Evolutionary games on the lattice: best-response dynamics

Nicolas Lanchier
Joint work with Stephen Evilsizor
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \longrightarrow \{1, 2\} = \blacksquare \whitesquare$$
Markov chain whose state at time $t$ is

$$
\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \blackbox \whitebox
$$

Dynamics – Payoff matrix

$$
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
  0 & 3 \\
  5 & 1
\end{pmatrix}
$$

$a_{ij}$ = payoff of a type $i$ player
    interacting with a type $j$ player
Markov chain whose state at time $t$ is
\[ \eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \square \boxdot \]

Dynamics – Payoff matrix
\[ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix} \]

$a_{ij} = \text{payoff of a type } i \text{ player}$
interacting with a type $j$ player

The player at $x$ has payoff
\[ \phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta) \]
Markov chain whose state at time $t$ is

$$\eta : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{array}{c}
\square \\
\blacksquare 
\end{array}$$

Dynamics – Payoff matrix

$$\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 3 \\
5 & 1
\end{pmatrix}$$

$a_{ij} = \text{payoff of a type } i \text{ player interacting with a type } j \text{ player}$

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \square \blacksquare$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix} \quad \square \blacksquare$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff
Markov chain whose state at time $t$ is
\[ \eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \blackbox\whitebox \]

Dynamics – Payoff matrix
\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
  0 & 3 \\
  5 & 1
\end{pmatrix}
\]

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff
\[
\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)
\]

Players update their strategy at rate one in order to maximize their payoff
Markov chain whose state at time $t$ is
\[ \eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \square \]  

Dynamics – Payoff matrix
\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 3 \\
5 & 1
\end{pmatrix}
\]

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff
\[
\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)
\]

Players update their strategy at rate one in order to maximize their payoff
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{bmatrix} \bullet & \square \end{bmatrix}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij} =$ payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff
Best-response dynamics

Model description

Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{bmatrix} \begin{array}{c} \bullet \end{array} & \begin{array}{c} \square \end{array} \end{bmatrix}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

- gets payoff $(5, 1) \cdot (1, 3)^T = 8$
Markov chain whose state at time \( t \) is
\[
\eta: \mathbb{Z}^d \rightarrow \{1, 2\} = \blacksquare \square
\]

Dynamics – Payoff matrix
\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}
\]

\( a_{ij} = \text{payoff of a type } i \text{ player}\)
interacting with a type \( j \) player

The player at \( x \) has payoff
\[
\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)
\]

Players update their strategy at rate one
in order to maximize their payoff

\( \blacksquare \text{ gets payoff} \quad (5, 1) \cdot (1, 3)^T = 8 \)
Markov chain whose state at time $t$ is

$$
\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \triangleleft \triangleright
$$

Dynamics – Payoff matrix

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} =
\begin{pmatrix}
0 & 3 \\
5 & 1
\end{pmatrix}
$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$
\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)
$$

Players update their strategy at rate one in order to maximize their payoff

- gets payoff $(5, 1) \cdot (1, 3)^T = 8$
- gets payoff $(0, 3) \cdot (1, 3)^T = 9$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{cases} 1, & \text{gets payoff } (5, 1) \\ 2, & \text{gets payoff } (0, 3) \end{cases}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij} =$ payoff of a type $i$ player interacting with a type $j$ player.

The player at $x$ has payoff

$$\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff.

- $\begin{cases} 1, & \text{gets payoff } (5, 1) \cdot (1, 3)^T = 8 \\ 2, & \text{gets payoff } (0, 3) \cdot (1, 3)^T = 9 \end{cases}$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \boxed{\text{■ □}}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix} \boxed{\text{■}}$$

$a_{ij} =$ payoff of a type $i$ player
interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

- \boxed{□} gets payoff \ $(5, 1) \cdot (1, 3)^T = 8$
- \boxed{■} gets payoff \ $(0, 3) \cdot (1, 3)^T = 9$

\boxed{■} $\rightarrow$ \boxed{□} at rate

$$1\{a_{11} N_1 + a_{12} N_2 < a_{21} N_1 + a_{22} N_2\}$$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij} =$ payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

