Stochastic spatial model for the division
of labor in social insects

Alesandro Arcuri* and Nicolas Lanchier†
School of Mathematical and Statistical Sciences,
Arizona State University, Tempe, AZ 85287, USA
* alesandro.arcuri@asu.edu
† nicolas.lanchier@asu.edu

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Motivated by the study of social insects, we introduce a stochastic model based on
interacting particle systems in order to understand the effect of communication on the
division of labor. Members of the colony are located on the vertex set of a graph repre-
senting a communication network. They are characterized by one of two possible tasks,
which they update at a rate equal to the cost of the task they are performing by either
defecting by switching to the other task or cooperating by anti-imitating a random
neighbor in order to balance the amount of energy spent in each task. We prove that, at
least when the probability of defection is small, the division of labor is poor when there
is no communication, better when the communication network consists of a complete
graph, but optimal on bipartite graphs with bipartite sets of equal size, even when both
tasks have very different costs. This shows a non-monotonic relationship between the
number of connections in the communication network and how well individuals organize
themselves to accomplish both tasks equally.

Keywords: Interacting particle systems; anti-voter model; social insects; division of labor;
task allocation.

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1. Introduction

This work is primarily motivated by the study of social insects such as ants, honey
bees, wasps and termites but applies more generally to any population whose individu-
als have the ability to exchange information in order to get organized socially.

* Current address: Department of Economics, Cornell University, 455 Uris Hall, Ithaca, NY 14853,
USA.
† Corresponding author
More specifically, we are interested in the following two fundamental components of social insects colonies.

**Communication system.** Social insects are characterized by their well-developed communication system. For instance, ants communicate with each other using some pheromones that they most often leave on the soil surface to mark a trail leading from the colony to a food source. But pheromones are also used by ants to let other colony members know what task group they belong to. Ants can also communicate by direct contact, using for instance their antennae.

**Division of labor.** The well-developed communication system of social insects is central to complex social behaviors, including division of labor,\(^{13,56}\) i.e. cooperative work. Returning to the example of ants, except for the queens and reproductive males, the other individuals work together to create a favorable environment for the colony and the brood. The repertoire of tasks includes nest maintenance, foraging, brood care and nest defense, which have different costs. While some ants may specialize on a given task, which depends on their age class and morphology, most of the workers are totipotent,\(^{13,55}\) meaning that they are able to perform all tasks and can therefore switch from one task to another in response to the need of the colony.

For a review on the various models of division of labor, we refer to Ref. 13. There, the authors distinguish six classes of models: response threshold, integrated threshold-information transfer, self-reinforcement, foraging for work, social inhibition, and network task allocation models. These six different classes of models are built on six different assumptions about the causes of division of labor in social insects. The stochastic model introduced in this paper belongs to the last class of models: network task allocation models.\(^{29,53}\) Our objective is not to speculate on the communication system or on the division of labor in social insects, but instead to understand how the communication system may affect the division of labor. More precisely, we think of the colony as being spread out on the vertex set of a graph representing a communication network in the sense that two individuals are given the opportunity to communicate if and only if the corresponding vertices are connected by an edge, and study the effects of the topology of the communication network on the division of labor.

**2. Modeling Approach and Related Works**

As previously mentioned, Ref. 13 gives a review of the various modeling approaches for the division of labor in social insects. Another notable contribution where mathematical models are designed to understand the behavior of social insects is Ref. 35. This paper is the first one introducing kinetic theory methods to model social dynamics. To understand the relationships between communication system and division of labor, network task allocation models,\(^{29,53}\) one of the classes of models discussed in Ref. 13, seem to be more appropriate. To design such a model, it is
natural to use the mathematical framework of interacting particle systems.\textsuperscript{25,48,50} These models describe the interactions among particles or individuals that are located on the vertex set of a graph and can only interact with their neighbors, which models explicit space. The main objective of research in the field of interacting particle systems is to deduce the macroscopic behavior and spatial patterns at the population level that emerge from the microscopic rules at the individual level, which usually strongly depends on the topology of the network of interactions.

To our knowledge, the present paper is the first one studying division of labor from the point of view of interacting particle systems. However, there have been some works using this theoretical framework to study other aspects of social dynamics.

**The voter model.** The first historical example is the voter model.\textsuperscript{17,34} In this model, individuals are characterized by their opinion that they update at rate one by imitating one of their neighbors chosen uniformly at random. The first important result about this model is that the process clusters on the one- and two-dimensional integer lattices, meaning that any finite region sees eventually a local consensus with probability converging to one as time goes to infinity, whereas there is convergence in distribution to a non-trivial equilibrium in which both opinions coexist in higher dimensions. This result follows from a certain duality relationship between the voter model and a certain system of coalescing random walks together with the recurrence property of symmetric random walks. Using again this duality relationship, how fast the clusters expand in low dimensions has been studied analytically in Refs. 15 and 22 while how strong the spatial correlations are at equilibrium in higher dimensions has been investigated in Refs. 14 and 57. Another natural problem is to look at the so-called occupation time, the fraction of time a given individual holds opinion 1, and a remarkable result is that the occupation time converges almost surely to the initial density of type 1 individuals in two and higher dimensions.\textsuperscript{21} This means that, in two dimensions, clusters keep growing but also move around and give the impression of local coexistence though strictly speaking both opinions cannot coexist at equilibrium.

Just after the framework of interacting particle systems has been introduced in the early 1970s, only a couple of simple models, including the voter model, were studied, mostly to develop new mathematical tools. Most of these early techniques are reviewed in Refs. 48 and 50. This is only recently that the number of models suddenly exploded, first in physics and biology, and later in the field of social sciences. Models of social dynamics introduced and studied heuristically or numerically by statistical physicists are reviewed in Ref. 16. Of these models, however, only few have been studied analytically, mostly because the techniques in the field of particle systems are model-specific rather than universal. In particular, it is not clear whether the techniques developed in this paper can be used to study even a slight generalization of our model. We now give a very brief review of the models in Ref. 16 that have been studied analytically by probabilists.
The Galam model. The Galam’s majority rule model is introduced in Ref. 28. Like in the voter model, individuals are characterized by one of two possible opinions but now interact by blocks representing discussion groups. Each interaction results in all the individuals in the discussion group to switch to the majority opinion of the group, with opinion 1 being always favored in case of a tie. This model is studied analytically in Ref. 46 while a spatial version based on interacting particle systems is investigated in Ref. 41. Threshold voter models introduced in Ref. 20 are modifications of the classical voter model in which individuals change their strategy at rate one when the number of opponents in their neighborhood exceeds some threshold. Choosing the threshold equal to half of the neighborhood size results in the so-called majority vote model which is closely related to the Galam’s majority rule model. Threshold voter models are studied in Refs. 4, 27 and 49.

The Deffuant model. This model is introduced in Ref. 23 and assumes that individuals’ possible opinions are initially chosen uniformly at random in the unit interval. Neighbors interact only if their opinion distance does not exceed some confidence threshold $\theta \in [0, 1]$, which models homophily, while each interaction results in the opinions’ neighbors getting closer to each other. When the threshold is set equal to one, one recovers the averaging process reviewed in Ref. 3. The main conjecture about the Deffuant model is that the system converges to a consensus when the threshold exceeds one-half whereas disagreements persist when the threshold is smaller. This has first been proved for the process on the integers in Ref. 39 using a geometric approach while shortly after Ref. 30 gave an alternative simpler proof. For related results, we also refer to Refs. 31 and 33.

The Axelrod model. The Axelrod model for the dynamics of cultures is introduced in Ref. 5. Individuals are now characterized by a vector of opinions called cultural features and interact at a rate proportional to the number of cultural features they have in common, which again models homophily. Each interaction results in one more agreement between the two neighbors in case they do not already completely agree. The process on the integers either clusters, driving the system to a global consensus like in the voter model, or fixates, meaning that individuals update their culture only a finite number of times, which leads to a fragmented configuration in which disagreements persist. The outcome depends on the number of cultural features and the number of possible states per cultural feature. The consensus phase is studied analytically in Refs. 38 and 45 while the fixation regime is studied in Refs. 40 and 42. For consensus results, see also Ref. 47. A closely-related model is the so-called vectorial Deffuant model which is studied analytically in Ref. 44. See also Ref. 43 for a more general class of opinion models with confidence threshold.

