Lectures 16, 17: Directional Derivatives, Gradient, N.L.

Let $f(x,y,z)$ be a differentiable function, for example $f(x,y,z) =$ temperature at $(x,y,z)$.

Suppose we are at a point $P_0 = (x_0, y_0, z_0)$ and have a unit vector specifying a direction $\hat{u} = \langle u_1, u_2, u_3 \rangle$.

Parametric equations of the line $l$ are

$$\begin{align*}
x &= x_0 + u_1 t \\
y &= y_0 + u_2 t \\
z &= z_0 + u_3 t
\end{align*}$$

We want to compute the rate of change of $f(x,y,z)$ at that point in that direction. This is called the directional derivative and denoted $f_{\hat{u}}(x_0, y_0, z_0)$. Its definition is:

$$f_{\hat{u}} = f_{\hat{u}}(x_0,y_0,z_0) = \frac{d}{dt} f(x_0,y_0,z_0) \bigg|_{t=0}$$

$$= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \bigg|_{t=0}$$

$$f_{\hat{u}} = f_x(x_0,y_0,z_0) \cdot u_1 + f_y(x_0,y_0,z_0) \cdot u_2 + f_z(x_0,y_0,z_0) \cdot u_3$$

The corresponding 2D formula is:

$f(x,y)$ function of two variables, $\hat{u} = \langle u_1, u_2 \rangle$

$$f_{\hat{u}}(x_0,y_0) = f_x(x_0,y_0) u_1 + f_y(x_0,y_0) u_2$$
Let's look again at the formula for \( f \hat{u} (x_0, y_0, z_0) \)
\[
f \hat{u} (x_0, y_0, z_0) = f_x (x_0, y_0, z_0) u_1 + f_y (x_0, y_0, z_0) u_2 + f_z (x_0, y_0, z_0) u_3
\]
This looks like a dot product
\[
= \langle f_x (x_0, y_0, z_0), f_y (x_0, y_0, z_0), f_z (x_0, y_0, z_0) \rangle \cdot \langle u_1, u_2, u_3 \rangle
\]
**Def:** For a function \( f(x, y, z) \) we define the gradient of \( f \) by
\[
\nabla f = \langle f_x, f_y, f_z \rangle
\]
Then
\[
f \hat{u} (x_0, y_0, z_0) = \nabla f (x_0, y_0, z_0) \cdot \hat{u}
\]
The gradient is often written in terms of the "Del" operator, which is a "vector" partial derivative operator
\[
\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle
\]
This "vector" can be "multiplied" on the right by a scalar function:
\[
\nabla f (x, y, z) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f (x, y, z) = \langle f_x, f_y, f_z \rangle
\]
So we can write \( \nabla f = \nabla f \) and our formula for directional derivative is:
\[
\frac{df}{d\hat{u}} \bigg|_{u (x_0, y_0, z_0)} \hat{u} (x_0, y_0, z_0) = \nabla f (x_0, y_0, z_0) \cdot \hat{u}
\]
In summary:
3D: For \( f(x, y, z) \), \( \nabla f = \langle f_x, f_y, f_z \rangle \) function of \( x, y, z \).
2D: For \( f(x, y) \), \( \nabla f = \langle f_x, f_y \rangle \) function of \( x, y \).
In either case, we can express the formula for directional derivative in the compact form \( \nabla f \cdot \hat{u} \).

Example: Compute \( \nabla f \) for the following functions \( f(x, y) \):
(a) \( f(x, y) = x^2 + y^2 \)
(b) \( f(x, y) = xe^{xy} \)
(c) \( f(x, y) = \tan^{-1}(2x+3y) \)

\[ \text{Solv. (a)} \quad \nabla f = \langle 2x, 2y \rangle \]
\[ \text{(b)} \quad \nabla f = \langle e^{xy} + xy e^{xy}, x^2 e^{xy} \rangle \]
\[ \text{(c)} \quad \nabla f = \langle \frac{2}{1 + (2x+3y)^2}, \frac{3}{1 + (2x+3y)^2} \rangle \]

