The Mollifier Theorem

Definition of the Mollifier

The function

\[ T(x) = \begin{cases} \frac{K \exp\left(-\frac{1}{1-|x|^2}\right)}{1-|x|^2} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}, \quad x \in \mathbb{R}^n \]

where the constant K is chosen such that \( \int_{\mathbb{R}^n} T(x) \, dx = 1 \) is a test function on \( \mathbb{R}^n \). Note that \( T(x) \) vanishes, together with all its derivatives as \( |x| \to 1^- \), so \( T(x) \) is infinitely differentiable and has compact support. The graph of \( T(x) \) is sketched in the following figure.

For \( n = 1 \) and \( \varepsilon > 0 \), let

\[ S_\varepsilon(x) = \frac{1}{\varepsilon^n} T\left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad P_\varepsilon(x) = T\left( \frac{x}{\varepsilon} \right). \]

Then

\[ S_\varepsilon(x) \geq 0 \quad \text{and} \quad P_\varepsilon(x) \geq 0 \quad \text{for all } x \]

\[ S_\varepsilon(x) = 0 \quad \text{and} \quad P_\varepsilon(x) = 0 \quad \text{for } |x| > \varepsilon \]

\[ \int_{\mathbb{R}} S_\varepsilon(x) \, dx = 1 \quad \forall \varepsilon > 0, \quad S_\varepsilon(0) \to +\infty \text{ as } \varepsilon \to 0, \]

\[ \int_{\mathbb{R}} P_\varepsilon(x) \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad P_\varepsilon(0) = K/\varepsilon \quad \forall \varepsilon > 0, \]

Evidently, \( S_\varepsilon(x) \) becomes thinner and higher as \( \varepsilon \) tends to zero but the area under the graph is constantly equal to one. On the other hand, \( P_\varepsilon(x) \) has constant height but grows thinner as \( \varepsilon \) tends to zero. These test functions can be used as the "seeds" from which an infinite variety of other test functions can be constructed by using a technique called regularization which we will now describe.

For \( n \geq 1 \) we have

\[ S_\varepsilon(x) = \frac{1}{\varepsilon^n} T\left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad P_\varepsilon(x) = T\left( \frac{x}{\varepsilon} \right). \]
For $U$ a bounded open set in $\mathbb{R}^n$, and for $u \in L^1_{\text{loc}}(U)$, define for any $\varepsilon > 0$ and any $x \in U_\varepsilon = \{ x \in U : \text{dist}(x, \partial U) > \varepsilon \}$,

\[ J_\varepsilon u(x) = \int_{|x-y| \leq \varepsilon} S_\varepsilon(x-y) u(y) \, dy \quad (1.1a) \]

\[ = \int_{|z| \leq \varepsilon} S_\varepsilon(z) u(x-z) \, dz \quad (1.1b) \]

\[ = \int_{|z| \leq \varepsilon} s_1(z) u(x-\varepsilon z) \, dz. \quad (1.1c) \]

We refer to $J_\varepsilon u(x)$ as the mollified $u(x)$. This mollified function, $J_\varepsilon u(x)$, is a smoothed version of the original function, $u(x)$.

**Properties of the Mollifier**

Note first that $J_\varepsilon u(x)$ is infinitely differentiable; i.e., for any $\varepsilon > 0$ and any $x \in U_\varepsilon$, it is clear from (1.1a) that

\[ \frac{J_\varepsilon u(x + \varepsilon e_i) - J_\varepsilon u(x)}{\varepsilon} = \int_{|y| \leq \varepsilon} \left[ S_\varepsilon(x + \varepsilon e_i - y) - S_\varepsilon(x - y) \right] u(y) \, dy \]

i.e.,

\[ J_\varepsilon u(x + \varepsilon e_i) - J_\varepsilon u(x) \to \int_{|y| \leq \varepsilon} \delta_{e_i} S_\varepsilon(x-y) u(y) \, dy \quad \text{as} \ \varepsilon \to 0. \]

Since $S_\varepsilon(x)$ is infinitely differentiable, it follows that $J_\varepsilon u(x)$ is infinitely differentiable on the open set $U_\varepsilon$.

