Lecture #19: Lagrange Multipliers.

Suppose we want the extrema value of an objective function $f(x,y)$ subject to some constraint $g(x,y) = c$.

This is the same question as finding the largest $k$ value of $k$ for which the level curve $f(x,y) = k$ intersects $g(x,y) = c$.

This would appear to happen when $f(x,y) = k$ is tangent to $g(x,y) = c$.

So at the point $f$ and $g$ share parallel normal vectors:

At $(x_0, y_0)$ \[ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \]

Similarly, if we want the maximum of $f(x,y,z)$ subject to a constraint $g(x,y,z) = c$, we have the level surface for $f$ tangent to $g(x,y,z) = c$, so at $(x_0, y_0, z_0)$ they have parallel normal vectors:

\[ \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \]
To give a more mathematical argument, let \( P(t) = (x(t), y(t), z(t)) \) a curve on \( g(x, y, z) = c \) passing through \( P \).

Since \( f(x, y, z) \) has an extreme at \( P \), then
\[
\frac{df}{dt} \bigg|_{t=0} = 0 = \nabla f \cdot \frac{dP}{dt}
\]
by the chain rule.

\[
\Rightarrow \nabla f \perp g(x, y, z) = c 
\]
Since \( \nabla g \) is also \( \perp \) to \( g = c \) surface, we have
\[
\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)
\]

In summary, for the two variable case we need to solve
\[
f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = c
\]
for \( \lambda, x, y \).

For the 3 variable case:
\[
f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = c
\]
solved for \((x, y, z)\) and \( \lambda \).

Note: A constrained max or min must occur at one of \((x_0, y_0, z_0)\) or \((x, y, z)\) if there is one. But not every solution necessarily yields a max or min.
Example: Find the max and min values of
\[ f(x, y) = x^2 + y^2 \text{ on } x^2 + y^2 = 4 \]
You can see what the answer is going to be...

\[ 1 = 2x\lambda \]
\[ 1 = 2y\lambda \]
\[ x^2 + y^2 = 4 \]
\[ 2x = 4 \]
\[ x = \pm \sqrt{2} \]
\[ y = \pm \sqrt{2} \]
\[ (\sqrt{2}, \sqrt{2}) \text{ and } (-\sqrt{2}, -\sqrt{2}) \]
\[ f_{\text{max}} = 2\sqrt{2} \quad f_{\text{min}} = -2\sqrt{2} \]

Example: A rectangular box, no lid, 12 m² cardboard.
Find the max volume.
\[ V = xyz \quad \text{constraint: } xy + 2zx + 2zy = 12 \]
\[ x = \lambda(y + 2z) \]
\[ y = \lambda(x + 2z) \]
\[ z = \lambda(2x + 2y) \]
\[ xy + 2zx + 2zy = 12 \]

Multiply:
\[ xy^2 = \lambda(xy + 2x) \]
\[ xy^2 = \lambda(xy + 2y) \]
\[ xy^2 = \lambda(2xy + 2y) \]
\[ 2x = 2y \Rightarrow x = y \]
\[ xy = 2z \Rightarrow y = 2z \]
\[ xy = 2z \Rightarrow y = 2z \]

Put \( x = y = 2z \) in last eqn:
\[ 4z^2 + 4z + 4z^2 = 12 \]
\[ 2^2 = 1 \quad 2 = 1 \]
\[ x = y = 2 \]
When we have a closed bounded region, we may use a combination of critical point / Lagrange.

**Ex:** Find the extrema of \( f = x^2 + 2y^2 \) on \( x^2 + y^2 \leq 1 \)

\[
\begin{align*}
  f_x &= 2x, & f_y &= 4y \\
  \text{(0,0)} &\text{ is a c.p.} & f(0,0) &= 0
\end{align*}
\]

Now the identity, \( x^2 + y^2 = 1 \)

\[
\nabla f = \langle 2x, 4y \rangle \\
\nabla g = \langle 2x, 2y \rangle
\]

\[
\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle
\]

\[
2x = 2\lambda x \\
2y = 2\lambda y
\]

\[
\begin{align*}
x &= 0 & x(1-\lambda) &= 0 & \Rightarrow & x = 0 \text{ or } \lambda = 1 \\
y &= \frac{1}{2} y & 2y(2-\lambda) &= 0
\end{align*}
\]

If \( x = 0 \) we can have \( y = \pm 1 \) \( (0, \pm 1) \text{ (appropriate)} \)

At \( \lambda = 1 \), then \( y = 0 \Rightarrow x = \pm 1 \) \( (\pm 1, 0) \)

\[
\begin{align*}
f(0, \pm 1) &= 2 < \text{Abz Max} \\
f(\pm 1, 0) &= 1 \\
f(0,0) &= 0 < \text{Abz Min}
\end{align*}
\]