Lecture 36

Divergence & Curl of a vector field.

\[ \mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle = \langle P, Q, R \rangle \]

\[ \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \]

**Def:** The curl of a vector field is \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} \)

\[
\left( \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
Q & R & P
\end{array} \right)
\]

\[ \nabla \times \mathbf{F} = \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle \]

**Example:** \( \mathbf{F} = \langle xy-z, x^2+2yz, x^2+y^2+z^2 \rangle \)

\[ \nabla \times \mathbf{F} = \left( \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
xy & z & \frac{\partial}{\partial x} + \frac{\partial}{\partial y}
\end{array} \right) = \left\langle \frac{\partial}{\partial y} - \frac{\partial}{\partial z}, \frac{\partial}{\partial z} - \frac{\partial}{\partial x}, \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right\rangle = \left\langle 2y - 2z, z - 2x, 2x - 2z \right\rangle = \left\langle 0, xy - 2x, 2x - z \right\rangle \]

**Example:** \( \mathbf{F} = \langle x, y, z \rangle \)

\[ \nabla \times \mathbf{F} = \left( \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & z & x
\end{array} \right) = \left\langle 0, 0, 0 \right\rangle = \mathbf{0} \]

A field with \( \nabla \times \mathbf{F} = \mathbf{0} \) is called curl free or irrotational.
Theorem: Any gradient field in $C^1$ is curl-free.

Suppose $F$ is a gradient field so $F = \nabla f = (f_x, f_y, f_z)$

\[ \nabla \times F = \left( \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial f_y}{\partial x} & \frac{\partial f_z}{\partial x} & \frac{\partial f_x}{\partial x} \\
f_y & f_z & f_x
\end{array} \right) = (f_{yz} - f_{zy}, f_{zx} - f_{xz}, f_{xy} - f_{yx}) \]

$= 0$ by Clairaut's theorem.

Let's look at this situation in 2D.

$\vec{F} = \langle P(x,y), Q(x,y) \rangle$, only express it in 3D:

$\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0 \rangle$

\[ \nabla \times \vec{F} = \left( \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial Q}{\partial x} & \frac{\partial P}{\partial x} & \frac{\partial Q}{\partial x} \\
Q & P & Q
\end{array} \right) = (0, 0, Q_x - P_y) \]

$\therefore \nabla \times \vec{F} = 0$ is the same criteria as $Q_x = P_y$ which is why we called that curl-free.

The Big Theorem is true for 3-D vectors if the condition

$Q_x = P_y$ is replaced by $\nabla \times \vec{F} = 0$

$\therefore$ The test for independence of path and potential function is $\nabla \times F = 0$. 

Example: Show $\mathbf{F} = \langle y, \ x+z, \ y+6z \rangle$
has a potential function and find it.

$\nabla \mathbf{F} = \left( \begin{array}{c}
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array} \right) \mathbf{F} = \langle 1-1, 0, 1-1 \rangle = \langle 0, 0, 0 \rangle$

\[ \therefore \text{exists } f(x,y,z) \text{ such that } \mathbf{F} = \nabla f \]

\[ f_x = y \]
\[ f = yx + h(y,z) \]

\[ f_y = x + h_y(y,z) = x + z \]
\[ h_y = z \]
\[ h = zy + g(z) \]

\[ f = yx + zy + g(z) \]

\[ f_z = y + g'(z) = y + 6z \]
\[ g' = 6z \]
\[ g = 3z^2 + C \]

\[ \therefore f = xy + yz + 3z^2 + C \]

(\text{can take } C = 0)
Curl and rotation

Consider an object rotating about a with angular velocity \( \omega \text{ rad/sec} \).

The speed at \((x, y, z)\) is
\[
\mathbf{v} = \mathbf{a} \times \mathbf{\omega} = (1R^2 \sin \alpha) \mathbf{\omega}
\]

The angular velocity vector \( \mathbf{\omega} \) is defined by:
\[
|\mathbf{\omega}| = \omega
\]
Direction = right hand rule (along axis of rotation)

Note that \( |\mathbf{\omega} \times \mathbf{R}| = |\mathbf{\omega}| R \sin \alpha = \omega R \sin \alpha = \mathbf{v} \)

Also note \( \mathbf{\omega} \times \mathbf{R} \) is in the direction of \( \mathbf{v} \).

By the duck theorem \( \mathbf{v} = \mathbf{\omega} \times \mathbf{R} \)

If \( \mathbf{\omega} = \langle \Omega_1, \Omega_2, \Omega_3 \rangle \), \( \mathbf{R} = \langle x, y, z \rangle \) then
\[
\mathbf{v} = \left( \begin{array}{c}
-\Omega_2 z - \Omega_3 y \\
\Omega_3 x - \Omega_1 z \\
\Omega_1 y - \Omega_2 x
\end{array} \right) = \langle -\Omega_2 z - \Omega_3 y, \Omega_3 x - \Omega_1 z, \Omega_1 y - \Omega_2 x \rangle
\]

Let's see what the curl of \( \mathbf{v} \) gives:
\[
\nabla \times \mathbf{v} = \left( \begin{array}{c}
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} \\
\frac{\partial}{\partial x}
\end{array} \right) \mathbf{v} = \left( \begin{array}{c}
\Omega_3 x - \Omega_1 z \\
\Omega_1 y - \Omega_2 x \\
\Omega_2 z - \Omega_3 y
\end{array} \right)
\]
\[
= \langle 2\Omega_1, 2\Omega_2, 2\Omega_3 \rangle
\]

\[
\nabla \times \mathbf{v} = 2\mathbf{\omega}
\]

\[
\mathbf{\omega} = \frac{1}{2} \nabla \times \mathbf{v}
\]

The curl of the velocity is proportional to angular velocity. If \( \nabla \times \mathbf{v} = 0 \), there is no rotation tendency at the point, hence \textbf{irrotational},
The divergence of a vector field \( \mathbf{F} = < p, q, r > \) is
\[
\text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right> \cdot < p, q, r > \\
= p_x + q_y + r_z \quad \text{(This is a scalar)}
\]

Example: \( \mathbf{R} = < e^{xyz}, yz \sin x, x^2 + y^2 + z^2 > \)
\begin{align*}
\nabla \cdot \mathbf{F} &= yz e^{xyz} + z \sin x + 2z \\
\end{align*}

Example
\[
\mathbf{F} = \mathbf{R} = < x, y, z > \\
\nabla \cdot \mathbf{F} &= 1 + 1 + 1 = 3.
\]

**Theorem:** If \( \mathbf{F} \) is a \( C^2 \) vector field then \( \text{div} \left( \text{curl} \mathbf{F} \right) = 0 \)
(The divergence of a curl is zero)

Proof:
\[
\mathbf{F} = < p, q, r > \\
\n\nabla \times \mathbf{F} = \left| \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{array} \right| = < r_y - q_z, p_z - r_x, q_x - p_y > \\
\n\nabla \cdot \nabla \times \mathbf{F} &= r_y - q_z + p_z - r_x + q_x - p_y = 0 \text{ by Clairaut's theorem.}
\]

Note: Therefore if \( \nabla \cdot \mathbf{F} \neq 0 \), \( \mathbf{F} \) is not a curl field.

Example: \( \mathbf{F} = < x, y, z > \) is not the curl of some vector \( \mathbf{G} \) because \( \nabla \cdot \mathbf{F} = 3 \neq 0 \)