- □ gets payoff $(5, 1) \cdot (1, 3)^T = 8$
- ■ gets payoff $(0, 3) \cdot (1, 3)^T = 9$
- • → □ at rate

$$1\{a_{11} N_1 + a_{12} N_2 < a_{21} N_1 + a_{22} N_2\}$$

$$= 1\{(a_{11} - a_{21}) N_1 < (a_{22} - a_{12}) N_2\}$$
Markov chain whose state at time $t$ is
\[ \eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \square \square \]

Dynamics – Payoff matrix
\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} =
\begin{pmatrix}
0 & 3 \\
5 & 1
\end{pmatrix}
\]

$a_{ij} = \text{payoff of a type } i \text{ player}$
interacting with a type $j$ player

The player at $x$ has payoff
\[
\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)
\]

Players update their strategy at rate one
in order to maximize their payoff

- gets payoff $5, 1 \cdot (1, 3)^T = 8$
- gets payoff $0, 3 \cdot (1, 3)^T = 9$

- at rate
\[
1\{a_{11} N_1 + a_{12} N_2 < a_{21} N_1 + a_{22} N_2\}
= 1\{(a_{11} - a_{21}) N_1 < (a_{22} - a_{12}) N_2\}
= 1\{a_1 N_1(x, \eta) < a_2 N_2(x, \eta)\}
\]

where $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij}$ = payoff of a type $i$ player
interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

- Gets payoff $(5, 1) \cdot (1, 3)^T = 8$
- Gets payoff $(0, 3) \cdot (1, 3)^T = 9$

- at rate $1\{a_{11} N_1 + a_{12} N_2 < a_{21} N_1 + a_{22} N_2\}$

where $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$

- at rate $1\{a_1 N_1(x, \eta) > a_2 N_2(x, \eta)\}$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{cases} \text{ } & \text{ } \\ \text{ } & \text{ } \end{cases}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player

The player at $x$ has payoff

$$\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

1 → 2 at rate 1\{a_1 N_1(x, \eta) < a_2 N_2(x, \eta)\}
2 → 1 at rate 1\{a_1 N_1(x, \eta) > a_2 N_2(x, \eta)\}
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{array}{c} \blacksquare \cr \blacksquare \end{array}$$

**Dynamics – Payoff matrix**

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij} = \text{payoff of a type } i \text{ player interacting with a type } j \text{ player}$

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

1 → 2 at rate $1\{a_1 N_1(x, \eta) < a_2 N_2(x, \eta)\}$

2 → 1 at rate $1\{a_1 N_1(x, \eta) > a_2 N_2(x, \eta)\}$

**Non-spatial deterministic counterpart**

Let $u_i = \text{frequency of strategy } i \text{ players}$
Markov chain whose state at time $t$ is

$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \begin{cases} \square & \text{for } 1 \\ \blacksquare & \text{for } 2 \end{cases}$$

Dynamics – Payoff matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij}$ = payoff of a type $i$ player interacting with a type $j$ player.

The player at $x$ has payoff

$$\phi(x | \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one in order to maximize their payoff

$$1 \rightarrow 2 \text{ at rate } 1\{a_{11} N_1(x, \eta) < a_{22} N_2(x, \eta)\}$$

$$2 \rightarrow 1 \text{ at rate } 1\{a_{11} N_1(x, \eta) > a_{22} N_2(x, \eta)\}$$

Non-spatial deterministic counterpart

Let $u_i = \text{frequency of strategy } i$ players

$$u'_i = u_2 1\{a_1 u_1 > a_2 u_2\} - u_1 1\{a_1 u_1 < a_2 u_2\}$$
Markov chain whose state at time $t$ is
$$\eta_t : \mathbb{Z}^d \rightarrow \{1, 2\} = \blacksquare \square$$

**Dynamics – Payoff matrix**

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}$$

$a_{ij} =$ payoff of a type $i$ player
interacting with a type $j$ player

The player at $x$ has payoff
$$\phi(x \mid \eta(x) = i) = a_{i1} N_1(x, \eta) + a_{i2} N_2(x, \eta)$$