Rumor processes. Other models of interest in social sciences based on interacting particle systems are rumor processes. Individuals are again located on the vertex set of a graph but now characterized by whether or not they are aware of some rumor. The process evolves in discrete time and individuals learn about the rumor
from individuals who became aware of it at the previous time step and are located within some radius of influence. The main question is whether the rumor can spread without bound or not. This problem is studied in Refs. 10 and 36 and we also refer to Ref. 37 for a related work. Such a dynamics is obviously reminiscent of epidemics and forest fire models whose spatial version using interacting particle systems has first been studied in Refs. 19 and 26.

Mathematical models of interest in social sciences, not based on interacting particle systems as defined in Refs. 48 and 50 but on the closely-related framework of statistical mechanics and kinetic theory, have also been extensively studied. The literature in this field is rather copious. For a review of statistical mechanics models of opinion, cultural and language dynamics using heuristic arguments and/or numerical simulations, we again refer to Ref. 16. Looking at the literature in kinetic theory, we refer to Refs. 12 and 54 for models of opinion formation. For references on animal behavior like the present paper, see for instance Ref. 24. For models in socio-economics, we refer to Refs. 1 and 52. For a survey of mathematical models in kinetic theory covering topics such as traffic flow, swarms and crowd behavior, see Refs. 6–9 and 18. See also Refs. 2 and 11 for models at the intersection of kinetic theory and game theory.

Returning to our original problem, to obtain a model for the dynamics of tasks, we now think of individuals as being characterized by the task they are performing, and use a variant of the anti-voter model, also called the dissonant voting model, where individuals anti-imitate a randomly chosen neighbor. The general idea is that individuals have no information about the overall division of labor beyond knowing which tasks their neighbors on the communication network are performing so, in order to balance the amount of energy spent in each task, they will try to perform a task different from their neighbors. Though our model has some similarities with previous network task allocation models, this paper relies on analytical results rather than just numerical simulations and is therefore more mathematical in spirit than previous works on this topic.

3. Model Description

To construct our model for the dynamics of tasks, we first let $G = (V, E)$ be a graph representing a communication network. The colony is spread out on the vertex set of this graph, with exactly one individual at each of the vertices, and two individuals are given the opportunity to communicate if and only if there is an edge connecting the corresponding two vertices. Individuals are characterized by the task they are performing, which they update at random times based on the information they get from their neighbors on the graph. For simplicity, we assume that the communication network is static, that there are only two tasks to be performed and that each individual is always performing exactly one task. As previously mentioned, it is natural to employ the framework of interacting particle systems in which we keep track of the task performed by each individual using a
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continuous-time Markov chain whose state at time $t$ is a configuration

$$\xi_t : V \rightarrow \{1, 2\} = \text{set of tasks to be performed},$$

with $\xi_t(x)$ denoting the task performed at time $t$ by the individual at vertex $x$. The evolution rules depend on a couple of parameters. First, we assume that performing task $i$ has a cost $c_i > 0$, which is included in the dynamics by interpreting the cost of a task as the rate at which an individual performing this task attempts to switch to the other task. We also assume that both tasks are equally important for the survival of the colony so, in the best case scenario, each time an individual communicates with a neighbor, it will cooperate by anti-imitating this neighbor in order to balance the amount of energy spent in each task. We consider an additional parameter $\epsilon$ that we interpret as probability of defection and is included in the dynamics by assuming that, each time an individual wants to switch task:

- with probability $\epsilon$, the individual defects by switching to the other task without communicating with any of its neighbors whereas
- with probability $1-\epsilon$, the individual cooperates by communicating with a random neighbor and anti-imitating this neighbor.

Note that individuals with no neighbor have no other choice than always defecting due to their lack of knowledge. To describe the evolution rules formally, we let $N_x := \{y \in V : (x, y) \in E\}$ for all $x \in V$ be the interaction neighborhood of vertex $x$ and

$$f_i(x, \xi) := \text{card}\{y \in N_x : \xi(y) = i\}/\text{card}(N_x) \text{ for } i = 1, 2$$

be the fraction of neighbors of $x$ that are performing task $i$ when the system is in configuration $\xi$, which we assume to be zero when $x$ has no neighbor. Then, the transition rates described above verbally can be written into equations as:

$$1 \rightarrow 2 \text{ at rate } c_1(\epsilon + (1-\epsilon)(1-f_2(x, \xi))),$$

$$2 \rightarrow 1 \text{ at rate } c_2(\epsilon + (1-\epsilon)(1-f_1(x, \xi))). \quad (3.1)$$

By our convention (the fraction of neighbors is zero when there is no neighbor), when a vertex has no neighbor, it switches from task $i$ to the other task at rate $c_i$ in agreement with our verbal description of the model. Note also that the anti-voter model\textsuperscript{51} is simply obtained by setting $c_1 = c_2$ and $\epsilon = 0$, though this special case is not of particular interest in our biological context.

**Division of labor.** The main objective is to understand how the topology of the communication network affects the division of labor, which we model using the random variable

$$\phi(s) := \frac{1}{s \text{ card}(V)} \int_0^s X_t \, dt \quad \text{where} \quad X_t := \sum_{x \in V} 1\{\xi_t(x) = 1\}. \quad (3.2)$$

The process $X_t$ keeps track of the number of individuals performing task 1 at time $t$. We point out that this is not a Markov process in general because the rate
at which the number of individuals performing a given task varies depends on the specific location of these individuals on the communication network. The random variable \( \phi(s) \) represents the fraction of time up to time \( s \) and averaged across all the colony task 1 has been performed. Lemma 5.1 will show that, at least when the defection probability is positive, this random variable converges almost surely to a limit that does not depend on the initial configuration of the system. Under our assumption that both tasks are equally important for the survival of the colony, the division of labor is optimal when this limit is one-half and poor when the limit is close to either zero or one.

4. Main Results

To fix the ideas, we assume from now on without loss of generality that the first task is the less costly therefore we have \( c_1 < c_2 \). To start with a reference value, assume that the communication network is completely disconnected, i.e. there is no edge, meaning no communication. In this case, an individual performing task \( i \) switches to the other task at rate \( c_i \) so, by independence,

\[
\lim_{s \to \infty} \phi(s) = \lim_{t \to \infty} P(\xi_t(x) = 1) = \bar{v}_1 := c_2(c_1 + c_2)^{-1} > \frac{1}{2}
\]

almost surely for all \( x \in V \). Note that the division of labor converges almost surely to the same limit \( \bar{v}_1 \) for all communication networks when the defection probability is equal to one, since in this case the individuals never communicate with their neighbors.

Looking now at more complex communication networks, Fig. 1 shows the limit of the division of labor as a function of the defection probability obtained from numerical simulations of the process on each of the first three graphs of Fig. 2 but with 1000 instead of eight vertices like in the picture. As expected, the simulation results suggest that the division of labor gets improved, moving closer to one-half, as the defection probability decreases and converges almost surely to the reference value \( \bar{v}_1 \) as the defection probability increases to one. In addition, at least for small defection probabilities, the division of labor is better for the process on large complete graphs than for the process on completely disconnected graphs. However, the simulation results also suggest something less obvious: the division of labor is much better on the one-dimensional torus where individuals can only communicate with their two nearest neighbors than on the complete graph where individuals can communicate with every other individual. This reveals a non-monotonic relationship between the number of connections and how well individuals organize themselves to accomplish both tasks equally, which we now prove analytically.