It is evident from (1.1a) that for $1 \leq p < \infty$, $\varepsilon > 0$, and $x \in U_\varepsilon$,

\[ J_\varepsilon u(x) = \int_{|y| \leq \varepsilon} S_\varepsilon(x-y)^{1-p} S_\varepsilon(x-y)^{1/p} u(y) \, dy. \]

Then, using Holder's inequality, we get

\[ |J_\varepsilon u(x)|^p = \left( \int_{|y| \leq \varepsilon} S_\varepsilon(x-y) \, dy \right)^{p-1} \int_{|y| \leq \varepsilon} S_\varepsilon(x-y) |u(y)|^p \, dy \]

and since $\int \varepsilon S_\varepsilon(x) \, dx = 1 \quad \forall \varepsilon > \delta$,

\[ \int W |J_\varepsilon u(x)|^p \, dx \leq \int W \left( \int_{|y| \leq \varepsilon} S_\varepsilon(x-y) |u(y)|^p \, dy \right) dx \]

\[ = \int W |u(y)|^p \int_{|y| \leq \varepsilon} S_\varepsilon(x-y) \, dx \, dy = \int W |u(y)|^p \, dy \]

for open sets $W = U_\varepsilon$, and $V = W_\varepsilon$. This result is just that assertion that

\[ \|J_\varepsilon u\|_{L^p(V)} \leq \|u\|_{L^p(W)} \quad \text{for} \quad V \subseteq W \subseteq U \quad (1.2) \]

Next, use (1.1c) to write

\[ J_\varepsilon u(x) - u(x) = \int_{|z| \leq 1} s_1(z) \left[ u(x-\varepsilon z) - u(z) \right] \, dz. \]

If the function $u = u(x)$ is, in fact, continuous on $U$, then this last result shows that

\[ \max \|J_\varepsilon u(x) - u(x)\| \leq \max \| \left[ u(x-\varepsilon z) - u(z) \right] \| \to 0 \quad \text{as} \ \varepsilon \to 0; \quad (1.3) \]

i.e., $J_\varepsilon u(x)$ converges uniformly to $u(x)$ for $x \in \bar{V}$ when $u(x)$ is continuous on $\bar{U}$.

For $u \in L^1_{\text{loc}}(U)$, $W = U_\varepsilon$, and arbitrary $\delta > 0$, use the fact that the continuous functions
are dense in $L^p(W)$ to choose $v \in C(W)$ such that
$$
\|u - v\|_{L^p(W)} \leq \delta.
$$
Then for $V = W_{\varepsilon}$,
$$
\|J_\varepsilon u - u\|_{L^p(V)} \leq \|J_\varepsilon u - J_\varepsilon v\|_{L^p(V)} + \|J_\varepsilon v - v\|_{L^p(V)} + \|v - u\|_{L^p(V)} \\
\leq \|u - v\|_{L^p(W)} + \|J_\varepsilon v - v\|_{L^p(V)} + \|v - u\|_{L^p(W)} \leq 2\delta + \|J_\varepsilon v - v\|_{L^p(V)}
$$
It follows now from (1.3) that for $u \in L^p_{loc}(U)$,
$$
\forall \varepsilon > 0, \quad \|J_\varepsilon u - u\|_{L^p(V)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (1.4)
$$
We can summarize these results in the following,

**Theorem (Local Approximation)** Suppose $U$ is open and bounded in $\mathbb{R}^n$, $1 \leq p < \infty$, and for $\varepsilon > 0$, let $U_\varepsilon$ denote the subset $\{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$.

