Chapter 1
Operator semigroups: main concepts

Strongly continuous semigroups constitute the core of the theory of operator semigroups.

1.1 Notion of a strongly continuous semigroup

1.1.1. Definition. A family \( \{S_t\}_{t \in \mathbb{R}_+} \) of bounded linear operators on a Banach space \( X \) is said to be a strongly continuous (one-parameter) semigroup (or \( C_0 \)-semigroup) if it satisfies the semigroup property

\[
S_0 = I, \quad S_{t+\tau} = S_t S_\tau \quad \text{for all} \quad t, \tau \in \mathbb{R}_+,
\]

and the semitrapejctory maps \( t \mapsto S_t x \) are strongly continuous from \( \mathbb{R}_+ \) into \( X \) for every \( x \in X \). If these properties hold for \( \mathbb{R} \) instead of \( \mathbb{R}_+ \), we call \( S_t \) a strongly continuous (one-parameter) group (or \( C_0 \)-group) on \( X \).

1.1.2. Remark. The algebraic relation in Definition 1.1.1 implies almost immediately several important continuity properties. For instance, one can see that the continuity property in Definition 1.1.1 is equivalent to the requirement of strong continuity of the mapping \( t \mapsto S_t x \) at \( t = 0 \), see, e.g., [16] or [29]. It is also possible to prove that a weak continuity of \( t \mapsto S_t x \) implies its strong continuity, [16, 29].

The following exercise shows that a one-parameter semigroup is some kind of generalization of exponential function.

1.1.3. Exercise. Let \( f(s) \) be a scalar continuous function on \( \mathbb{R}_+ \) such that the semigroup property \( f(t+s) = f(t) f(s) \) holds and \( f(0) \neq 0 \). Show that \( f(t) = e^{at} \) for some \( a \in \mathbb{R} \). Hint: First show that \( f(1/k) = [f(1)]^{1/k} \) for every natural \( k \) and then that \( f(r) = [f(1)]^r \) for every rational \( r > 0 \).

1.1.4. Exercise. Let \( \{S_t\}_{t \in \mathbb{R}_+} \) be a semigroup of linear bounded operators on \( X \). If there is \( \tau > 0 \) such that the operator \( S_\tau \) is invertible, then the semigroup \( \{S_t\}_{t \in \mathbb{R}_+} \)
can be extended to a group \( \{ \tilde{S}_t \}_{t \in \mathbb{R}} \). Moreover, \( \{ \tilde{S}_t \}_{t \in \mathbb{R}} \) is a strongly continuous group, provided the semigroup \( \{ S_t \}_{t \in \mathbb{R}} \) possesses the same property. Hint: we note that 
\[ S^{-1}_t \] 
commutes with every operator \( S_t \) and define \( \tilde{S}_t \) for negative \( t \) by the formula 
\[ \tilde{S}_t = S_{t+n\tau} S^{-n}_t, \]
where \( n \in \mathbb{Z}^+ \) is chosen such that \( t+n\tau \geq 0 \).

In the following exercises we collect several standard (see any textbook mentioned above on operator semigroups) examples of strongly continuous semigroups.

1.1.5. Exercise. Let \( A \) be a bounded operator in a Banach space \( X \). Show that the series 
\[ S_t = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = e^{At}, \]
converges for each \( t \in \mathbb{R} \) in the operator norm and and define a strongly continuous group \( \{ S_t \}_{t \in \mathbb{R}} \) on \( X \).

1.1.6. Exercise. Show that the left translation semigroup \( \{ S_t \}_{t \in \mathbb{R}^+} \) defined by the relation
\[ [S_t f](\tau) = f(\tau + t), \quad t, \tau > 0, \tag{1.1.1} \]
is strongly continuous on each of the Banach spaces
\[ C_{ub}(\mathbb{R}^+) := \{ f \in C(\mathbb{R}^+) : f \text{ is bounded and uniformly continuous on } \mathbb{R}^+ \} \]
and
\[ C_0(\mathbb{R}^+) := \{ f \in C(\mathbb{R}^+) : f(\tau) \to 0 \text{ as } \tau \to +\infty \}. \]
Both spaces are endowed with the sup-norm: \( \| f \| = \sup_{\tau \geq 0} |f(\tau)| \). Hint: (a) strong continuity of \( S_t f \) is equivalent to uniform continuity of the function \( t \mapsto f(t) \) on \( \mathbb{R} \),
(b) \( C_0(\mathbb{R}^+) \subset C_{ub}(\mathbb{R}^+) \).

