \text{card}(\mathcal{E})(a' + 1)(b' + 1) p(\xi', \xi).\]

\[\Rightarrow \text{reversible distribution} \]

\[\pi(\xi) = \frac{\mu(\xi)}{\sum_{\eta} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{x \in \mathcal{V}} (\xi(x) + 1).\]

By induction, the denominator is

\[\Gamma_{N,M} = \sum_{c_1 + \cdots + c_N = M} (c_1 + 1)(c_2 + 1) \cdots (c_N + 1)\]

\[= \binom{M + 2N - 1}{2N - 1}.\]
Immediate exchange (2014)

Discrete-time Markov chain \( \xi_t : \mathcal{V} \rightarrow \mathbb{N} \)
where \( \xi_t(x) = \) number of coins agent \( x \) has.

Conjecture – When \( N \) and \( T \) are large and the graph is complete
\[
P(\xi_\infty(x) = c) \approx \frac{4}{T^2} e^{-2c/T}.
\]

Theorem – For all \( N \) and \( M \) and all finite connected graphs
\[
P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c+1) \left( \frac{M-c+2N-3}{2N-3} \right).
\]

Key: reversibility. Assume \( p(\xi,\xi') > 0, \)
\[
\xi(x) = a, \ \xi(y) = b, \ \xi'(x) = a', \ \xi'(y) = b'.
\]

Number of outcomes such that \( \xi'(x) = a' \) is
\[
a' + 1 \quad \text{for} \quad a' \leq \min(a,b) \leq b' \\
\min(a,b) + 1 \quad \text{for} \quad \min(a,b) < a', b' < \max(a,b) \\
b' + 1 \quad \text{for} \quad a' \geq \max(a,b) \geq b'.
\]

This implies that
\[
\text{card}(\mathcal{E})(a+1)(b+1) p(\xi,\xi') = \min(a,b,a',b') + 1 \\
= \text{card}(\mathcal{E})(a'+1)(b'+1) p(\xi',\xi).
\]

\( \Rightarrow \) reversible distribution
\[
\pi(\xi) = \frac{\mu(\xi)}{\sum_{\eta} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{x \in \mathcal{V}} (\xi(x) + 1).
\]

By induction, the denominator is
\[
\Gamma_{N,M} = \sum_{c_1+\ldots+c_N=M} (c_1 + 1)(c_2 + 1) \cdots (c_N + 1) \\
= \begin{pmatrix} M + 2N - 1 \\ 2N - 1 \end{pmatrix}.
\]

\[
P(\xi_\infty(x) = c) = \frac{\sum_{\xi(x) = c} \mu(\xi)}{\sum_{\eta} \mu(\eta)} = \frac{(c+1) \Gamma_{N-1,M-c}}{\Gamma_{N,M}}.
\]
Immediate exchange (2014)

Discrete-time Markov chain $\xi_t: \mathcal{Y} \rightarrow \mathbb{N}$ where $\xi_t(x) =$ number of coins agent $x$ has.

\[
\begin{array}{ccc}
a & b & a - U_1 + U_2 \\
\rightarrow & \rightarrow & \rightarrow \\
b + U_1 - U_2 & & \\
\end{array}
\]

where $U = \text{Uniform} \left( \left\{ 0, 1, \ldots, a \right\} \times \left\{ 0, 1, \ldots, b \right\} \right)$.

Conjecture – When $N$ and $T$ are large and the graph is complete

\[
P(\xi_\infty(x) = c) \approx \frac{4}{T^2} c e^{-2c/T}.
\]

Theorem – For all $N$ and $M$ and all finite connected graphs

\[
P(\xi_\infty(x) = c) = \frac{1}{\Gamma_{N,M}} (c + 1) \left( M - c + 2N - 3 \right) \left( 2N - 3 \right).
\]

Key: reversibility. Assume $p(\xi, \xi') > 0$,

$\xi(x) = a, \xi(y) = b, \xi'(x) = a', \xi'(y) = b'$.

