Lecture 33  Green's theorem

Let \( C \) be a piecewise smooth curve not crossing itself, surrounding a region \( R \) in the \( x-y \) plane.

Let \( \mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle \) be a vector field with continuous partial derivatives. Then,
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (Q_x - P_y) \, dA
\]

where \( \oint \) means a line integral around a closed curve in the counterclockwise direction.

In other words,
\[
\oint_C P \, dx + Q \, dy = \iint_R Q_x - P_y \, dA
\]

The proof is given on the next page.
\[ \int \int (p \cdot a - p \cdot b) = \int \int p \, dy + \int \int q \, dy \]

\[ \oint p \, dx = (\int_{c_1} + \int_{c_2}) p_1(x, y) \, dx \]

\[ c_1: \quad x = a \rightarrow b \quad y = g(x) \]

\[ \int_{c_1} = \int_{a}^{b} p_1(x, g(x)) \, dx \]

\[ c_2: \quad x = b \
\quad y = f(x) \]

\[ \int_{c_2} = \int_{a}^{b} p_1(x, f(x)) \, dx = -\int_{a}^{b} p_1(x, f(x)) \, dx \]

\[ \oint p \, dx = \int_{a}^{b} p_1(x, g(x)) - p_1(x, f(x)) \, dx \]

\[ = -\int_{a}^{b} p_1(x, y) \left| \frac{y - g(x)}{y - f(x)} \right| \, dy = \int_{a}^{b} p_1(x, y) \, dy \]

\[ \oint q \, dy = \int_{c_3} + \int_{c_4} q(x, y) \, dy \]

\[ c_3: \quad x = a \rightarrow b \quad y = y \]

\[ \int_{c_3} = \int_{c} q(x, y) \, dy \]

\[ c_4: \quad x = a \rightarrow b \quad y = y \]

\[ \oint q \, dy = \int_{c} q(x, y) \, dy = \int_{c} q_1(x, y) \, dy \]

\[ = \int_{c} \int_{y(x)}^{y(y)} q_2(x, y) \, dx \, dy \]
More complicated

\[ \oint_{C_1} G_y \, dx + G_x \, dy = \oint_{C_2} G_y \, dx + G_x \, dy \]

\[ = \int_{R_1} \int_{R_2} \left( \int C_1 \right) (G_x - G_y) \, dA \]

Example:

\[ \oint_C x^2 \, dy + (x+y) \, dy \]

Would take 3 integrations,

\[ = \int_0^2 \int_0^{2-x} (1-x^2) \, dy \, dx \]

\[ = \int_0^2 (1-x^2)(4-x^2) \, dx = \int_0^2 4 - 5x^2 + x^4 \, dx \]

\[ = \left[ 4x - \frac{5}{3}x^3 + \frac{1}{5}x^5 \right]_0 \]

\[ = 8 - \frac{40}{3} + \frac{32}{5} = \frac{40 - 200 + 96}{15} = \frac{16}{15} \]

Example:

\[ \oint_C y^2 \, dx + 3xy \, dy \]

\[ = \oint_{\frac{x^2+y^2}{2}} y^2 \, dx + 3xy \, dy \]

\[ = \int_{R_1} \int_{R_2} (0) \, dA = \int_{R_1} \int_{R_2} y \, dA \]

\[ = \int_0^2 \int_0^\pi r^2 \sin \theta \, r \, d \theta \, dr \]

\[ = \int_0^2 \int_0^\pi \frac{r^3}{3} \sin \theta \, d \theta \]

\[ = \frac{7}{3} (-u^2 \theta) \]

\[ = \frac{14}{3} \]
Region with holes:

$$\oint F \cdot dr = \oint_{c_1} + \oint_{c_2} + \oint_{c_3} F \cdot dr$$

where we go clockwise around the hole.

Example:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the area

$$A = \frac{1}{2} \oint xdy - ydx \quad (\alpha x - \beta y = 2)$$

$$x = a \cos t \quad dx = -a \sin t \, dt$$
$$y = b \sin t \quad dy = b \cos t \, dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab \cos^2 t + ab \sin^2 t \, dt$$

$$= \pi ab$$