MAT 576  
Fall 2008  
Homework set 4

1. In class we looked at the wave equation $u_{xx} - u_{yy} = 0$ on a bounded domain $\Omega \subset \mathbb{R}^2$ that is partitioned into two (open) subdomains $\Omega_1$ to the left of a smooth curve $C : x = s(t)$ and $\Omega_2$ to the right. Letting $[w] := w|_{\Omega_2} - w|_{\Omega_1}$ we saw that a function $u$ that satisfies $u_{xx} - u_{yy} = 0$ in $\Omega_1 \cup \Omega_2$ is a weak solution only if for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_C ([u_x] N_1 \phi - [u_y] N_2 \phi) \, d\ell + \int_C ([u] N_1 \phi_x - [u] N_2 \phi_y) \, d\ell = 0,$$

where $(N_1, N_2)$ is the (continuously varying) unit normal to $C$. Show that the function $H(x - y)$ satisfies this condition.

2. This is an exercise in using Green’s theorem (also known as Gauss’s theorem or Stokes’s theorem, or Ostrogradskii’s theorem or by various combinations of these names):

$$\int_\Omega u_x v \, dx = - \int_\Omega w_x v \, dx + \int_{\partial\Omega} u v N_i \, dS,$$

where $N := (N_1, \cdots, N_n)$ is the unit outward normal to $\partial\Omega$ and $\Omega$ a bounded domain in $\mathbb{R}^n$, $n > 1$, with piecewise smooth boundary. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and suppose that $u$ is a solution to the nonlinear elliptic problem

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Suppose that $f$ is continuous and define $F(s) := \int_0^s f(\sigma) \, d\sigma$. We employ the summation convention.

(a) Show that

$$\int_\Omega x_j \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i} \, dx = -\frac{n}{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \, dx + \frac{1}{2} \int_{\partial\Omega} (N \cdot x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \, dS.$$

(b) Show that

$$\int_\Omega f(u) x_i \frac{\partial u}{\partial x_i} \, dx = \int_{\partial\Omega} F(u)x_i N_i \, dS - n \int_\Omega F(u) \, dx.$$

(c) Show that

$$\frac{n - 2}{2} \int_\Omega \| \nabla u \|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} \left( \frac{\partial u}{\partial N} \right)^2 (x \cdot N) \, dS + n \int_\Omega F(u) \, dx \, dx = 0.$$

Here $\partial u / \partial N = \nabla u \cdot N$ is the outward normal derivative. Hint: use the fact that $u = 0$ on the boundary - at least twice!

3. Consider the conservation law with 2 components (1 space variable) $u_t + f(u)_x = 0$, or equivalently $u_t + A(u)u_x = 0$. Let $\lambda_1(u)$ and $\lambda_2(u)$ be the eigenvalues of $A(u)$ with respective eigenvectors $r_1$ and $r_2$ and let $w_1(u)$ and $w_2(u)$ be corresponding Riemann invariants. Let $v(x, t) := (v^1(x, t), v^2(x, t)) = w(u(x, t))$. Show that

$$v_t^1 + \lambda_2(u)v_x^1 = 0$$

$$v_t^2 + \lambda_1(u)v_x^2 = 0$$
4. One-dimensional isentropic flow of a gas is modeled by the system
\[ \begin{align*}
\rho_t + v \rho_x + \rho v_x &= 0 \\
v_t + v v_x + c^2 \rho_x / \rho &= 0,
\end{align*} \]
where \( c \), which is positive, is the speed of sound and is a function of the density \( \rho \), \( c^2 = K \rho^{\gamma-1} \), \( \gamma > 1 \) is the ratio of the heat capacities (\( v \) is the velocity of the flow).

(a) Write this system as a 2-dimensional conservation law \( u_t + f(u)_x = 0 \) for \( u = (\rho, v)^T \).

(b) Write the system as \( u_t + A u_x = 0 \) and find the eigenvalues and eigenvectors of \( A(u) \).

(c) Introduce the function \( L(\rho) = \int_0^\rho (c(\sigma)/\sigma) \, d\sigma \) and find Riemann invariants \( q(u) \) corresponding to the eigenvalue \( \lambda_1 \) and \( s(u) \) corresponding to the eigenvalue \( \lambda_2 \). Pick the labeling so that \( \lambda_1 > \lambda_2 \).

(d) Use problem 3 to find the system of equations for \( q \) and \( s \).

(e) Show that the system can be “inverted” to obtain:
\[ \begin{align*}
x_q &= (v - c) t_q \\
x_s &= (v + c) t_s
\end{align*} \]

Hint:
\[ \left( \begin{array}{cc}
\frac{\partial x}{\partial q} & \frac{\partial x}{\partial s} \\
\frac{\partial x}{\partial q} & \frac{\partial x}{\partial s}
\end{array} \right)^{-1} = \frac{1}{D} \left( \begin{array}{cc}
\frac{\partial q}{\partial x} & -\frac{\partial q}{\partial s} \\
-\frac{\partial s}{\partial x} & \frac{\partial s}{\partial s}
\end{array} \right), \quad D = \frac{\partial(q, s)}{\partial(x, t)} \]

(f) Show that \( v + c \) and \( v - c \) can be written as functions of \( q \) and \( s \) and that the last system is therefore a fully linear system.