Cylindrical and Spherical Coordinates

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

Also \[ x^2 + y^2 = r^2 \]
\[ \tan \theta = \frac{y}{x} \]

Given cylindrical coordinates \( (r, \theta, z) = (2, \frac{2\pi}{3}, 1) \)

Use the first set to get \( x, y, z \):
\[ x = 2 \cos \frac{2\pi}{3} = 2 \left(-\frac{1}{2}\right) = -1 \]
\[ y = 2 \sin \frac{2\pi}{3} = 2 \left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \]
\[ z = 1 \]

Conversely, given rectangular \( (3, -3, -7) \)
\[ \tan \theta = -1, \text{ 4th quadrant} \]
\[ \theta = -\frac{\pi}{4} + 2\pi n \]
\[ r = \sqrt{9 + 9} = 3\sqrt{2} \]

so for an example \( (3\sqrt{2}, -\frac{\pi}{4}, -7) \) is one.

Surfaces involving \( x^2 + y^2 \) are frequently well represented in cylindrical coordinates.
Note the graphs:

\[ r = c \quad (\theta \text{ and } z \text{ missing}) \]

\[ \theta = c \quad (r \geq 0, \text{ and } z \text{ missing}) \]

\[ z = c \quad (r \neq 0 \text{ missing}) \]
Example: \( x^2 + y^2 = a^2 \) (z missing)

is just \( r = a \), which is where the name comes from.

Example: The cone \( z^2 = x^2 + y^2 \)

This becomes \( z^2 = r^2 \) or \( z = \pm r \)

\( z = r \) is the top, \( z = -r \) is the bottom

(\( \theta \) variable missing)

Example: Convert the ellipsoid \( 4x^2 + 4y^2 + z^2 = 1 \)
to cylindrical.

\[
4(x^2 + y^2) + z^2 = 1
\]

\[
4r^2 + z^2 = 1
\]

\( z^2 = 1 - 4r^2 \) \(<\text{implicit, two functions}\)

\( z = \pm \sqrt{1-4r^2} \)

Example: Convert \( 4x + 3y - z = 5 \) to cylindrical

\( x = r \cos \theta \) \( y = r \sin \theta \)

\( z = 4x + 3y - 5 = 4r \cos \theta + 3r \sin \theta - 5 \)

(usually not done unless there is a reason).
Spherical Coordinates

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= \rho \cos \theta \\
\rho &= \rho \sin \phi \\
\phi &= \rho \sin \phi \cos \theta \\
\theta &= \rho \cos \phi
\end{align*}
\]

Also note \( \rho^2 = x^2 + y^2 + z^2 \)

Example. Convert \((\rho, \theta, \phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)\) to \((x, y, z)\)

\[
\begin{align*}
x &= 2 \cos \frac{\pi}{3} \cos \frac{\pi}{4} = 2 \left(\frac{\sqrt{3}}{2}\right) \frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} \\
y &= 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2 \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} \\
z &= 2 \cos \frac{\pi}{3} = 2 \left(\frac{1}{2}\right) = 1
\end{align*}
\]
Example: Convert \((0, 2\sqrt{3}, -2)\) to spherical.

\[
\rho^2 = 0^2 + (2\sqrt{3})^2 + (-2)^2 = 12 + 4 = 16
\]

\[
\rho = 4 \quad \cos \phi = \frac{2}{4} = \frac{1}{2}
\]

\[
\theta = \frac{\pi}{2} \quad \Rightarrow \phi = \frac{2\pi}{3}
\]

\((\rho, \theta, \phi) = (4, \frac{\pi}{2}, \frac{2\pi}{3})\)

Example: Write the equation of the plane \(z = 5\) in spherical coordinates.

Solve \(\rho \cos \phi = 5\) \(\rho = 5 \sec \phi\)

Example: Write the cone \(z^2 = x^2 + y^2\) in spherical.

\[
\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta
\]

\[
\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi
\]

\[
1 = \tan^2 \phi
\]

\[
\tan \phi = \pm 1 \quad 0 \leq \phi \leq \pi
\]

\[
\phi = \frac{\pi}{4}, \frac{3\pi}{4}
\]

Example: The sphere \(x^2 + y^2 + z^2 = r^2\)

becomes \(\rho = r\).