Example: Compute \( \nabla f \) for the 3D functions \( f(x, y, z) \):
(a) \( f(x, y, z) = x^2 + 3z^2x \)
(b) \( f(x, y, z) = x\tan yz \)
(c) \( f(x, y, z) = \sin(xye^z) \)

\[ \text{Solv.:} \]
(a) \( \nabla f = \langle 2xy + 3z^2, x^2, 6z \rangle \)
(b) \( \nabla f = \langle \tan yz, xz \sec^2 yz, xy \sec^2 yz \rangle \)
(c) \( \nabla f = \langle ye^{x} \omega (xye^z), xe^{x} \omega (xye^z), xy e^{x} \omega (xye^z) \rangle \)
2. Example: \( f(x,y) = x^2 + y^2 \). Find the directional derivatives in the \( \hat{i} \), \( \hat{j} \) and \( 3\hat{i} - 4\hat{j} \) direction.
We use \( \frac{\partial f}{\partial u} = \nabla f \cdot \hat{u} \). Well, \( \nabla f = <2x, 2y> \)
(a.) \( \hat{u} = \hat{i} = <1, 0> \) so \( \frac{\partial f}{\partial \hat{i}} = <2x, 2y> \cdot <1, 0> = 2x \)
(b.) \( \hat{u} = \hat{j} = <0, 1> \) so \( \frac{\partial f}{\partial \hat{j}} = <2x, 2y> \cdot <0, 1> = 2y \)
(Note the derivatives in the \( \hat{i} \) and \( \hat{j} \) direction are just \( f_x, f_y \).
(c.) \( \hat{u} = 3\hat{i} - 4\hat{j} \) is not a unit vector.
\( \hat{u} = <\frac{3}{5}, \frac{-4}{5}> \) \( \frac{\partial f}{\partial \hat{u}} = <2x, 2y> \cdot \frac{3}{5}, \frac{-4}{5} > = \frac{6x - 8y}{5} \)
If we are interested in a particular \( x \) and \( y \), we can put those values in.

3-D example: The temp at \((x,y,z)\) is \( T = x^2 + 2y^2 + 3z^2 \). At the point \((2,1,2)\), what is the rate of change of the temp in the direction of \( \vec{V} = <2, -1, -1> \).

Solu: We use \( \frac{\partial T}{\partial \vec{V}} = \nabla T \cdot \vec{V} \), here we will evaluate \( \nabla T \) at the point \((2,1,2)\).

\( \nabla T = <2x, 4y, 6z> \) \( \nabla T (2,1,2) = <4, 4, 12> \)
\( ||\vec{V}|| = \sqrt{6} \) \( \vec{V} = \frac{1}{\sqrt{6}} <2, -1, -1> \)

\( \frac{\partial T}{\partial \vec{V}} (2,1,2) = \nabla T (2,1,2) \cdot \vec{V} = \frac{1}{\sqrt{6}} (4 - 4 - 12) = -\frac{8}{\sqrt{6}} \text{ deg/ft} \)
Temp is decreasing.
Let's look again: \[ f_\hat{u}(p) = \nabla f(p) \cdot \hat{u} \]
\[ = \| \nabla f(p) \| \cdot \| \hat{u} \| \cdot \cos \theta \]
\[ \nabla f(p) \text{ is fixed} \]
\[ \| \hat{u} \| = 1 \text{ The only thing we can change} \]
\[ \theta \text{ by varying the direction of } \hat{u}. \]

\[ f_\hat{u}(p) \text{ is max when } \cos \theta = 1 \text{ i.e., when } \hat{u} \text{ points in the same direction as } \nabla f(p): \]
\[ \max f_\hat{u}(p) = \| \nabla f(p) \| \]

\[ f_\hat{u}(p) \text{ is min when } \cos \theta = -1 \text{ i.e., } \hat{u} \text{ points in the direction of } \nabla f : \]
\[ \min f_\hat{u}(p) = -\| \nabla f(p) \| \]

The gradient points in the direction of max rate of change of \( f \), and that max is \( \| \nabla f(p) \|. \) The min rate of change occurs opposite the direction of the gradient, and \( \min f_{\hat{u}}(p) = -\| \nabla f(p) \|. \)
Example 2.1: $f(x,y) = xe^y$; Draw a

level curve. $c = 1, 2, 3, 4$.

