Review of conics

Lecture #17

The general quadratic equation \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \) generates the conic sections - Hyperbolas, Ellipses, and Parabolas, or one of their degenerate forms, if there is any graph at all. The \( xy \) term causes the axes of the conic to be rotated. We will consider the case when \( B = 0 \):

\[
Ax^2 + Cy^2 + Dx + Ey + F = 0, \quad A \text{ and } C \text{ not both } 0.
\]

If the graph is not degenerate, we have the following cases:

1. \( A \) and \( C \) both nonzero and of the same sign. This gives an ellipse and if \( A = C \), a circle.

2. \( A \) and \( C \) both nonzero and of opposite sign. This gives a hyperbola, or in a degenerate case, the asymptotes of a hyperbola.

3. If one of \( A \) or \( C \) is zero, we get a parabola.

The standard forms are:

- **Ellipse**: \( \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \)

- **Hyperbola**: \( \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \) or \( \frac{(y-y_0)^2}{b^2} - \frac{(x-x_0)^2}{a^2} = 1 \)

- **Parabola**: \( (y-y_0) = k(x-x_0)^2 \) or \( (x-x_0)^2 = k(y-y_0) \)
Example: Identify and sketch.

\[ 9x^2 + 4y^2 - 36x + 8y + 4 = 0 \]

This is an ellipse.

\[ \frac{9(x^2 - 4x)}{9} + \frac{4(y^2 + 2y + 1)}{4} = -4 \]

\[ \frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{4} = 1 \]

Example:

\[ -25x^2 + 4y^2 - 50x - 16y - 109 = 0 \]

This is a hyperbola.

\[ -25(x^2 + 2x + 1) + 4(y^2 - 4y + 4) = 109 - 25 + 16 \]

\[ -25(x + 1)^2 + 4(y - 2)^2 = 100 \]

\[ \frac{(x + 1)^2}{4} + \frac{(y - 2)^2}{25} = 1 \]
\[ 2x^2 - 8x - y + 7 = 0 \]

This is a parabola

\[ 2(x^2 - 4x + 4) = y - 7 + 8 \]

\[ 2(x-2)^2 = y + 1 \]

\[ 2(x-2)^2 - 1 = y \]

**Example:** \[ x^2 - 4y^2 - 2x + 1 = 0 \]

This should be a hyperbola

\[ (x^2 - 2x + 1) - 4y^2 = -1 + 1 \]

\[ (x-1)^2 - 4y^2 = 0 \]

\[ \frac{(x-1)^2}{4} - \frac{y^2}{1} = 0 \]

\[ (\frac{x-1}{2} + y) (\frac{x-1}{2} - y) = 0 \]

\[ y = -\frac{x-1}{2} = -\frac{1}{2}x + \frac{1}{2} = -\frac{1}{2}(x-1) \]

\[ y = \frac{1}{2}x - \frac{1}{2} = \frac{1}{2}(x-1) \]

As just the asymptotes.
Exercises on Conics

For each problem below, identify the conic, put the equation in standard form, and do a sketch by hand.

Problem 1
\[ x^2 - 6x + 8y - 7 = 0. \]

Problem 2
\[ 2x + y^2 - 8y + 12 = 0. \]

Problem 3
\[ 9x^2 - 18x + 4y^2 = 27. \]

Problem 4
\[ 16x^2 - 9y^2 + 64x - 90y = 305. \]

Problem 5
\[ 2y^2 - 3x^2 - 4y + 12x + 8 = 0. \]

Problem 6
\[ x^2 + 2y^2 - 6x + 4y + 7 = 0. \]

Problem 7
\[ 4x^2 - 25y^2 = 0. \]