Players update their strategy at rate one
in order to maximize their payoff

1 $\rightarrow$ 2 at rate 1\{$a_1 N_1(x, \eta) < a_2 N_2(x, \eta)$\}
2 $\rightarrow$ 1 at rate 1\{$a_1 N_1(x, \eta) > a_2 N_2(x, \eta)$\}

**Non-spatial deterministic counterpart**

Let $u_i =$ frequency of strategy $i$ players

$$u'_i = u_2 1\{a_1 u_1 > a_2 u_2\} - u_1 1\{a_1 u_1 < a_2 u_2\}$$

**Fixed points:** $u_1 = 0$ (all type 2)
$u_1 = 1$ (all type 1)
$u_1 = a_2/(a_1 + a_2)$ (mixture?)
**Best-response dynamics**

**Main results**

- **White wins**
  - No interior fixed point
- **Bistability**
  - Interior fixed point unstable
- **Coexistence**
  - Interior fixed point globally stable
- **Black wins**
  - No interior fixed point

**Equations**

\[
a_1 = a_{11} - a_{21} < 0
\]

\[
a_1 = a_{11} - a_{21} > 0
\]

\[
a_2 = a_{22} - a_{12} > 0
\]

\[
a_2 = a_{22} - a_{12} < 0
\]
Best-response dynamics

Main results

1 ESS
white wins
no interior fixed point

1 ESS
black wins
no interior fixed point

0 ESS
coexistence
interior fixed point globally stable

2 ESS
bistability
interior fixed point unstable

\[
a_1 = a_{11} - a_{21} < 0
\]

\[
a_1 = a_{11} - a_{21} > 0
\]
Best-response dynamics

**Main results**

1. **ESS white wins**
   - No interior fixed point
   - 1 ESS

2. **ESS bistability**
   - Interior fixed point unstable
   - 2 ESS

3. **ESS black wins**
   - No interior fixed point
   - 1 ESS

4. **ESS coexistence**
   - Convergence to a nontrivial invariant measure
   - 0 ESS

### Equations

\[ a_2 = a_{22} - a_{12} > 0 \]

\[ a_2 = a_{22} - a_{12} < 0 \]

\[ a_1 = a_{11} - a_{21} < 0 \]

\[ a_1 = a_{11} - a_{21} > 0 \]
Best-response dynamics

Main results

**1 ESS**
- white wins
- convergence to the all white configuration
- bistability
- interior fixed point unstable

**2 ESS**

**0 ESS**
- coexistence
- convergence to a nontrivial invariant measure

**1 ESS**
- black wins
- convergence to the all black configuration

\[
a_2 = a_{22} - a_{12} > 0
\]

\[
a_2 = a_{22} - a_{12} < 0
\]

\[
a_1 = a_{11} - a_{21} < 0
\]

\[
a_1 = a_{11} - a_{21} > 0
\]
**Main results**

- **1 ESS**
  - white wins
  - convergence to the all white configuration

- **0 ESS**
  - coexistence
  - convergence to a nontrivial invariant measure

- **1 ESS**
  - black wins
  - convergence to the all black configuration

Mathematical expressions:

\[ a_1 = a_{11} - a_{21} < 0 \]

\[ a_2 = a_{22} - a_{12} > 0 \]
Best-response dynamics

Main results

**white wins**
- convergence to the all white configuration
- 1 ESS

**coexistence**
- convergence to a nontrivial invariant measure
- 0 ESS

**black wins**
- convergence to the all black configuration
- 1 ESS

**clustering**

\[
a_2 = a_{22} - a_{12} > 0
\]
\[
a_2 = a_{22} - a_{12} < 0
\]
\[
a_1 = a_{11} - a_{21} < 0
\]
\[
a_1 = a_{11} - a_{21} > 0
\]
Initial density = 4/24
Best-response dynamics

Infinite time limits

Initial density = 4/24

Initial density = 4/25

Initial density = 4/26
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive

$$P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0$$
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive

\[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]