To begin with, we give the exact limit of the division of labor for the process on complete graphs for all values of the defection probability. In particular, we obtain an explicit expression for the equation of the curve in solid line in Fig. 1. Then, we give lower and upper bounds for the limit of the division of labor when the defection probability is small and for the process on bipartite graphs, which will also give a
proof that the dashed curve in Fig. 1 indeed starts at one-half using that the ring with an even number of vertices is an example of a bipartite graph. Finally, we will prove one more result for the process on the one-dimensional integer lattice to extend to infinite graphs the results obtained for finite bipartite graphs.

Complete graphs. To begin with, we examine the process on a complete graph. We recall that the complete graph with $N$ vertices, denoted by $K_N$, is such that $(x, y) \in E$ if and only if $x, y \in V$ with $x \neq y$,

indicating that all the individuals can communicate with each other. From a modeling perspective, this is realistic for well-mixing colonies. In this case, we have the following result.

**Theorem 4.1.** Let $G = K_N$ and $\epsilon > 0$. Let $B := (1 - \epsilon)(1 - 1/N)^{-1}$.

- Regardless of the initial state,

\[
\lim_{s \to \infty} \phi(s) = \bar{u}_1(B) := \frac{1}{2} - \frac{1}{2B} \left( \frac{c_1 + c_2}{c_1 - c_2} + \sqrt{(B - 1)^2 + \frac{4c_1c_2}{(c_1 - c_2)^2}} \right). \quad (4.1)
\]
The function $B \mapsto \bar{u}_1(B)$ is decreasing on $(0, 2)$ and
\[ \lim_{B \to 0} \bar{u}_1(B) = \bar{v}_1 = \frac{c_2}{c_1 + c_2} > \bar{u}_1(1) = \frac{\sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}} > \bar{u}_1(2) = \frac{1}{2}. \]

Noticing that $B(\epsilon, N)$ is decreasing with respect to both the probability of defection and the colony size, the monotonicity of the function $\bar{u}_1$ implies the following:

- The smaller the probability of defection $\epsilon$, the better the division of labor, i.e. the fraction of individuals performing any given task gets closer to one-half. This conclusion is expected from the definition of our model: cooperation improves the division of labor.
- The smaller the colony size $N$, the better the division of labor. The intuition behind this result is that, at each update of an individual, the knowledge this individual gets about the tasks performed by the rest of the colony is only through a single random individual therefore the amount of information obtained from each interaction about the overall division of labor gets smaller as the size of the colony increases.
The three different values of the limiting fraction of individuals performing task 1 in the second part of the theorem are also interesting from a biological point of view.

- The limit as $B \to 0$, i.e. $\epsilon \to 1$, is not obvious because (4.1) is not defined at $B = 0$. However, the process is well defined when $\epsilon = 1$. In this case, there is no cooperation so the process evolves as when the communication network is completely disconnected, which explains the value of the limit: $\tilde{v}_1 = c_2(c_1 + c_2)^{-1}$.

- The value at $B = 1$ is the most interesting one biologically because it gives the limiting behavior when the colony is large and the probability of defection is small. In this case, cooperation acts positively by making the fraction of individuals performing any given task closer to one-half compared to the scenario where $\epsilon = 1$.

- The value at $B = 2$ corresponds to $\epsilon = 0$ and $N = 2$ and can be easily guessed: there is perfect cooperation and only two individuals so the two states where one individual performs task 1 and the other task 2 are absorbing states, which gives the value $1/2$.

**Bipartite graphs.** Theorem 4.1 shows that, at least for complete graphs, the division of labor is optimal, i.e. converges to one-half, when there are only two vertices and the defection probability converges to zero. We now extend this result to finite, connected, bipartite graphs. Recall that a graph is bipartite when there exists a partition $\{V_1, V_2\}$ of the vertex set into so-called bipartite sets such that vertices in the same bipartite set cannot be neighbors, i.e.

$$(x, y) \in E \implies (x, y) \in V_1 \times V_2 \text{ or } (x, y) \in V_2 \times V_1.$$  

In other words, individuals are separated into two groups and only communicate with members of the opposite group (see Fig. 2(d)). At least when the defection probability is small, the dynamics of the process tends to separate the two tasks in each of the two bipartite sets. More precisely, the division of labor is bounded from above and from below as follows.

**Theorem 4.2.** Assume that $G$ has $N$ vertices, is connected and bipartite. Then,

- for all $\rho > 0$, there exists $\epsilon_0 > 0$ such that

$$(1 - \rho) \min(N_1/N, N_2/N) \leq \lim_{s \to \infty} \phi(s) \leq (1 + \rho) \max(N_1/N, N_2/N)$$

almost surely for all $\epsilon < \epsilon_0$, where $N_i = \text{card}(V_i)$ for $i = 1, 2$.

Note that a complete graph is a bipartite graph if and only if it has only two vertices. In this case, the previous theorem implies that, for all $\epsilon < \epsilon_0$,

$$\frac{1}{2}(1 - \rho) \leq \lim_{s \to \infty} \phi(s) \leq \frac{1}{2}(1 + \rho),$$

(4.2)

showing that the division of labor converges to one-half as $\epsilon \to 0$. In particular, Theorem 4.2 indeed extends the last limit in the statement of Theorem 4.1. Figure 1(a) and more generally any torus with degree two and an even number of vertices are
other examples of bipartite graphs where both bipartite sets have the same cardinality. In particular, the theorem implies that (4.2) holds for all these graphs, which gives a proof of the simulation results of Fig. 1 for the dashed curve in the neighborhood of $\epsilon = 0$. Another example of communication network more relevant for ecologists is to assume that the colony is spread out on the two-dimensional grid with vertex set

$$V := \mathbb{Z}^2 \cap \{[0, L] \times [0, H]\} \text{ where } L, H \in \mathbb{N}^*$$

and where there is an edge between vertices at Euclidean distance one from each other. In this context, the individuals are mostly static and can only communicate with individuals which are close to them. This is an example of bipartite graph whose bipartite sets are given by:

$$V_1 := \{ x \in V : x_1 + x_2 \text{ is odd} \} \quad \text{and} \quad V_2 := \{ x \in V : x_1 + x_2 \text{ is even} \}.$$ 

These two sets differ in cardinality by at most one so

$$(1 - \rho)(1/2 - 1/N) \leq \lim_{s \to \infty} \phi(s) \leq (1 + \rho)(1/2 + 1/N),$$

for all $\epsilon > 0$ small according to Theorem 4.2, showing that, at least when $\epsilon$ is small and the colony size is large, the division of labor again approaches one-half. Since these bipartite graphs have more edges than completely disconnected graphs but less edges than complete graphs, this shows a non-monotonic relationship between the degree distribution of the communication network and how well individuals organize themselves to accomplish both tasks equally.

**One-dimensional lattice.** The last communication network we consider is the one-dimensional integer lattice: individuals are arranged in a line and can only communicate with their two nearest neighbors on the left and on the right. Our main motivation is to show that more sophisticated techniques can be used to study the process on a simple infinite graph. In this case, the existence of the process follows from a general result due to Harris.\textsuperscript{32} Because the graph is infinite, the random variables in (3.2) are no longer well defined therefore we study instead the division of labor in an increasing sequence of spatial intervals

$$\phi_N(s) := \frac{1}{s(2N + 1)} \int_0^s \sum_{|x| \leq N} 1\{\xi_t(x) = 1\} \, dt$$

as time $s$ and space $N$ both go to infinity. Then, for the process on the infinite one-dimensional lattice, we have the following result that extends in part Theorem 4.2.

**Theorem 4.3.** Assume that $G$ is the one-dimensional integer lattice and that the initial distribution of the process is a Bernoulli product measure. Then,

- for all $\rho > 0$, there exists $\epsilon_0 > 0$ such that

  $$\frac{1}{2}(1 - \rho) \leq \lim_{s,N \to \infty} \phi(s, N) \leq \frac{1}{2}(1 + \rho) \quad \text{for all } \epsilon < \epsilon_0.$$
The theorem states that, when the defection probability is small, the fraction of individuals performing task 1 in a large spatial interval approaches one-half. Even though the one-dimensional lattice is a bipartite graph, the result does not simply follow from Theorem 4.2 because the two bipartite sets are infinite. Instead, the proof relies on a coupling between the dynamics of tasks and a system of annihilating random walks together with large deviation estimates.