Number of outcomes such that $\xi'(x) = a'$ is

\[
\begin{align*}
a' + 1 & \quad \text{for } a' \leq \min(a, b) \leq b' \\
\min(a, b) + 1 & \quad \text{for } \min(a, b) < a', b' < \max(a, b) \\
b' + 1 & \quad \text{for } a' \geq \max(a, b) \geq b'.
\end{align*}
\]

This implies that

\[
\text{card} (\mathcal{E})(a + 1)(b + 1) p(\xi, \xi') = \min(a, b, a', b') + 1 = \text{card} (\mathcal{E})(a' + 1)(b' + 1) p(\xi', \xi).
\]

$\Rightarrow$ reversible distribution

\[
\pi(\xi) = \frac{\mu(\xi)}{\sum_{\eta} \mu(\eta)} \quad \text{where } \mu(\xi) = \prod_{x \in \mathcal{Y}} (\xi(x) + 1).
\]

By induction, the denominator is

\[
\Gamma_{N,M} = \sum_{c_1 + \cdots + c_N = M} (c_1 + 1)(c_2 + 1) \cdots (c_N + 1)
\]

\[
= \binom{M + 2N - 1}{2N - 1}.
\]

$\text{card} (\mathcal{E})(c + 1) p(\xi, \xi') = \frac{(c + 1) \Gamma_{N-1,M-c}}{\Gamma_{N,M}}$

$P(\xi_\infty(x) = c) = \frac{\sum_{\xi(x) = c} \mu(\xi)}{\sum_{\eta} \mu(\eta)} \frac{(c + 1) \Gamma_{N-1,M-c}}{\Gamma_{N,M}}$

\[
= \frac{1}{\Gamma_{N,M}} (c + 1) \left( M - c + 2N - 3 \right) \left( 2N - 3 \right).
\]
**Uniform reshuffling**

\[
P(\xi_\infty(x) = c) = \binom{M - c + N - 2}{N - 2} \bigg/ \binom{M + N - 1}{N - 1}
\]
**Uniform reshuffling**

\[
P(\xi_\infty(x) = c) = \frac{\binom{M - c + N - 2}{N - 2}}{\binom{M + N - 1}{N - 1}}
\]

\[
= \frac{(M - c + N - 2)!}{(M - c)! (N - 2)!} \frac{M! (N - 1)!}{(M + N - 1)!}
\]
Uniform reshuffling

\[ P(\xi_\infty(x) = c) = \frac{\binom{M - c + N - 2}{N - 2}}{\binom{M + N - 1}{N - 1}} \]

\[ = \frac{(M - c + N - 2)!}{(M - c)! (N - 2)!} \frac{M! (N - 1)!}{(M + N - 1)!} \]

\[ = \frac{N - 1}{M + N - 1} \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)} \]
Uniform reshuffling

\[ P(\xi_\infty(x) = c) = \binom{M - c + N - 2}{N - 2} / \binom{M + N - 1}{N - 1} \]

\[ = \frac{(M - c + N - 2)! \cdot M! \cdot (N - 1)!}{(M - c)! \cdot (N - 2)! \cdot (M + N - 1)!} \]

\[ = \frac{N - 1}{M + N - 1} \cdot \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)} \]

\[ \approx \left( \frac{N}{M} \right) \left( 1 - \frac{c}{M} \right)^N = \frac{1}{T} \left( 1 - \frac{c}{NT} \right)^N \approx \frac{1}{T} e^{-c/T} \]
Uniform reshuffling

\[
P(\xi_\infty(x) = c) = \left( \frac{M - c + N - 2}{N - 2} \right) / \left( \frac{M + N - 1}{N - 1} \right)
= \frac{(M - c + N - 2)!}{(M - c)! (N - 2)!} \frac{M! (N - 1)!}{(M + N - 1)!}
= \frac{N - 1}{M + N - 1} \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)}
\approx \left( \frac{N}{M} \right) \left( 1 - \frac{c}{M} \right)^N = \frac{1}{T} \left( 1 - \frac{c}{NT} \right)^N \approx \frac{1}{T} e^{-c/T}
\]