1.1.7. Exercise. Show that the left translation semigroup defined by (1.1.1) is strongly continuous on each space \( L^p(\mathbb{R}^+), 1 \leq p < \infty \). Hint: consider first continuity of \( S_t \) on step-functions.

The following example of an operator semigroup is borrowed from [16, p.8]

1.1.8. Exercise. Show that the relation
\[ [S_t f](\tau) = e^{-t^2 - 2\tau} f(\tau + t), \quad t, \tau > 0, \]
defines a strongly continuous semigroup on \( C_0(\mathbb{R}^+) \).

The following assertion provides uniform bounds for strongly continuous semigroups.

1.1.9. Proposition (Exponential bound). Let \( S_t \) be a strongly continuous semigroup. Then there exist \( M \geq 1 \) and \( \omega \geq 0 \) such that
\[ \| S_t \| \leq Me^{\omega t} \text{ for all } t \in \mathbb{R}^+. \tag{1.1.2} \]
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**Proof.** For each \( x \in X \) and \( T > 0 \) the set \( \Gamma_x^T = \{ S_x : t \in [0, T] \} \) is a continuous image of a compact interval. This implies that the set \( \Gamma_x^T \) is bounded for each \( x \in X \). Thus by Uniform Boundedness Principle (see, e.g, [31] or [14, 39]) we have

\[
1 \leq M \equiv \sup \{ \|S_t\| : t \in [0, 1] \} < \infty.
\]

Let \([t]\) be the integer part of \( t \in \mathbb{R}_+ \) and \( \{t\} = t - [t] \). Then it follows from the semigroup property that

\[
\|S_t\| \leq \|S_{\{t\}}\| \|S_{[t]}\| \leq M \|S_{[t]}\|^{[t]} \leq M \exp\{[t]\log M\}.
\]

Thus (1.1.2) holds with \( \omega = \log M \).

1.1.10. **Exercise.** Let \( S_t \) be a \( C_0 \)-semigroup. Show that the mapping \( (t; x) \mapsto S_t x \) is (jointly) continuous from \( \mathbb{R}_+ \times X \) to \( X \).

Now we define a growth bound of a \( C_0 \)-semigroup by the formula:

\[
\omega_0 = \inf_{t > 0} \left( \frac{1}{t} \log \|S_t\| \right).
\]

(1.1.3)

By Proposition 1.1.9 we have that \( -\infty \leq \omega_0 < \infty \). (the case \( \omega_0 = -\infty \) is possible, see Exercise 1.1.12 below). It is also clear that

\[
\|S_t\| \geq Me^{\omega_0 t} \text{ for all } t \in \mathbb{R}_+.
\]

Thus \( \omega_0 \) gives us a bound from below for \( \|S_t\| \). The following assertion shows that \( \omega_0 \) also provides the sharp estimate from below for the best growth exponent for \( \|S_t\| \).

1.1.11. **Proposition (Sharp growth bound).** Let \( S_t \) be a strongly continuous semigroup and \( \omega_0 \) be given by (1.1.3). Then

\[
\omega_0 = \lim_{t \to +\infty} \left( \frac{1}{t} \log \|S_t\| \right).
\]

(1.1.4)

Moreover, for every \( \omega > \omega_0 \) there exists \( M = M_{\omega_0} \) such that (1.1.2) is valid.

**Proof.** Let \( \tau > 0 \) be fixed. For every \( t \geq \tau \) there is \( n \in \mathbb{Z}_+ \) such that \( n\tau \leq t < (n + 1)\tau \). We obviously have that

\[
t^{-1} \log \|S_t\| = t^{-1} \log \|S_{n\tau}S_{t-n\tau}\| \leq \frac{n}{t} \log \|S_{\tau}\| + \frac{1}{t} \log \left( \sup_{t' \in [0, \tau]} \|S_{t'}\| \right)
\]

\[
\leq \frac{1}{\tau} \log \|S_{\tau}\| + \frac{1}{t} \log \left( \sup_{t' \in [0, \tau]} \|S_{t'}\| \right).
\]

Therefore
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\[ \limsup_{t \to +\infty} \left( \frac{1}{t} \log \| S_t \| \right) \leq \frac{1}{\tau} \log \| S_\tau \| \quad \forall \tau > 0. \]

This implies that

\[ \limsup_{t \to +\infty} \left( \frac{1}{t} \log \| S_t \| \right) \leq \inf_{\tau > 0} \left( \frac{1}{\tau} \log \| S_\tau \| \right) \leq \liminf_{t \to +\infty} \left( \frac{1}{t} \log \| S_t \| \right) \]

Thus the limit in (1.1.4) exists and the corresponding relation holds.