5. Proof of Theorem 4.1 (Complete Graphs)

This section is devoted to the proof of Theorem 4.1 which assumes that the communication network is a complete graph. To prove that the division of labor converges almost surely to a limit that does not depend on the initial state, we start with a general lemma about finite graphs that will be used in this section and the next one. Now, in the special case of complete graphs, the process $X_t$ that keeps track of the number of individuals performing task 1 turns out to be a continuous-time birth and death process. From this, we will deduce that the expected fraction of individuals performing task 1 satisfies a certain ordinary differential equation. The exact value of the almost-sure limit of the division of labor is then obtained by showing that this limit is the unique fixed point in the biologically relevant interval $(0,1)$ of the differential equation.

**Lemma 5.1.** Let $G$ be finite and $\epsilon > 0$. Then $\phi(s)$ converges almost surely to a limit that does not depend on the initial configuration of the stochastic process.

**Proof.** Let $\xi$ be a configuration and $x$ be a vertex, and assume that $\xi(x) = i$. Since $\epsilon > 0$, the rate at which vertex $x$ switches task is bounded from below by

$$c_i(\epsilon + (1 - \epsilon)(1 - f_j(x, \xi))) \geq cc_i \geq c \xi > 0 \quad \text{where} \quad \{i, j\} = \{1, 2\}.$$

This shows that the process is irreducible. Since in addition the communication network is finite, the state space of the process is finite as well. In particular, standard Markov chain theory implies that there is a unique stationary distribution $\pi$ to which the process converges weakly starting from any initial configuration. Moreover, we have the almost-sure convergence

$$\lim_{s \to \infty} \phi(s) = E_\pi \left( \frac{1}{N} \sum_{x \in V} 1\{\xi(x) = 1\} \right),$$

where the integer $N$ denotes the number of vertices. \hfill $\Box$

In the next lemma, we compute the limit (5.1) for complete graphs. Following the notation of the theorem, we let $\bar{u}_1(B)$ denote this limit where

$$B = (1 - \epsilon) \left( 1 - \frac{1}{N} \right)^{-1}.$$
Lemma 5.2. For all $\epsilon \in (0,1]$ and $N \geq 2$, we have
\[
\bar{u}_1(B) = \frac{1}{2} - \frac{1}{2B} \left( \frac{c_1 + c_2}{c_1 - c_2} + \sqrt{(B - 1)^2 + \frac{4c_1c_2}{(c_1 - c_2)^2}} \right).
\] (5.2)

Proof. The communication network being a complete graph, the transition rates (3.1) are invariant by permutation of the vertices therefore the spatial configuration of individuals is no longer relevant and the process $X_t$ itself is Markov. More precisely, since the state can only increase or decrease by one unit at a positive rate, this process is a continuous-time birth and death process, thus characterized by its birth and death parameters:
\[
\beta_j := \lim_{h \to 0} \frac{1}{1/h} P(X_{t+h} = j + 1 | X_t = j),
\]
\[
\delta_j := \lim_{h \to 0} \frac{1}{1/h} P(X_{t+h} = j - 1 | X_t = j).
\]

By standard properties of Poisson processes, the birth parameter $\beta_j$ is obtained by multiplying the number of individuals performing task 2 by the common rate at which each of these individuals switches to task 1. Together with (3.1), this gives
\[
\beta_j = c_2(N - j) \left( \epsilon + (1 - \epsilon) \left( \frac{N - j - 1}{N - 1} \right) \right) \quad \text{for } j = 0, 1, \ldots, N.
\] (5.3)

By symmetry, the death parameters are given by
\[
\delta_j = c_1 j \left( \epsilon + (1 - \epsilon) \left( \frac{N - j - 1}{N - 1} \right) \right) \quad \text{for } j = 0, 1, \ldots, N.
\] (5.4)

The next step is to use the transition rates (5.3) and (5.4) to derive a differential equation for the expected value of the fraction of individuals performing a given task at time $t$, namely:
\[
u_1(t) := E(X_t/N) \quad \text{and} \quad \nu_2(t) := E(1 - X_t/N).
\]

Even though these functions depend on the initial state, by Lemma 5.1, their limit as time goes to infinity does not. To compute the limit, we first observe that
\[
u'_1 = \lim_{h \to 0} (1/h)(u_1(t+h) - u_1(t))
\]
\[
= \lim_{h \to 0} (1/Nh)E(X_{t+h} - X_t | X_t = Nu_1(t))
\]
\[
= (1/N) \lim_{h \to 0} (1/h)P(X_{t+h} = Nu_1(t) + 1 | X_t = Nu_1(t))
\]
\[
- (1/N) \lim_{h \to 0} (1/h)P(X_{t+h} = Nu_1(t) - 1 | X_t = Nu_1(t))
\]
\[
= (1/N)\beta_{Nu_1(t)} - (1/N)\delta_{Nu_1(t)}.
\]
Using also (5.3) and (5.4), we deduce that
\[ u'_1 = c_2 u_2 \left( \epsilon + (1 - \epsilon) \left( 1 - \frac{N u_1}{N - 1} \right) \right) - c_1 u_1 \left( \epsilon + (1 - \epsilon) \left( 1 - \frac{N u_2}{N - 1} \right) \right) \]
\[ = c_2 u_2 - c_1 u_1 + (1 - \epsilon) \left( \frac{1}{1 - 1/N} \right) (c_1 - c_2) u_1 u_2 =: Q(u_1, u_2). \]

Recalling that \( B = (1 - \epsilon)(1 - 1/N)^{-1} \) and \( u_1 + u_2 = 1 \), we get
\[ Q(u_1, u_2) = Q(u_1, 1 - u_1) \]
\[ = c_2(1 - u_1) - c_1 u_1 + B(c_1 - c_2)(1 - u_1) u_1 \]
\[ = c_2 - (c_1 + c_2) u_1 + B(c_1 - c_2) u_1 - B(c_1 - c_2) u_1^2. \]

Since this is a polynomial with degree two in \( u_1 \) and that
\[ Q(0, 1) = c_2 > 0 \quad \text{and} \quad Q(1, 0) = -c_1 < 0, \]
we obtain the existence of a unique fixed point in \((0, 1)\), namely \( \bar{u}_1(B) \), which is globally asymptotically stable. Basic algebra shows that the discriminant is
\[ \Delta = \left( B(c_1 - c_2) - (c_1 + c_2) \right)^2 + 4Bc_2(c_1 - c_2) \]
\[ = B^2(c_1 - c_2)^2 + (c_1 + c_2)^2 - 2B(c_1 + c_2)(c_1 - c_2) + 4Bc_2(c_1 - c_2) \]
\[ = B^2(c_1 - c_2)^2 + (c_1 + c_2)^2 - 2B(c_1 - c_2)^2 \]
\[ = B^2(c_1 - c_2)^2 + (c_1 - c_2)^2 - 2B(c_1 - c_2)^2 + 4c_1 c_2 \]
\[ = (B - 1)^2(c_1 - c_2)^2 + 4c_1 c_2, \]
from which it follows that \( u_1(t) = E(X_t/N) \) converges to
\[ \bar{u}_1(B) = \frac{B(c_1 - c_2) - (c_1 + c_2)}{2B(c_1 - c_2)} = \frac{1}{2B} \sqrt{\frac{\Delta}{(c_1 - c_2)^2}} \]
\[ = \frac{1}{2} - \frac{1}{2B} \left( \frac{c_1 + c_2}{c_1 - c_2} + \sqrt{(B - 1)^2 + \frac{4c_1 c_2}{(c_1 - c_2)^2}} \right). \]

This completes the proof. \( \square \)

The first part of Theorem 4.1 follows from Lemmas 5.1 and 5.2. We now deal with the second part, i.e. we study the function \( \bar{u}_1(B) \), which gives some insight into the role of the probability of defection and the size of the colony in the division of labor. The monotonicity of this function and its value at biologically relevant values of the parameter \( B \) are established in the next four lemmas.

