1. Let $S$ be a collection of subsets of $X$ whose union is $X$ and such that $\emptyset \in S$. Let $B$ be the collection of all finite intersections of members of $S$ and let $T$ be the collection of arbitrary unions of members of $B$. Show that $T$ is a topology on $X$.

2. Let $L(X,Y)$ be the linear space of all linear transformations from the normed space $X$ with norm $\| \cdot \|_X$ to the normed space $Y$ with norm $\| \cdot \|_Y$ and define

$$\| L \| = \sup \{ \| Lx \|_Y / \| x \|_X : x \in X, x \neq 0 \}.$$ 

(a) Show that this defines a norm on $L(X,Y)$.

(b) Show that if $X$ and $Y$ are Banach spaces then, with this norm, $L(X,Y)$ is complete.

3. Show that if the Hilbert space $H$ is over the field $\mathbb{R}$ then a bounded positive operator is not necessarily self-adjoint (i.e. give a simple counterexample). But then show that if the Hilbert space $H$ is over the field $\mathbb{C}$ then a bounded positive operator is always self-adjoint.

4. Let $(a_{ij})$ be the matrix representation of $A \in B(H)$ and let $(a_{ij}^*)$ be the matrix representation of $A^*$. Show that $a_{ij}^* = a_{ji}$.

5. Let $H$ be a Hilbert space with orthonormal basis $\{ e_1, e_2, \cdots \}$ and let $P_n$ be the projection operator onto the span of $\{ e_1, e_2, \cdots, e_n \}$. Show that this sequence of operators converges to the identity operator $I$ in the strong sense. Also $\| P_n \| = \| I \| = 1$ for all $n$. Show that the sequence does not converge in the uniform topology.

6. Show that if the sequence of bounded linear operators $\{ T_n \}$ on a Hilbert space has the property that the numerical sequences $\{ \langle T_n x, y \rangle \}$ all converge, then there is a bounded linear operator $T$ such that $\{ T_n \}$ converges weakly to $T$.

7. (a) Prove the map $A \rightarrow A^*$ is continuous on $B(H)$ in the weak topology.

(b) Let $GL(H)$ denote the invertible bounded linear operators on $H$. Prove the following map is not continuous from $GL(H)$ to $B(H)$ in the strong topology: $A \rightarrow A^{-1}$.

8. The Abel transform, $A$, is defined as follows:

$$Au = f \text{ if } f(y) = \int_0^y \frac{u(\eta)}{\sqrt{y-\eta}} d\eta.$$ 

It can be seen that this map can be inverted, at least on continuously differentiable functions:

$$u(x) = \int_0^x \frac{f'(y)}{\pi \sqrt{x-y}} dy.$$ 

(a) Use the inequality $\| f * q \|_r \leq \| f \|_p \| q \|_q$ ($1 \leq p, q, r < \infty$ with $1/r = 1/p + 1/q - 1$) to show that $A$ is a bounded linear operator on $L_2(0,a)$ for any $0 < a < \infty$.

(b) Show that $A^2$ is a compact linear operator on $L_2[0,\pi]$. (Actually also for $L_2[0,a]$ for any $0 < a < \infty$.)

(c) Find $A^*$ for $A : L_2[0,a] \rightarrow L_2[0,a]$