To prove the second part of the statement we note that if \( \omega > \omega_0 \), then there exists \( \tau > 0 \) such that

\[ \frac{1}{t} \log \| S_t \| \leq \omega \quad \forall t \geq \tau. \]

and thus \( \| S_t \| \leq e^{\omega t} \) for \( t \geq \tau \). Therefore

\[ \| S_t \| \leq \max \{ N_\tau, e^{\omega t} \} \leq M_\tau \quad \forall t \geq 0, \]

where \( N_\tau = \sup_{\tau' \in [0, \tau]} \| S_{\tau'} \| \) and \( M_\tau = \max \{ N_\tau, e^{-\omega \tau} \} \).

1.1.12. Exercise. Show that \( \omega_0 = -\infty \) for the semigroup in Exercise 1.1.8.

1.1.13. Exercise. Show that for the matrix semigroup

\[ S_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ in } \mathbb{R}^2 \]

we have that \( \omega_0 = 0 \) and \( \| S_t \| \to \infty \) as \( t \to \infty \), i.e., the estimate in (1.1.2) is not valid with \( \omega = \omega_0 \).

1.1.14. Exercise. Give an example of a semigroup showing that (1.1.2) can be valid with \( \omega = \omega_0 \), where \( \omega_0 \) is a growth bound defined in (1.1.4).

1.1.15. Exercise. Using the same idea as in the proof of Proposition 1.1.11 show that for any bounded linear operator \( L \) on a Banach space the limit below exists and the relation

\[ r(L) := \lim_{n \to \infty} \| T^n \|^{1/n} = \inf_{n \geq 0} \| T^n \|^{1/n} \quad (1.1.5) \]

holds. This characteristic is called (see, e.g., [31]) the \textit{spectral radius} of the operator \( L \). Hint: pay attention that \( S_n = L^n \) forms a discrete semigroup.

1.1.16. Exercise. Let \( S_t \) be a \( C_0 \)-semigroup on a Banach space \( X \). Prove the following relation between the growth bound \( \omega_0 \) (see (1.1.4)) and the spectral radius (see (1.1.5)) of \( S_t \) with fixed \( t: r(S_t) = \exp \{ \omega_0 t \} \).

The following integral continuity property of \( C_0 \)-semigroups is important in the further considerations.

1.1.17. Exercise. Let \( S_t \) be a strongly continuous semigroup. Show that

\[ \lim_{h \to 0} \left[ \frac{1}{h} \int_{t}^{t+h} S_{\tau} \, d\tau \right] = S_t x, \quad \forall t \in \mathbb{R}_+, \ x \in X, \quad (1.1.6) \]
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in the strong topology of \(X\). The integral in (1.1.6) is understood in the Riemann sense.

1.2 Uniformly continuous semigroups

This class of semigroups is related mainly to ordinary differential equations.

1.2.1. Definition. A \(C_0\)-semigroup \(S_t\) is called uniformly continuous (or norm-continuous) if the mapping \(t \mapsto S_t\) is continuous in the operator norm.

The main example of a uniformly continuous semigroup is given in Exercise 1.1.5.

1.2.2. Exercise. Show that a \(C_0\)-semigroup \(S_t\) is uniformly continuous if \(S_t\) is uniformly continuous at \(t = 0\), i.e., \(\lim_{t \to +0} \|S_t - I\| = 0\) or in the equivalent form \(\lim_{t \to +0} \sup_{\{x \in X : \|x\| \leq 1\}} \|S_t x - x\| = 0\).

1.2.3. Exercise. Let \(S_t\) be a uniformly continuous semigroup. Prove the uniform analog of (1.1.6):

\[
\lim_{h \to 0} \frac{1}{h} \int_0^{t+h} S_\tau d\tau = S_t, \quad \forall t \in \mathbb{R}_+,
\]

in the operator norm.

The following exercises show that there exist semigroups which are not uniformly continuous.