**Lemma 5.3.** The function \( B \mapsto \bar{u}_1(B) \) is decreasing on \((0, 2)\).
To begin with, we observe that

\[
\sqrt{(B-1)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}} \leq \sqrt{1 + \frac{4c_1c_2}{(c_1-c_2)^2}} = \sqrt{\left(\frac{c_1-c_2}{c_1-c_2}\right)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}} = -\frac{c_1+c_2}{c_1-c_2}, \tag{5.5}
\]

Then, we distinguish two cases.

Assume first that \( B < 1 \). Inequality (5.5) implies that

\[
\bar{u}_1(B) = \frac{1}{2} - \frac{1}{2B}\left(\frac{c_1+c_2}{c_1-c_2}\right) = \frac{1}{2B}\sqrt{(B-1)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}} \geq \frac{1}{2},
\]

showing that \( \bar{u}_1(B) \) is the unique solution \( X \in [1/2, 1] \) of

\[
P(B, X) = c_2 - (c_1 + c_2)X + B(c_1 - c_2)X - B(c_1 - c_2)X^2 = 0.
\]

Since in addition, for all \( B \in (0, 1) \) and \( X \in [1/2, 1] \),

\[
\partial_B P(B, X) = (c_1-c_2)X - (c_1 - c_2)X^2 = (c_1 - c_2)(1-X)X < 0,
\]

\[
\partial_X P(B, X) = -(c_1 + c_2) + B(c_1 - c_2) - 2B(c_1-c_2)X
\]

\[
= -(c_1 + c_2) + B(c_1 - c_2)(1-2X)
\]

\[
\leq -(c_1 + c_2) - B(c_1 - c_2) = -(B+1)c_1 + (B-1)c_2 < 0,
\]

the limit \( \bar{u}_1(B) \) is decreasing on \( (0, 1) \).

Assume now that \( B \geq 1 \). Since

\[
\bar{u}_1'(B) = \frac{1}{2B^2}\left(\frac{c_1+c_2}{c_1-c_2} + \sqrt{(B-1)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}}\right)
\]

\[
- \frac{B-1}{2B}\sqrt{(B-1)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}},
\]

using again (5.5) gives

\[
\bar{u}_1'(B) = -\frac{B-1}{2B}\sqrt{(B-1)^2 + \frac{4c_1c_2}{(c_1-c_2)^2}} \leq 0,
\]

showing that \( \bar{u}_1(B) \) is also decreasing on \( [1, 2) \).

As explained in Sec. 1, the value in the limit as \( B \to 0 \) and at \( B = 2 \) can be understood returning to the stochastic model. We now give a proof using the expression (5.2).

**Lemma 5.4.** We have \( \lim_{B \to 0} \bar{u}_1(B) = \bar{v}_1 \). 

\[\square\]
Proof. Using a Taylor expansion and (5.5), we get
\[
\sqrt{(B - 1)^2 + \frac{4c_1c_2}{(c_1 - c_2)^2}} = \sqrt{1 + \frac{4c_1c_2}{(c_1 - c_2)^2}} - B\left(1 + \frac{4c_1c_2}{(c_1 - c_2)^2}\right)^1\right) + o(B)
\]
when \(B\) is small. In particular,
\[
\lim_{B \to 0} \bar{u}_1(B) = \frac{1}{2} - \lim_{B \to 0} \frac{1}{2B} \left(\frac{c_1 + c_2}{c_1 - c_2} - \sqrt{\frac{4c_1c_2}{(c_1 - c_2)^2}}\right)
\]
\[
= \frac{1}{2} - \frac{1}{2} \frac{c_1 - c_2}{c_1 + c_2} = \frac{c_2}{c_1 + c_2} = \bar{v}_1.
\]
This completes the proof.

Lemma 5.5. We have \(\bar{u}_1(1) = \sqrt{c_2(\sqrt{c_1} + \sqrt{c_2})^{-1}}\).

Proof. Taking \(B = 1\) in the expression (5.2), we get
\[
\bar{u}_1(1) = \frac{1}{2} - \frac{1}{2} \left(\frac{c_1 + c_2}{c_1 - c_2}\right) - \frac{1}{2} \sqrt{\frac{4c_1c_2}{(c_1 - c_2)^2}}
\]
\[
= \frac{c_2}{c_1 - c_2} - \sqrt{\frac{c_1c_2}{(c_1 - c_2)^2}} = \frac{\sqrt{c_2(\sqrt{c_2} - \sqrt{c_1})}}{c_2 - c_1} = \frac{\sqrt{c_2}}{\sqrt{c_2 + \sqrt{c_1}}}
\]
This completes the proof.

Lemma 5.6. We have \(\bar{u}_1(2) = 1/2\).

Proof. Using (5.5) once more, we get
\[
\bar{u}_1(2) = \frac{1}{2} + \frac{1}{4} \left(\frac{c_1 + c_2}{c_1 - c_2} + \sqrt{1 + \frac{4c_1c_2}{(c_1 - c_2)^2}}\right)
\]
\[
= \frac{1}{2} + \frac{1}{4} \left(\frac{c_1 + c_2}{c_1 - c_2} - \frac{c_1 + c_2}{c_1 - c_2}\right) = \frac{1}{2}.
\]
This completes the proof.

The combination of Lemmas 5.3–5.6 gives the second part of the theorem.

6. Proof of Theorem 4.2 (Bipartite Graphs)

This section is devoted to the proof of Theorem 4.2. Throughout this section, we assume that the communication network is a finite, connected, bipartite graph.
Recall that a graph is bipartite when there exists a partition \( \{V_1, V_2\} \) of the vertex set such that

\[
(x, y) \in E \quad \text{implies that} \quad (x, y) \in V_1 \times V_2 \quad \text{or} \quad (x, y) \in V_2 \times V_1. \quad (6.1)
\]

First, we prove that, when \( \epsilon = 0 \), the two configurations:

\[
\xi_+ := 1_{V_1} + 2 \times 1_{V_2} \quad \text{and} \quad \xi_- := 1_{V_2} + 2 \times 1_{V_1}
\]

are the only two absorbing states, from which the theorem easily follows when \( \epsilon = 0 \).

Note that these configurations are simply the configurations in which all the vertices in the same bipartite set perform the same task and all the individuals in the other bipartite set perform the other task. To deal with positive defection probability, in which case the process becomes irreducible and converges weakly to a unique stationary distribution according to the proof of Lemma 5.1, the key ingredient is to prove that the fraction of time the process spends in one of the two configurations \( \xi_\pm \) in the long run can be made arbitrarily close to one by choosing \( \epsilon > 0 \) small enough.