Immediate exchange

\[
P(\xi_\infty(x) = c) = (c + 1) \left( \frac{M - c + 2N - 3}{2N - 3} \right) / \left( \frac{M + 2N - 1}{2N - 1} \right)
\]
Uniform reshuffling

\[
P(\xi_\infty(x) = c) = \binom{M - c + N - 2}{N - 2} \bigg/ \binom{M + N - 1}{N - 1} \\
= \frac{(M - c + N - 2)! M! (N - 1)!}{(M - c)! (N - 2)! (M + N - 1)!} \\
= \frac{N - 1}{M + N - 1} \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)} \\
\approx \left( \frac{N}{M} \right) \left( 1 - \frac{c}{M} \right)^N = \frac{1}{T} \left( 1 - \frac{c}{NT} \right)^N \approx \frac{1}{T} e^{-c/T}
\]

Immediate exchange

\[
P(\xi_\infty(x) = c) = (c + 1) \binom{M - c + 2N - 3}{2N - 3} \bigg/ \binom{M + 2N - 1}{2N - 1} \\
= (c + 1) \frac{(M - c + 2N - 3)! M! (2N - 3)!}{(M - c)! (2N - 3)! (M + 2N - 1)!}
\]
Uniform reshuffling

\[ P(\xi_\infty(x) = c) = \frac{\binom{M - c + N - 2}{N - 2}}{\binom{M + N - 1}{N - 1}} \]

\[ = \frac{(M - c + N - 2)! \ M! \ (N - 1)!}{(M - c)! \ (N - 2)! \ (M + N - 1)!} \]

\[ = \frac{N - 1}{M + N - 1} \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)} \]

\[ \approx \left( \frac{N}{M} \right) \left( 1 - \frac{c}{M} \right)^N = \frac{1}{T} \left( 1 - \frac{c}{NT} \right)^N \approx \frac{1}{T} e^{-c/T} \]

Immediate exchange

\[ P(\xi_\infty(x) = c) = (c + 1) \left( \frac{M - c + 2N - 3}{2N - 3} \right) / \left( \frac{M + 2N - 1}{2N - 1} \right) \]

\[ = (c + 1) \frac{(M - c + 2N - 3)! \ M! \ (2N - 3)!}{(M - c)! \ (2N - 3)! \ (M + 2N - 1)!} \]

\[ \approx (c + 1) \left( \frac{2N}{M} \right)^2 \frac{(M - c + 2N - 3) \cdots (M - c + 1)}{(M + 2N - 3) \cdots (M + 1)} \]
Uniform reshuffling

\[
P(\xi_\infty(x) = c) = \left(M - c + N - 2\right) \binom{N}{N-2} / \binom{M + N - 1}{N-1}
\]

\[
= (M - c + N - 2)! M! (N - 1)! \\
(M - c)! (N - 2)! (M + N - 1)!
\]

\[
= \frac{N - 1}{M + N - 1} \frac{(M - c + N - 2) \cdots (M - c + 1)}{(M + N - 2) \cdots (M + 1)}
\]

\[
\approx \left(\frac{N}{M}\right) \left(1 - \frac{c}{M}\right)^N = \frac{1}{T} \left(1 - \frac{c}{NT}\right)^N \approx \frac{1}{T} e^{-c/T}
\]

Immediate exchange

\[
P(\xi_\infty(x) = c) = (c + 1) \binom{M - c + 2N - 3}{2N - 3} / \binom{M + 2N - 1}{2N - 1}
\]

\[
= (c + 1) \frac{(M - c + 2N - 3)! M! (2N - 3)!}{(M - c)! (2N - 3)! (M + 2N - 1)!}
\]

\[
\approx (c + 1) \left(\frac{2N}{M}\right)^2 \frac{(M - c + 2N - 3) \cdots (M - c + 1)}{(M + 2N - 3) \cdots (M + 1)}
\]

\[
\approx (c + 1) \left(\frac{2N}{M}\right)^2 \left(1 - \frac{c}{M}\right)^{2N} \approx \frac{4}{T^2} c e^{-2c/T}
\]