- Let \( H_z = 2z + \{0,1\}^d \) and define

\[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \} \]
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive

$$P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0$$

- Let $H_z = 2z + \{0, 1\}^d$ and define

$$\bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0\}$$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  $P(x \in \bar{\eta}_t) \leq P(x \in \eta_t)$ \text{ when } \bar{\eta}_0 \subset \eta_0$
- Let $H_z = 2z + \{0, 1\}^d$ and define
  
  $\bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  $P(x \in \bar{\eta}_t) \leq P(x \in \eta_t)$ when $\bar{\eta}_0 \subset \eta_0$

- Let $H_z = 2z + \{0, 1\}^d$ and define
  
  $\bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \, \text{and} \, H_z \subset \eta_0\} \]
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \tilde{\eta}_t) \leq P(x \in \eta_t) \text{ when } \tilde{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \tilde{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \} \]
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  
  \[
P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0
  \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define
  
  \[
  \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \}
  \]
Interacting particle system: \(a_1 > a_2 > 0\)

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let \(H_z = 2z + \{0, 1\}^d\) and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level

\[ \blacksquare \rightarrow \square \text{ at rate 1 only if } N_1 < N_2 \iff N_1 < d \]
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0,1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level

- \(\square \rightarrow \blacksquare\) at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- \(\blacksquare \rightarrow \square\) at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[
  P(x \in \eta_t) \leq P(x \in \bar{\eta}_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0
  \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define \( \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \)

At the player level

- ■ → □ at rate 1 only if \( N_1 < N_2 \Leftrightarrow N_1 < d \)
- □ → ■ at rate 1 if \( N_1 \geq N_2 \Leftrightarrow N_1 \geq d \)
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define
  
  \( \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \)

At the player level

- \( \square \rightarrow \blacksquare \) at rate 1 only if \( N_1 < N_2 \Leftrightarrow N_1 < d \)
- \( \blacksquare \rightarrow \square \) at rate 1 if \( N_1 \geq N_2 \Leftrightarrow N_1 \geq d \)
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0, 1\}^d$ and define
  
  \[ \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0 \} \]

At the player level

- ■ → □ at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- □ → ■ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level

- $\bullet$ at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- $\blacksquare$ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  $$P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0$$

- Let $H_z = 2z + \{0, 1\}^d$ and define
  
  $$\bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}$$

At the player level

- $\blacksquare \rightarrow \square$ at rate 1 only if $N_1 < N_2 \leftrightarrow N_1 < d$
- $\square \rightarrow \blacksquare$ at rate 1 if $N_1 \geq N_2 \leftrightarrow N_1 \geq d$
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[
P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0
  \]
- Let \( H_z = 2z + \{0, 1\}^d \) and define
  \[
  \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}
  \]

At the player level

- \( \blackBox \rightarrow \whiteBox \) at rate 1 only if \( N_1 < N_2 \iff N_1 < d \)
- \( \whiteBox \rightarrow \blackBox \) at rate 1 if \( N_1 \geq N_2 \iff N_1 \geq d \)
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0,1\}^d$ and define
  \[ \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0 \} \]

At the player level

- □ → □ at rate 1 if $N_1 < N_2 \iff N_1 < d$
- □ → ■ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0,1\}^d$ and define
  
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0 \} \]

At the player level

- ■ → ■ at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- ■ → □ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]
- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0}\]

At the player level

- $\bullet$ at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- $\blacksquare$ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]

- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level

- ■ → □ at rate 1 only if $N_1 < N_2 \Leftrightarrow N_1 < d$
- □ → ■ at rate 1 if $N_1 \geq N_2 \Leftrightarrow N_1 \geq d$
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define
  \[ \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \} \]

At the player level
- \( \square \rightarrow \square \) at rate 1 only if \( N_1 < N_2 \leftrightarrow N_1 < d \)
- \( \square \rightarrow \blacksquare \) at rate 1 if \( N_1 \geq N_2 \leftrightarrow N_1 \geq d \)
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level

- ■ → □ at rate 1 only if \( N_1 < N_2 \leftrightarrow N_1 < d \)
- □ → ■ at rate 1 if \( N_1 \geq N_2 \leftrightarrow N_1 \geq d \)