**Lemma 6.1.** Let \( \epsilon = 0 \). Then \( \xi_- \) and \( \xi_+ \) are the only two absorbing states.

**Proof.** Since \( \epsilon = 0 \) and \( G \) is connected, (3.1) becomes:

\[
1 \rightarrow 2 \quad \text{at rate} \quad c_1(1 - f_2(x, \xi)) = c_1f_1(x, \xi),
\]

\[
2 \rightarrow 1 \quad \text{at rate} \quad c_2(1 - f_1(x, \xi)) = c_2f_2(x, \xi).
\]

It follows that \( \xi \) is an absorbing state if and only if

\[
f_i(x, \xi) = 0 \quad \text{for all} \quad x \in V \quad \text{such that} \quad \xi(x) = i \quad (6.2)
\]

for \( i = 1, 2 \), indicating that no two neighbors perform the same task. In view of (6.1), the two configurations \( \xi_- \) and \( \xi_+ \) clearly satisfy this property so they are absorbing states. To prove that there are no other absorbing states, fix \( \xi \) different from both \( \xi_\pm \). In such a configuration, there are two individuals in the same bipartite set that are performing different tasks. By obvious symmetry, we may assume without loss of generality that

\[
\xi(x) \neq \xi(y) \quad \text{for some} \quad x, y \in V_1. \quad (6.3)
\]

Now, since the graph is connected and (6.1) holds, there exists a path with an even number of edges connecting vertices \( x \) and \( y \), i.e. there exist \( x_0, x_1, \ldots, x_{2n} \) such that:

\[
x_0 = x \quad \text{and} \quad x_{2n} = y \quad \text{and} \quad (x_0, x_1), (x_1, x_2), \ldots, (x_{2n-1}, x_{2n}) \in E. \quad (6.4)
\]

It follows from (6.3) and (6.4) that

\[
\xi(x_j) = \xi(x_{j+1}) \quad \text{for some} \quad j = 0, 1, \ldots, 2n - 1,
\]

showing that there are two neighbors performing the same task. In particular, (6.2) fails, which implies that configuration \( \xi \) is not an absorbing state. \( \square \)
It follows from Lemma 6.1 and the fact that the graph is connected that, when \( \epsilon = 0 \), the process has exactly three communication classes, namely:

\[
C_- := \{\xi_-\}, \quad C_0 := \{\xi \in \{1, 2\}^V : \xi \notin \{\xi_-, \xi_+\}\}, \quad C_+ := \{\xi_+\}.
\]

The classes \( C_- \) and \( C_+ \) are closed but \( C_0 \) is not. Since in addition the graph is finite, the process gets trapped eventually in one of its two absorbing states.

By the proof of Lemma 5.1, when \( \epsilon > 0 \), there is a unique stationary distribution \( \pi \) to which the process converges weakly starting from any initial state. To establish the theorem in this case, the idea is to prove that the fraction of time spent in \( \xi \pm \) can be made arbitrarily large by choosing the defection probability sufficiently small. Using the stationary distribution \( \pi \), this statement can be expressed as follows: for all \( \rho > 0 \), there exists \( \epsilon_0 > 0 \) such that

\[
P_\pi(\xi = \xi_+) + P_\pi(\xi = \xi_-) > 1 - \rho \quad \text{for all} \ \epsilon < \epsilon_0. \tag{6.5}
\]

To show (6.5), we define the stopping times:

\[
T_{\text{in}} := \inf\{t : \xi_t \in \{\xi_-, \xi_+\}\} = \inf\{t : \xi_t \notin C_0\}, \quad T_{\text{out}} := \inf\{t : \xi_t \notin \{\xi_-, \xi_+\}\} = \inf\{t : \xi_t \in C_0\}
\]

and give upper/lower bounds for the expected values:

\[
\tau_{\text{in}} := \sup_{\epsilon, \xi_0} E(T_{\text{in}} | \xi_0 \notin \{\xi_-, \xi_+\}) = \sup_{\epsilon, \xi_0} E(T_{\text{in}} | \xi_0 \in C_0),
\]

\[
\tau_{\text{out}} := \inf_{\xi_0} E(T_{\text{out}} | \xi_0 \in \{\xi_-, \xi_+\}) = \inf_{\xi_0} E(T_{\text{out}} | \xi_0 \notin C_0).
\]

This is done in the next two lemmas.

**Lemma 6.2.** We have \( \tau_{\text{in}} < \infty \).

**Proof.** Fix \( \xi \in C_0 \). Then (6.2) does not hold so, when the system is in configuration \( \xi \), there exist two neighbors \( x, y \) performing the same task, say task \( i \). In particular, the rate at which vertex \( x \) switches task is bounded from below by

\[
c_i(\epsilon + (1 - \epsilon)f_i(x, \xi)) \geq c_1(\epsilon + (1/N)(1 - \epsilon)) \geq c_1/N > 0.
\]

Since this lower bound does not depend on \( \epsilon \) and since the communication class \( C_0 \) is finite and not closed, we deduce that

\[
c(\xi) := \sup_\epsilon E(T_{\text{in}} | \xi_0 = \xi) < \infty \quad \text{for all} \ \xi \in C_0.
\]

Using again that \( C_0 \) is finite, we conclude that

\[
\tau_{\text{in}} = \sup_{\xi \in C_0} c(\xi) < \infty.
\]

This completes the proof. \( \square \)

**Lemma 6.3.** We have \( \tau_{\text{out}} \geq (\epsilon Nc_2)^{-1} \).
Proof. Intuitively, the result follows from the fact that the transition rates of the process are continuous with respect to $\epsilon$ and the fact that, when $\epsilon = 0$, the expected time $\tau_{out}$ is infinite because the process cannot leave its absorbing states. To make this precise, let $x$ be a vertex performing say task $i$. Then, according to (6.2), the rate at which this vertex switches its task given that the system is in one of the two configurations $\xi_{\pm}$ is bounded by

$$c_i (\epsilon + (1 - \epsilon)f_i(x, \xi_{\pm})) = \epsilon c_i \leq \epsilon c_2.$$ 

Since there are $N$ vertices, standard properties of independent Poisson processes imply that the rate at which the process jumps in the set $C_0$ is bounded by

$$\lim_{h \to 0} \frac{1}{h} P(\xi_t + h \in C_0 | \xi_t \notin C_0) \leq \epsilon N c_2.$$ 

In conclusion, we have

$$\tau_{out} = \inf_{\xi_0} E(T_{out} | \xi_0 \notin C_0) \geq E(\text{Exponential}(\epsilon N c_2)) = (\epsilon N c_2)^{-1},$$

which completes the proof.

With Lemmas 6.1–6.3, we are now ready to prove the theorem.

Proof of Theorem 4.2. The result is obvious for $\epsilon = 0$. Indeed, in this case, Lemma 6.1 implies that the process gets trapped in one of its two absorbing states therefore

$$\lim_{s \to \infty} \phi(s) = \min(N_1/N, N_2/N)1\{\xi_t = \xi_+ \text{ for some } t\} + \max(N_1/N, N_2/N)1\{\xi_t = \xi_- \text{ for some } t\}.$$

In particular, the limit belongs to

$$\{\min(N_1/N, N_2/N), \max(N_1/N, N_2/N)\} \subset ((1 - \rho) \min(N_1/N, N_2/N), (1 + \rho) \max(N_1/N, N_2/N)).$$

Now, assume that $\epsilon > 0$. Then, according to the proof of Lemma 5.1, the process converges weakly to a unique stationary distribution $\pi$ so standard Markov chain theory implies that the fraction of time the process spends in one of the configurations $\xi_{\pm}$ converges almost surely to

$$\pi(\xi_-) + \pi(\xi_+) = P_\pi(\xi_t \notin C_0) \geq \tau_{out}(\tau_{in} + \tau_{out})^{-1}. \quad (6.6)$$

But according to Lemmas 6.2 and 6.3, there exists $\epsilon_0 > 0$ such that

$$\tau_{out}(\tau_{in} + \tau_{out})^{-1} \geq (1 + \epsilon N c_2 \tau_{in}) \geq 1 - \rho \quad \text{for all } \epsilon < \epsilon_0. \quad (6.7)$$

Combining (6.6) and (6.7) together with (5.1), we get

$$\lim_{s \to \infty} \phi(s) \geq \min(N_1/N, N_2/N)P_\pi(\xi_t \notin C_0) \geq (1 - \rho) \min(N_1/N, N_2/N).$$
Similarly, since \( \max(\frac{N_1}{N}, \frac{N_2}{N}) \geq \frac{1}{2} \), we have
\[
\lim_{s \to \infty} \phi(s) \leq P_\pi(\xi_t \in C_0) + \max(\frac{N_1}{N}, \frac{N_2}{N})P_\pi(\xi_t \notin C_0)
\leq \rho + (1 - \rho) \max(\frac{N_1}{N}, \frac{N_2}{N})
\leq (1 + \rho) \max(\frac{N_1}{N}, \frac{N_2}{N}).
\]
This completes the proof of the theorem.

