Homework set 1

Do the marked problems in chapter 1 of the handout: 22, 24, 26, 28, 37, 47.

1. Let \( X \) be a topological space whose topology is induced by a metric \( d \). Let \( S \subset X \). Prove the following three statements are equivalent:
   (i) \( x_0 \) is an accumulation point of \( S \), i.e. for every open set \( G \) that contains \( x_0 \) we have \( (G \cap S) \setminus \{x_0\} \neq \emptyset \).
   (ii) Every open set containing \( x_0 \) contains infinitely many members of \( S \).
   (iii) There exists a sequence \( \{x_n\}_{n=1}^{\infty} \) of points in \( S \) that converges to \( x_0 \) such that \( \forall n \in \mathbb{N} \) we have \( x_n \neq x_0 \).

One of the important applications of functional analysis, and in particular the theory of linear operators, is to the theory of boundary value problems for linear partial differential equations. Often, in the study of these problems, the convolution arises. We will see examples later on. The following 2 problems prepare the way a little. The definition of the convolution \( f \ast g \) of the functions \( f \) and \( g \), defined on the real line, is

\[
(f \ast g)(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy.
\]

2. Suppose that \( 1 \leq a, b, c < \infty \), such that

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1,
\]

and let \( \Omega \subset \mathbb{R}^n \) be an open set with smooth boundary.

(a) Suppose \( f, g, h \) are \( C(\Omega) \) functions with compact support in \( \Omega \). Extend Hölder’s inequality to

\[
\left| \int_{\Omega} f(x)g(x)h(x) \, dx \right| \leq \|f\|_a \|g\|_b \|h\|_c.
\]  

(b) Let \( \mathcal{D}(\Omega) \) denote the test functions on \( \Omega \), i.e. all the \( C^\infty(\Omega) \) functions with compact support in \( \Omega \). It is a known result that \( \mathcal{D}(\Omega) \) is dense in \( L_p(\Omega) \) for any \( 1 \leq p < \infty \). Use this to show that equation (1) is true for all \( f \in L_a(\Omega) \), \( g \in L_b(\Omega) \) and \( h \in L_c(\Omega) \).

3. The objective of this problem is to show that if \( f \in L_p(\mathbb{R}) \), \( g \in L_q(\mathbb{R}) \), \( 1 \leq p, q < \infty \) then \( f \ast g \in L_s(\mathbb{R}) \) where \( 1/p + 1/q = 1 + 1/s \). In what follows we will use the following notation: if \( 1 \leq t < \infty \) then \( t' \) denotes its “conjugate” defined by \( 1/t + 1/t' = 1 \).

(a) Given \( 1 \leq p, q < \infty \). Define \( r \) by \( 1/r = 2 - 1/p - 1/q \). If \( f, g, \) and \( h \) are test functions on \( \mathbb{R} \), show that

\[
\left| \int_{\mathbb{R}} h(x)(f \ast g)(x) \, dx \right| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} h(x)f(x - y)g(y) \, dy \, dx \leq \|f\|_p \|g\|_q \|h\|_r.
\]

Hint: define

\[
\alpha(x, y) := |h(x)|^{r/q'} |f(x - y)|^{p/q'} , \quad \beta(x, y) := |f(x - y)|^{p/r'} |g(y)|^{q/r'}, \quad |g(y)|^{q/r'} |h(x)|^{r/p'}
\]

and apply problem (1) to

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(x, y)\beta(x, y)\gamma(x, y) \, dy \, dx.
\]

(b) Choose \( h(x) \) in such a way that part a) above gives

\[
\int_{\mathbb{R}} |(f \ast g)(x)|^{r'} \, dx \leq \|f\|_p \|g\|_q \|f \ast g\|_{s'/r'}^{r'/r}.
\]

(c) Show that if \( f \in L_p(\mathbb{R}) \), \( g \in L_q(\mathbb{R}) \), then

\[
\|f \ast g\|_s \leq \|f\|_p \|g\|_q \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}.
\]