- Monotonicity: \( P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1 \)
  Infinite time limit \( \bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t \) exists
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]
- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \} \]

At the player level

- at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$

- Monotonicity: $P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1$
  Infinite time limit $\bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t$ exists

At the hypercube level

- not possible
Best-response dynamics
Bootstrap percolation
Best-response dynamics  Bootstrap percolation
Best-response dynamics

Bootstrap percolation
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  
  $P(x \in \bar{\eta}_t) \leq P(x \in \eta_t)$ when $\bar{\eta}_0 \subset \eta_0$

- Let $H_z = 2z + \{0, 1\}^d$ and define
  
  $\tilde{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z$ and $H_z \subset \eta_0\}$

At the player level

- $\blackBox \rightarrow \whiteBox$ at rate 1 only if $N_1 < N_2 \Leftrightarrow N_1 < d$
- $\whiteBox \rightarrow \blackBox$ at rate 1 if $N_1 \geq N_2 \Leftrightarrow N_1 \geq d$

- Monotonicity: $P(\tilde{\eta}_s \subset \tilde{\eta}_t$ for all $s < t) = 1$

  Infinite time limit $\tilde{\eta}_\infty = \lim_{t \rightarrow \infty} \tilde{\eta}_t$ exists

At the hypercube level

- $\blackBox \rightarrow \whiteBox$ not possible
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[
P(x \in \eta_t) \leq P(x \in \eta_t) \quad \text{when} \quad \eta_0 \subset \eta_0
  \]

- Let \( H_z = 2z + \{0, 1\}^d \) and define
  \[
  \eta_0 = \{x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0\}
  \]

At the player level

- \[ \square \rightarrow \blacksquare \quad \text{at rate 1 if} \quad N_1 \geq N_2 \iff N_1 \geq d \]
- \[ \square \rightarrow \blacksquare \quad \text{at rate 1 only if} \quad N_1 < N_2 \iff N_1 < d \]

- Monotonicity: \( P(\eta_s \subset \eta_t \quad \text{for all} \quad s < t) = 1 \)
  
  Infinite time limit \( \eta_\infty = \lim_{t \to \infty} \eta_t \) exists

At the hypercube level

- \[ \blacksquare \rightarrow \square \quad \text{not possible} \]
- \[ \square \rightarrow \blacksquare \quad \text{if at least one type 1 neighbor in each of the} \ d \ \text{directions} \]
Interacting particle system: \( a_1 > a_2 > 0 \)
- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0 \]
- Let \( H_z = 2z + \{0,1\}^d \) and define
  \[ \bar{\eta}_0 = \{ x \in \mathbb{Z}^d : x \in H_z \quad \text{and} \quad H_z \subset \eta_0 \} \]

At the player level
- \( \square \rightarrow \blacksquare \) at rate 1 only if \( N_1 < N_2 \Leftrightarrow N_1 < d \)
- \( \blacksquare \rightarrow \square \) at rate 1 if \( N_1 \geq N_2 \Leftrightarrow N_1 \geq d \)

- Monotonicity: \( P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1 \)
  Infinite time limit \( \bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t \) exists

At the hypercube level
- \( \blacksquare \rightarrow \square \) not possible
- \( \square \rightarrow \blacksquare \) if at least one type 1 neighbor in each of the \( d \) directions

Bootstrap percolation (Schonmann, 1992)
Discrete time process: \( z \in \xi_{t+1} \) if and only if
  \[ z \in \xi_t \quad \text{or} \quad \xi_t \cap \{ z - e_i, z + e_i \} \neq \emptyset \quad \text{for all} \quad i \]
Interacting particle system: $a_1 > a_2 > 0$

- The process is attractive
  \[ P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \text{ when } \bar{\eta}_0 \subset \eta_0 \]
- Let $H_z = 2z + \{0, 1\}^d$ and define
  \[ \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\} \]