7. Proof of Theorem 4.3 (One-Dimensional Lattice)

The final communication network we study is the infinite one-dimensional integer lattice. Throughout this section, we assume that the initial configuration of the process is a Bernoulli product measure. The main objective is to prove that the limiting probability
\[
\lim_{s \to \infty} P(\xi_s(x) = \xi_s(x+1)) \quad \text{where } x \in \mathbb{Z}
\]
(7.1)
can be made arbitrarily small by choosing \( \epsilon > 0 \) sufficiently small. Since both the initial distribution and the evolution rules are translation invariant, this limiting probability does not depend on the particular choice of vertex \( x \). The theorem will follow from our estimate of the limit in (7.1) and the ergodic theorem.

The process on the one-dimensional lattice, or any (infinite) graph whose degree is uniformly bounded, can be constructed using the following rules and collections of independent Poisson processes, called a graphical representation\(^3\): for each oriented edge \((x, y)\) in \( E \):

- we draw a solid arrow \( y \to x \) at the times of a Poisson process with rate \( (1 - \epsilon)c_1 \) to indicate that vertex \( x \) anti-imitates vertex \( y \),
- we draw a dashed arrow \( y \leftrightarrow x \) at the times of a Poisson process with rate \( (1 - \epsilon)(c_2 - c_1) \) to indicate that if vertex \( x \) performs task 2 then it anti-imitates vertex \( y \),
- we put a dot • at vertex \( x \) at the times of a Poisson process with rate \( \epsilon c_1 \) to indicate that vertex \( x \) switches task,
- we put a cross \( \times \) at vertex \( x \) at the times of a Poisson process with rate \( \epsilon(c_2 - c_1) \) to indicate that if \( x \) performs task 2 then it switches task.

Since our objective is to compare the tasks performed by neighbors, rather than studying the dynamics of tasks on the set of vertices, it is more convenient to keep track of the dynamics of agreements along the edges. To do this, we put a type \( i \) particle on edge \( e \) if and only if the two individuals connected by this edge perform the same task \( i \). Returning to the special case of the one-dimensional lattice, this results in a process \((\zeta_t)\) defined as
\[
\zeta_t((x, y)) := 1[\xi(x) = \xi(y) = 1] + 2 \times 1[\xi(x) = \xi(y) = 2]
\]
for all \( x = y \pm 1 \in \mathbb{Z} \). In particular, edges in state zero, that we call empty edges, connect individuals performing different tasks. The process \((\zeta_t)\) is not Markov because
the configuration in which all the edges are empty can result in two different configurations of tasks on the vertices. It can be proved, however, that the pair \((\zeta_t, \xi_t(0))\) is a Markov process, but since this is not relevant to establish our theorem, we do not show this result. To understand the dynamics on the edges, we now look at the effect of the graphical representation on the configuration of particles.

- Solid arrows \(y \rightarrow x\) only affect the task at \(x\), and so the state of the two edges incident to \(x\), if and only if both vertices \(x\) and \(y\) perform the same task before the interaction. This leads to the two cases illustrated on the left-hand side of Fig. 3 as well as the additional two cases obtained by exchanging the roles of tasks 1 and 2.
- Dashed arrows have the same effect as solid arrows but only if both vertices \(x\) and \(y\) perform task 2 before the interaction.
- Similarly, dots at vertex \(x\) only affect the task at \(x\), and so only the state of the two edges incident to vertex \(x\). This leads to the four cases illustrated on the right-hand side of Fig. 3 as well as the additional four cases obtained by exchanging the roles of the tasks.
- Crosses have the same effect as dots only if \(x\) performs task 2 before the interaction.

In view of the transitions in Fig. 3 and the rates of the Poisson processes used in the graphical representation, the dynamics on the edges can be summarized as follows:

- Type \(i\) particles jump at rate \((1 - \epsilon)c_i\) to the right or to the left with probability one-half and change their type at each jump.
• There are spontaneous births of pairs of particles of type $i$ at rate $\epsilon c_i$ at edges incident to a vertex not performing task $i$.

• When two particles occupy the same edge as the result of a jump or a spontaneous birth, both particles annihilate.

Using the dynamics of type 1 particles and type 2 particles, we now study the probability in (7.1), which is the probability that the edge connecting vertices $x$ and $x+1$ is empty.

To prove that the limiting probability in (7.1) can be made arbitrarily small, we use the construction shown in Fig. 4. The proof also requires three simple estimates for the probability of the three events illustrated in the picture. These three estimates are given in the following three lemmas, where $\rho$ is a small positive constant like in the statement of the theorem. First, we look at the special case when the defection probability is equal to zero.

**Lemma 7.1.** Let $\epsilon = 0$. Then, there exists $T_1 < \infty$ such that
\[
\sup_{\xi} P(\xi_t(x) = \xi_t(x+1) \mid \xi_{t-T} = \xi) < \rho/3 \quad \text{for all } t \geq T \geq T_1.
\]

**Proof.** Note that, when $\epsilon = 0$, there is no spontaneous birth so the system of particles reduces to a system of annihilating symmetric random walks on the one-dimensional lattice that jump either at rate $c_1$ or at rate $c_2$. Since such a system goes extinct, there is $T_1 < \infty$ such that
\[
\sup_{\xi} P(\xi_T(x) = \xi_T(x+1) \mid \xi_0 = \xi) = \sup_{\zeta} P(\zeta_T((x,x+1)) \neq 0 \mid \zeta_0 = \zeta) < \rho/3 \quad \text{for all } T \geq T_1,
\]
but since $(\xi_t)$ is a time-homogeneous Markov chain,
\[
\sup_{\xi} P(\xi_t(x) = \xi_t(x+1) \mid \xi_{t-T} = \xi) = \sup_{\xi} P(\xi_T(x) = \xi_T(x+1) \mid \xi_0 = \xi) < \rho/3 \quad \text{for all } t \geq T \geq T_1.
\]
This completes the proof.

The second step of the proof is to show that, with probability close to one, the set of space-time points that may have influenced the state of edge $(x, x+1)$ at time $t$, which is traditionally called the influence set in the field of interacting particle systems, cannot grow too fast. Referring to the graphical representation, the influence set is the set of space-time points from which there is an oriented path of solid and dashed arrows leading to either $x$ or $x+1$ at time $t$. More formally, the influence set is defined as
\[
I_t(x, x+1) := \{(y,s) \in \mathbb{Z} \times \mathbb{R}_+: (y,s) \leadsto (x,t) \text{ or } (y,s) \leadsto (x+1,t)\}, \quad (7.2)
\]
where \((y, s) \sim (z, t)\) means that there exist
\[ y = x_0, x_1, \ldots, x_n = z \in \mathbb{Z} \quad \text{and} \quad s_0 < s_1 < \cdots < s_{n+1}, \]
such that, for \(j < n\), there is an arrow \(x_j \rightarrow x_{j+1}\) or \(x_j \leftarrow x_{j+1}\) at time \(s_{j+1}\). To state our next two lemmas, we also introduce:
\[
J_T := \{(y, s) \in \mathbb{Z} \times \mathbb{R}_+ : -[c_2T] < y < [c_2T] \text{ and } s \leq T\},
\]
\[
K_T := \{(y, s) \in \mathbb{Z} \times \mathbb{R}_+ : -[c_2T] \leq y \leq [c_2T] \text{ and } s \leq T\},
\]
where \([r]\) is the smallest integer not less than \(r\).