1.2.4. Exercise. Show that the left translation semigroup \(\{S_t\}_{t \in \mathbb{R}_+}\) defined by (1.1.1) on \(C_{ub}([0, \infty))\) (see the definition in Exercise 1.1.6) is not uniformly continuous. Hint: Use the fact that the function \(f_n(s) = (1 - ns)_+\), (by the definition \(x_+ = x\) for \(x \geq 0\) and \(x_+ = 0\) when \(x < 0\)) possesses the property \(\|S_{1/n} f_n - f_n\| = \|f_n\| = 1\).

1.2.5. Exercise. Consider ODE with delay

\[
x(t) = x(t - 1), \quad t > 0, \quad x|_{t \in [-1, 0]} = x_0(t).
\]

Show that this equation generates a strongly continuous semigroup in the space \(C([-1, 0])\) by the formula

\[
[S_t x_0](\xi) = x(t + \xi), \quad \xi \in [-1, 0], \quad t \geq 0,
\]

where \(x(t)\) is a solution. Show that this semigroup is not uniformly continuous.

Now we can state the main theorem of the theory of uniformly continuous semigroups.
1.2.6. **Theorem.** Every uniformly continuous semigroup $S_t$ on a Banach space $X$ has the form $S_t = e^{tA}$, $t \geq 0$, for some bounded operator $A$ in $X$, where $e^{tA}$ is defined as the series

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!},$$

which converges in the operator norm. Moreover:

1. The bound in (1.1.2) holds with $M = 1$ and $\omega = \|A\|$.
2. This semigroup can be extended to a uniformly continuous group $\{S_t = e^{tA}\}$ with $t \in \mathbb{R}$.
3. The function $t \mapsto S_t$ is differentiable in the operator norm and

$$\frac{d}{dt} S_t = AS_t = S_tA, \quad \forall t \in \mathbb{R}.$$  

The operator $A$ is called a generator of the semigroup $S_t = e^{tA}$. Since

$$\left. \frac{d}{dt} S_t \right|_{t=0} = A,$$

the generator is uniquely defined by the semigroup. As we will see below the formula in (1.2.2) can be made sensible in the case of strongly continuous semigroups. This fact plays the key role in the theory of $C_0$-semigroups.

To prove Theorem 1.2.6 we follow the scheme applied in [16] and present the argument as a sequence of exercises.

1.2.7. **Exercise.** Let $A$ be a bounded operator. Show that the statements (1)-(3) in Theorem 1.2.6 are valid for $S_t = e^{tA}$ defined as an operator series.

Thus to prove Theorem 1.2.6 we need only to show that any uniformly continuous semigroup admits exponential representation with a bounded operator.

1.2.8. **Exercise.** Using the semigroup property and the result of Exercise 1.2.3 show that for every $t \geq 0$ and $\tau > 0$ we have that

$$\frac{S_{t+h} - S_t}{h} \int_0^\tau S_\xi d\xi = \frac{1}{h} \left\{ \int_{t+\tau}^{t+\tau+h} S_\xi d\xi - \int_t^{t+h} S_\xi d\xi \right\} \rightarrow S_t[S_{\tau} - I], \quad h \rightarrow 0,$$

in the operator norm.

1.2.9. **Exercise.** Using Exercise 1.2.3 show that there exists $\rho > 0$ such that the operator $\left[\int_0^\tau S_\xi d\xi \right]^{-1}$ exists for every $0 < \tau \leq \rho$.

1.2.10. **Exercise.** Using the relation in (1.1.6) with $t = 0$ show that the function $t \mapsto S_t$ is continuously differentiable in the operator norm and

$$\frac{d}{dt} S_t = AS_t = S_tA, \quad \forall t \geq 0, \quad \text{with} \quad A = (S_\tau - I) \left[\int_0^{\tau} S_\xi d\xi \right]^{-1},$$

where $\tau > 0$ is small enough.
1.2.11. **Exercise.** Let $A$ be the same as in Exercise 1.2.10. Show that $\tau \mapsto S_{t-\tau}e^{\tau A}$ is differentiable on $[0,t]$ and

$$\frac{d}{d\tau} S_{t-\tau}e^{\tau A} = 0, \quad \forall \tau \in [0,t].$$

This means that $\tau \mapsto S_{t-\tau}e^{\tau A}$ is a constant. Conclude from this that $S_t = e^{At}$. 