What direction is the

max rate of change at $(2, 0)$? What is the max?

$x e^y = c \iff x = c e^{-y}$

$\nabla f = \langle e^y, xe^y \rangle$

$\nabla f(2, 0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$

$\max \, D_{\mathbf{u}} f(2, 0) = \sqrt{5}$

Example 3D: $T = \frac{800}{1 + x^2 + 2y^2 + 3z^2}$

What direction is the max $D_{\mathbf{u}} f$ and what is its value?

$\nabla T = \left\langle -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}, -\frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}, -\frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \right\rangle = \langle 160, -1, -2, 6 \rangle$

$\nabla T(1, 1, -1) = \left( \frac{160}{(1+1+2+12)^2} \right) = \langle -1, -2, 6 \rangle$

$\nabla T(1, 1, -1) = \frac{160}{256} = \frac{5}{8} \langle -1, -2, 6 \rangle$

The direction is $\langle -1, -2, 6 \rangle$

$\max \, D_{\mathbf{u}} f(1, 1, -1) = \frac{5}{8} \sqrt{1 + 4 + 36} = \frac{5\sqrt{41}}{8}$ °C/m
Now consider a function of 3 variables \( f(x,y,z) \) and consider a level surface \( f(x,y,z) = c \) and \( P \) a point on the surface. Let \( C \) be a curve on the surface passing through \( P \):

\[
C: \quad x = x(t), \quad y = y(t), \quad z = z(t)
\]

On that curve, \( f(x(t), y(t), z(t)) = c \) so \( \frac{df}{dt} = 0 \)

\[
\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = \nabla f \cdot <x', y', z'>
\]

We know \( <x', y', z'> \) is tangent to \( C \) hence to the surface. This says \( \nabla f \) is \( \perp \) to all curves in the surface through \( P \), so \( \nabla f \) is \( \perp \) to the surface and its tangent plane.

This fact can be used to write the equation of the tangent plane and normal line to a surface given implicitly.

**Example** \( z = x^2 + y^2 \)

\((1,2,5)\) is a point on this surface

\( \nabla f \) at \( f(x,y,z) = 0 \):

\[
\nabla f = <2x, 2y, -1> \quad \text{is} \quad \perp \text{to this surface}
\]

\( \nabla f (1,2,5) = <2, 4, -1> = N \)

This \( N \) is the normal vector for the tangent plane and the direction vector for the normal line at \((1,2,5)\).
T.P: \(<2, 4, -1>: <x-1, y-2, z-5> = 0\)
\[2(x-1) + 4(y-2) - (z-5) = 0\]
\[2x - 2 + 4y - 8 - z + 5 = 0\]
\[2x + 4y - z = 5\]

Normal Line:
\[
\begin{align*}
\alpha &= 1 + 2t \\
y &= 2 + 4t \\
z &= 5 - t
\end{align*}
\]

Find the equation of the tangent plane and normal line for the ellipse:
\[4x^2 + 2y^2 + 5z^2 = 17\]
at \((-1, 2, 1)\)

\[f = 4x^2 + 2y^2 + 5z^2\]
\[
\nabla f = <8x, 4y, 10z> \\
\n\nabla f (-1, 2, 1) = <-8, 8, 10> \\
\]

We can use this or we could use \(N = <-4, 4, 5>\) same direction:

T.P:
\[-4(x+1) + 4(y-2) + 5(z-1) = 0\]
\[-4x - 4 + 4y - 8 + 5z - 5 = 0\]
\[-4x + 4y + 5z = 17\]

Normal Line:
\[
\begin{align*}
\alpha &= -1 - 4t \\
y &= 2 + 4t \\
z &= 1 + 5t
\end{align*}
\]