**Lemma 7.2.** There exists \(T_2 < \infty\) such that, for all \(t \geq T \geq T_2\),
\[
P(I_t(x,x+1) \cap (\mathbb{Z} \times [t-T,t])) \not\subset (x,t-T) + J_T) < \rho/3.
\]

**Proof.** For all \(0 \leq s \leq t\), we write
\[
I_t(x,x+1) \cap (\mathbb{Z} \times \{t-s\}) = ([l_s, r_s] \cap \mathbb{Z}) \times \{t-s\}.
\]
In other words, \(l_s\) and \(r_s\) are the spatial location of the leftmost point and the spatial location of the rightmost point in the influence set at time \(t-s\). Since arrows of either type from a given vertex to a given neighbor appear in the graphical representation at rate at most \(c_2/2\), it follows from the definition of the influence set that
\[ l_s = x - \text{Poisson}(c_2s/2) \quad \text{and} \quad r_s = (x+1) + \text{Poisson}(c_2s/2) \]
in distribution for all \(0 \leq s \leq t\). In particular, standard large deviation estimates for the Poisson random variable imply that there exists \(T_2 < \infty\) such that
\[
P(I_t(x,x+1) \cap (\mathbb{Z} \times [t-T,t])) \not\subset (x,t-T) + J_T)
= P(I_t(x,x+1) \cap (\mathbb{Z} \times \{t-T\}) \not\subset (x-[c_2T],x+[c_2T]) \times \{t-T\})
\leq P(T \leq x-[c_2T]) + P(r_T \geq x+[c_2T])
\leq 2P(\text{Poisson}(c_2T/2) \geq [c_2T]-1) \leq \exp(-c_2T/8) < \rho/3
\]
for all \(t \geq T \geq T_2\). This completes the proof. \( \square \)

The third step is to show that, with probability close to one when the defection probability is small, there are no spontaneous births in a translation of \(K_T\).

**Lemma 7.3.** For all \(T\), there exists \(\epsilon_0 > 0\) such that, for all \(t \geq T\),
\[
P(\text{there is a } \bullet \text{ or a } \times \text{ in } (x,t-T) + K_T) < \rho/3 \quad \text{for all } \epsilon < \epsilon_0.
\]

**Proof.** Dots and crosses appear at each vertex altogether at rate \(c_2\) hence number of \(\bullet\)’s and \(\times\)’s in \((x,t-T) + K_T = \text{Poisson}(2[c_2T] + 1)T \epsilon c_2\)
Lemma 7.1 = no particle on \( (x, x + 1) \) when \( \epsilon = 0 \)

Lemma 7.2 = portion of the influence set included in this space-time region

Lemma 7.3 = no birth in this space-time region

\[ I_T \]

\[ I_T \cap (\mathbb{Z} \times [t - T, t]) \]

\( (x, t - T) + K_T \)

\( t - T \)

\( t \)

\( x - [c_2T] \)

\( x + [c_2T] \)

in distribution. This implies that, for all \( T \),

\[
P(\text{there is a } \bullet \text{ or a } \times \text{ in } (x, t - T) + K_T) = P(\text{Poisson}(2\lceil c_2T \rceil + 1)Tc_2) \neq 0
\]

\[
\leq 1 - \exp(-(2c_2T + 3)Tc_2) \leq (2c_2T + 3)Tc_2
\]

can be made smaller than \( \rho/3 \) by choosing \( \epsilon > 0 \) small.

\[ \square \]

Lemmas 7.1–7.3 identify three events with small probability. The complement of these events, which occur with high probability, is depicted in Fig. 4. With these three lemmas in hand, we are now ready to prove the theorem.

**Proof of Theorem 4.3.** Let \( \rho > 0 \) small, then:

- fix \( T_1 \) and \( T_2 \) such that the conclusions of Lemmas 7.1 and 7.2 hold and
- fix \( \epsilon_0 > 0 \) such that Lemma 7.3 holds for \( T := \max(T_1, T_2) \).

For these parameters, we will prove that

\[
P(\xi_t(x) = \xi_t(x + 1) < \rho \quad \text{for all } (x, t) \in \mathbb{Z} \times (T, \infty) \text{ and } \epsilon < \epsilon_0.
\]

(7.3)

To do this, let \( (\xi^0_t) \) be the process starting from the same initial configuration and constructed from the same graphical representation as \( (\xi_t) \) but ignoring the dots
and crosses in the graphical representation after time $t - T$, where time $t > T$ is fixed. In particular,
\[
\xi_s(y) = \xi_0^s(y) \quad \text{for all } (y, s) \in \mathbb{Z} \times [0, t - T].
\] (7.4)

Note also that the influence sets (7.2) of both processes coincide since both processes are constructed from the same collections of arrows and since the dots and the crosses are irrelevant in the construction of the influence sets. This, together with (7.4), implies that
\[
\xi_t(x) = \xi_0^t(x) \quad \text{and} \quad \xi_t(x + 1) = \xi_0^t(x + 1),
\] (7.5)

unless there is a dot or a cross after time $t - T$, when the two processes may disagree, in the neighborhood of the influence set, i.e.
\[
\text{there is a } \bullet \text{ or a } \times \text{ in } (I_t(x, x + 1) + [-1, 1]) \cap (\mathbb{Z} \times [t - T, t]),
\]
which occurs whenever
\[
I_t(x, x + 1) \cap (\mathbb{Z} \times [t - T, t]) \not\subset (x, t - T) + J_T.
\] (7.6)

Combining (7.5) and (7.6) and applying Lemmas 7.1–7.3, we deduce that
\[
P(\xi_t(x) = \xi_t(x + 1)) \leq P(\xi_0^t(x) = \xi_0^t(x + 1)) + P(\xi_t \neq \xi_0^t \text{ on } \{x, x + 1\})
\]
\[
\quad + P(\text{there is a } \bullet \text{ or a } \times \text{ in } (x, t - T) + K_T)
\]
\[
\quad + P(I_t(x, x + 1) \cap (\mathbb{Z} \times [t - T, t]) \not\subset (x, t - T) + J_T)
\]
\[
< \rho/3 + \rho/3 + \rho/3 = \rho.
\] (7.7)

This shows (7.3). In particular, for all $t > T$,
\[
P(\xi_t(x) = 1) = P(\xi_t(x) = 1 \mid \xi_t(x) = \xi_t(x + 1))P(\xi_t(x) = \xi_t(x + 1))
\]
\[
\quad + P(\xi_t(x) = 1 \mid \xi_t(x) \neq \xi_t(x + 1))P(\xi_t(x) \neq \xi_t(x + 1))
\]
\[
= P(\xi_t(x) = 1 \mid \xi_t(x) = \xi_t(x + 1))P(\xi_t(x) = \xi_t(x + 1))
\]
\[
\quad + \frac{1}{2}P(\xi_t(x) \neq \xi_t(x + 1))
\]
\[
\leq 1 \times \rho + \frac{1}{2}(1 - \rho) = \frac{1}{2}(1 + \rho).
\] (7.7)

Similarly, for all $t > T$, we have
\[
P(\xi_t(x) = 1) \geq 0 \times \rho + \frac{1}{2}(1 - \rho) = \frac{1}{2}(1 - \rho).
\] (7.8)

In addition, since the initial distribution of the process is a Bernoulli product measure and the Poisson processes in the graphical representation are independent, the
The ergodic theorem is applicable, from which it follows that, for all \( t > T \),
\[
\lim_{N \to \infty} \frac{1}{2N + 1} \sum_{|x| \leq N} 1\{\xi_t(x) = 1\} = P(\xi_t(x) = 1).
\] (7.9)

Combining (7.7)–(7.9), we conclude that
\[
\frac{1}{2}(1 - \rho) \leq \lim_{s, N \to \infty} \phi_N(s) = \lim_{s, N \to \infty} \frac{1}{s(2N + 1)} \int_0^s \sum_{|x| \leq N} 1\{\xi_t(x) = 1\} dt
\]
\[
= \lim_{s, N \to \infty} \frac{1}{(s - T)(2N + 1)} \int_T^s \sum_{|x| \leq N} 1\{\xi_t(x) = 1\} dt
\]
\[
\leq \frac{1}{2}(1 + \rho),
\]
which completes the proof of the theorem.

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