Arc length, velocity, acceleration.

\[
\vec{R}(t) = \langle f(t), g(t), h(t) \rangle \quad a \leq t \leq b
\]
\[
a = t_0 < t_1 < \ldots < t_n = b
\]

\[
S = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta R_i
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\Delta f_i^2 + \Delta g_i^2 + \Delta h_i^2}
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta f_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta g_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta h_i}{\Delta t_i}\right)^2}
\]
\[
= \int_{a}^{b} \sqrt{f'(t)^2 + (g'(t))^2 + (h'(t))^2} \, dt
\]

\[
S = \int_{0}^{b} |\vec{R}'(t)| \, dt
\]

Example: Circle of radius \( a \):

\[
\vec{R}(t) = \langle a \cos t, a \sin t \rangle
\]

\[
\vec{R}'(t) = \langle -a \sin t, a \cos t \rangle
\]

\[
S = \int_{0}^{2\pi} |\vec{R}'(t)| \, dt = \int_{0}^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \, dt
\]
\[
= \int_{0}^{2\pi} a \, dt = \int_{0}^{2\pi} a \, dt = 2\pi a
\]

Note: If we want arc length as a function of \( t \):

\[
S(t) = \int_{a}^{t} |\vec{R}'(t)| \, dt \quad \text{Note } s(a) = 0 \quad s(b) = \ell = \text{length of the curve.}
\]
Also note that we can calculate the speed:
\[
\frac{ds}{dt} = \| \vec{R}'(t) \|.
\]

We have seen that if $\vec{R}'(t) \neq \vec{0}$ it is tangent to the curve in the direction of motion. We have also seen that
\[
1 \| \vec{R}'(t) \| = \frac{ds}{dt} = v
\]

\[
\therefore \quad \vec{R}'(t) = \vec{v}(t) = \text{the velocity vector.} \quad \| \vec{v}(t) \| = v
\]

We define the acceleration vector:
\[
\vec{a}(t) = \vec{v}'(t) = \vec{R}''(t) = \langle f''(t), g''(t), h''(t) \rangle
\]

The acceleration vector points generally towards the direction the curve is bending.

The tangential component of the acceleration is
\[
a_t = \vec{a} \cdot \hat{T} = \vec{a} \cdot \frac{\vec{R}'(t)}{\| \vec{R}'(t) \|} = \frac{1}{v} \vec{a} \cdot \vec{R}'(t)
\]

**Example:** Motion on a helix:
\[
\vec{R}(t) = \langle a \cos wt, a \sin wt, bt \rangle
\]
\[
\vec{R}'(t) = \langle -aw \sin wt, aw \cos wt, b \rangle
\]
\[
v = \| \vec{R}'(t) \| = \sqrt{a^2 w^2 \sin^2 wt + a^2 w^2 \cos^2 wt + b^2} = aw
\]
\[
\vec{a} = \langle -aw^2 \cos wt, -aw^2 \sin wt, 0 \rangle
\]
\[
a_t = \frac{1}{v} \left( \frac{a^2 w^3 \sin wt \cos wt - a^2 w^3 \cos wt \sin wt}{aw} \right) = 0
\]
In this case, there is no tangential acceleration as the acceleration vector is $\perp$ to the curve, which is characteristic of circular motion.

$$ R(t) = a \cos \frac{\pi}{2}, a \sin \frac{\pi}{2}, \frac{b}{2} $$

$$ V = \left< -a \omega \sin \frac{\pi}{2}, a \omega \cos \frac{\pi}{2}, 0 \right> $$

$$ A(t) = \left< -a \omega^2 \cos \frac{\pi}{2}, -a \omega^2 \sin \frac{\pi}{2}, 0 \right> $$

Example: An object is moving with constant velocity $\vec{V}(t) = < -2, 3, -1 >$, at time $t = 0$ its position $\vec{R}(t) = < -4, -2, 5 >$, find $\vec{R}(t)$.

$$ \vec{R}(t) = \int \vec{V}(t) \, dt = \int < -2, 3, -1 > \, dt = \vec{c} $$

$$ \vec{R}(t) = < -2t, 3t, -t > + \vec{c}_1, \vec{c}_2, \vec{c}_3 > $$

Plug in $t = 0$:

$$ \vec{R}(0) = < -4, -2, 5 > = \vec{c}_1, \vec{c}_2, \vec{c}_3 > $$

$$ \Rightarrow \vec{c}_1 = -4, \vec{c}_2 = -2, \vec{c}_3 = 5 $$

$$ \Rightarrow \vec{R}(t) = < -2t, 3t, -t > + < -4, -2, 5 > $$

$$ = < -4, -2, 5 > + t < -2, 3, -1 > $$

Do you recognize this?
The projectile problem (2D)

A shell is fired at the origin with angle of elevation \( \alpha \) and muzzle velocity \( v_0 \).

- Find the trajectory and what \( \alpha \) gives max. range.

\[
\vec{R}(t) = \langle x(t), y(t) \rangle \quad \text{We are given:}
\]
\[
\vec{V}(0) = \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle \\
\vec{R}(0) = \langle 0, 0 \rangle \\
\vec{A}(t) = \langle 0, -g \rangle \\
\vec{V}(t) = \int \vec{V}(t) dt = \langle c_1, -gt + c_2 \rangle \\
\]

Put in \( t=0 \):
\[
\langle v_0 \cos \alpha, v_0 \sin \alpha \rangle = \langle c_1, c_2 \rangle \\
\Rightarrow \vec{V}(t) = \langle v_0 \cos \alpha, -gt + v_0 \sin \alpha \rangle
\]

\[
\vec{R}(t) = \int \vec{V}(t) dt = \langle (v_0 \cos \alpha) t + c_1, -\frac{gt^2}{2} + (v_0 \sin \alpha) t + c_2 \rangle \\
\vec{R}(0) = \langle 0, 0 \rangle = \langle c_1, c_2 \rangle \\
\Rightarrow \vec{R}(t) = \langle (v_0 \cos \alpha) t, -\frac{gt^2}{2} + (v_0 \sin \alpha) t \rangle
\]

At hits the ground when \( y=0 \):
\[
-\frac{gt^2}{2} + (v_0 \sin \alpha) t = t \left( -\frac{gt}{2} + v_0 \sin \alpha \right)
\]
\[
t = \frac{2v_0 \sin \alpha}{g} \quad \text{The range is} \quad x \quad \text{at this time:}
\]
\[
\text{Range} = \frac{(v_0 \cos \alpha)(2v_0 \sin \alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g} \\
\text{This is max when} \quad 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}