Extra problem set 5

1. on the bounded interval \((-L, L)\) consider the differential operator

\[ L := -i \frac{d}{dx}, \]

with boundary condition

\[ Bu = u(L) - u(-L). \]

(a) Show that \( L \) is formally self-adjoint and that the boundary condition that is adjoint to \( Bu = 0 \) is the boundary condition \( Bv = 0 \).

(b) Find the precise domain that makes \( L \) a selfadjoint operator.

(c) Find all the solutions to the eigenvalue problem

\[ Lu = \lambda u, \ B u = 0. \]

2. Consider the operator \( A := -\frac{d^2}{dx^2} + 1 \) in \( H := L^2[0, \pi] \) (i.e. \( Lu = -u'' + u \)) with domain

\[ D_A = \{ u \in C^2[0, \pi] | u'(0) = u'(\pi) = 0 \} \]

(a) Find Green’s function \( g(x, y) \) for this operator.

(b) Let \( G : H \to H \) be the integral operator defined using the kernel \( g \). How do you know it is compact?

(c) Find the spectrum of \( G \) (and hence of \( A \)) and the eigenfunctions. How do you know that the eigenfunctions form an orthogonal basis?

(d) Show that the range \( R_A \) is dense in \( H \) and hence that \( G := A^{-1} : H \to D_A \) where \( A \) is the closure of \( A \).

(e) Write down (no computation necessary!) a double Fourier series representation of the kernel \( g(x, y) \).

3. Consider the operator \( A := -\frac{d^2}{dx^2} + 1 \) in \( H := L^2(-\infty, \infty) \) with domain

\[ D_A = \{ u \in L^2(-\infty, \infty) | u' \text{ is absolutely continuous on bounded subintervals and } u'' \in L^2(-\infty, \infty) \}. \]

Let \( B \) be the operator in \( H \) defined by \( Bu = v \) where \( v(\omega) = \omega^2 f(\omega) \). The domain of \( B \) is

\[ D_B := \{ U \in H | BU \in H \}. \]

Note that \( A = F^{-1}BF \) where \( F \) denotes the Fourier transform.

(a) Determine the spectrum of \( B \) completely (point, residual, continuous). Is \( B \) selfadjoint?

(b) Using part (a) what can you say about the operator \( A \).

(c) Show that if \( f \) and \( f'' \) are both in \( L^2(-\infty, \infty) \) then so is \( f' \).

4. Let \( A \) be a selfadjoint operator on the Hilbert space \( H \) such that \( A^{-1} : H \to H \) exists and is compact. Suppose that the eigenvalues \( \mu_i \) of \( A^{-1} \) are all positive.

(a) Use the Spectral Theorem to show that \( A \) has the form \( \sum_{i=1}^{\infty} \lambda_i P_i \) where \( P_i \) are orthogonal projection operators.

(b) Define the operator

\[ e^{-At} := \sum_{i=1}^{\infty} e^{-\lambda_i t} P_i. \]

Show that for \( t > 0 \) this is well defined on \( H \), and that \( u(t) := e^{-At}u_0 \) is differentiable for all \( u_0 \in H \)

and \( t > 0 \), and that \( u'(t) = Au(t) \). Here differentiation is defined as usual, as the limit of the difference quotient.

(c) Show that if \( u_0 \in D_A \) then

\[ \lim_{t \to 0} u(t) = u_0. \]

5. Show that if \( \epsilon > 0 \) then there exists a positive number \( A_\epsilon \) such that for all \( f \in H^1[0, 1] \) we have

\[ \|f\|_\infty < \epsilon \|f\|_{H^1[0, 1]} + A_\epsilon \|f\|_{L^2[0, 1]}. \]

**Hint:** Consider

\[ \int_{x_0}^{x} (f(x) - f(x_0)) f'(x) \, dx. \]