At the player level
- \[
  \begin{array}{c}
    \square \rightarrow \square \text{ at rate 1 only if } N_1 < N_2 \Leftrightarrow N_1 < d \\
    \square \rightarrow \blacksquare \text{ at rate 1 if } N_1 \geq N_2 \Leftrightarrow N_1 \geq d
  \end{array}
\]
- Monotonicity: $P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1$
  Infinite time limit $\bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t$ exists

At the hypercube level
- \[
  \begin{array}{c}
    \blacksquare \rightarrow \square \text{ not possible} \\
    \square \rightarrow \blacksquare \text{ if at least one type 1 neighbor} \\
    \text{in each of the } d \text{ directions}
  \end{array}
\]

Bootstrap percolation (Schonmann, 1992)

Discrete time process: $z \in \xi_{t+1}$ if and only if
  \[ z \in \xi_t \text{ or } \xi_t \cap \{z - e_i, z + e_i\} \neq \emptyset \text{ for all } i \]

Conclusion: \[ \{z : H_z \subset \bar{\eta}_\infty\} \supset \lim_{t \to \infty} \xi_t \]
Interacting particle system: \( a_1 > a_2 > 0 \)

- The process is attractive
  \[
P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0
  \]
- Let \( H_z = 2z + \{0, 1\}^d \) and define
  \[
  \bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}
  \]

At the player level

- \( \square \rightarrow \square \) at rate 1 only if \( N_1 < N_2 \iff N_1 < d \)
- \( \square \rightarrow \blacksquare \) at rate 1 if \( N_1 \geq N_2 \iff N_1 \geq d \)

- Monotonicity: \( P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1 \)
  Infinite time limit \( \bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t \) exists

At the hypercube level

- \( \blacksquare \rightarrow \square \) not possible
- \( \square \rightarrow \blacksquare \) if at least one type 1 neighbor in each of the \( d \) directions

Bootstrap percolation (Schonmann, 1992)

Discrete time process: \( z \in \xi_{t+1} \text{ if and only if } \)

\[
z \in \xi_t \quad \text{or} \quad \xi_t \cap \{z - e_i, z + e_i\} \neq \emptyset \quad \text{for all } i
\]

Conclusion: \( \{z : H_z \subset \bar{\eta}_\infty\} \supset \lim_{t \to \infty} \xi_t = \mathbb{Z}^d \)
Interacting particle system: $a_1 > a_2 > 0$
- The process is attractive
  $$P(x \in \bar{\eta}_t) \leq P(x \in \eta_t) \quad \text{when} \quad \bar{\eta}_0 \subset \eta_0$$
- Let $H_z = 2z + \{0, 1\}^d$ and define
  $$\bar{\eta}_0 = \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0\}$$

At the player level
- ■ → □ at rate 1 only if $N_1 < N_2 \iff N_1 < d$
- □ → ■ at rate 1 if $N_1 \geq N_2 \iff N_1 \geq d$

- Monotonicity: $P(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) = 1$
  Infinite time limit $\bar{\eta}_\infty = \lim_{t \to \infty} \bar{\eta}_t$ exists

At the hypercube level
- ■ → □ not possible
- □ → ■ if at least one type 1 neighbor in each of the $d$ directions

Bootstrap percolation (Schonmann, 1992)
- Discrete time process: $z \in \xi_{t+1}$ if and only if $z \in \xi_t$ or $\xi_t \cap \{z - e_i, z + e_i\} \neq \emptyset$ for all $i$

Conclusion: $\{z : H_z \subset \bar{\eta}_\infty\} \supset \lim_{t \to \infty} \xi_t = \mathbb{Z}^d$ (1 ESS)
References

- **John Maynard Smith and George Price.**

- **Martin Nowak.**
  *Evolutionary dynamics: Exploring the equations of life.*

- **Stephen Evilsizor and Nicolas Lanchier.**
  Evolutionary games on the lattice: best-response dynamics.

- **Nicolas Lanchier.**
  Evolutionary games on the lattice: payoffs affecting birth and death rates.

- **Stephen Evilsizor and Nicolas Lanchier.**
  Evolutionary games on the lattice: death-birth updating process.

- **Eric Foxall and Nicolas Lanchier.**
  Evolutionary games on the lattice: death and birth of the fittest.