(a) For every $\varepsilon > 0$, $u \in L^p_{loc}(U)$ implies $J_\varepsilon u \in C^\infty(U_\varepsilon)$

(b) (i) $u \in C(U)$ implies $u_\varepsilon$ converges to $u$ uniformly on compact subsets of $U$; i.e.,
$$
\|J_\varepsilon u - u\|_{C(V)} = \max \|J_\varepsilon u(x) - u(x)\| \to 0 \quad \text{for all} \quad V \subset U
$$

(ii) $u_\varepsilon$ converges to $u$ in $L^p_{loc}(U)$; i.e., $u \in L^p_{loc}(U)$ implies that for all $V \subset W \subset U$,
$$
\|J_\varepsilon u\|_{L^p(V)} \leq \|u\|_{L^p(W)} \quad \text{and} \quad \|J_\varepsilon u - u\|_{L^p(V)} \to 0 \quad \text{as} \quad \varepsilon \to 0
$$

(c) $u_\varepsilon$ converges to $u$ in $W^{k,p}_0(U)$;

Result (c) follows from (b) by induction.

**Corollary (Global Approximation)** Suppose $U$ has a smooth boundary, and $1 \leq p < \infty$.

(a) For every $\varepsilon > 0$, $u \in L^p(U)$ implies $J_\varepsilon u \in C^\infty(U) \cap L^p(U)$.

(b) $u \in L^p(U)$ implies that
$$
\|J_\varepsilon u\|_{L^p(U)} \leq \|u\|_{L^p(U)} \quad \text{and} \quad \|J_\varepsilon u - u\|_{L^p(U)} \to 0 \quad \text{as} \quad \varepsilon \to 0
$$

(c) $u \in W^{k,p}(U)$ implies that there exists functions $\{\phi_m\} \in C^\infty(U) \cap W^{k,p}(U)$ such that
$$
\|\phi_m - u\|_{L^p} \to 0 \quad \text{as} \quad m \to \infty.
$$

The proof of the corollary makes use of a partition of unity (see theorem 2 pg 251 in Evans).

**Weak Equals Strong**

For $U$ a bounded open set in $\mathbb{R}^n$, we define $v = v(x)$ to be the weak derivative of order $\alpha$, of $u = u(x)$, $x \in U$ if
$$
\int_U u(x) \partial^\alpha \phi(x) \, dx = (-1)^\alpha \int_U v(x) \phi(x) \, dx \quad \text{for all} \quad \phi \in C^\infty_c(U)
$$
Similarly, we define $v = v(x)$ to be the \textit{strong $L_p$-derivative} of order $\alpha$, of $u = u(x)$, $x \in U$ if

for any $V \subset \subset U$, there exists a sequence $\{\phi_n\} \in C_c^\infty(U)$ such that

$$\int_V |\phi_n - u|^p \, dx \to 0 \quad \text{and} \quad \int_V |\partial^\alpha \phi_n - v|^p \, dx \to 0, \quad \text{as} \quad n \to \infty.$$ 

Using mollifiers, we can show that these two notions are equivalent.

Suppose first that $v = v(x)$ is the weak derivative of order $\alpha$, of $u = u(x)$. Then, since $S_\epsilon \in C_c^\infty(U),

$$\partial^\alpha J_\epsilon u(x) = \int_{|x-y| \leq \epsilon} \partial^\alpha S_\epsilon(x-y) u(y) \, dy = (-1)^{[a]} \int_{|x-y| \leq \epsilon} \partial^\alpha S_\epsilon(x-y) u(y) \, dy$$

$$= \int_{|x-y| \leq \epsilon} S_\epsilon(x-y) v(y) \, dy = J_\epsilon v(x) \quad \text{(by definition of weak derivative)}$$

Now apply (1.4) to write

$$\int_V |J_\epsilon u - u|^p \, dx \to 0 \quad \text{and} \quad \int_V |\partial^\alpha J_\epsilon u - v|^p \, dx = \int_V |J_\epsilon v - v|^p \, dx \to 0, \quad \text{as} \quad n \to \infty.$$ 

Thus every weak derivative is a strong $L_p$-derivative.

Conversely, suppose $v = v(x)$ is the strong $L_p$-derivative of order $\alpha$, of $u = u(x)$ with

$$\int_V |\phi_n - u|^p \, dx \to 0 \quad \text{and} \quad \int_V |\partial^\alpha \phi_n - v|^p \, dx \to 0, \quad \text{as} \quad n \to \infty,$$

for arbitrary $V \subset \subset U$, and $\{\phi_n\} \in C_c^\infty(U)$. Then for any $\psi \in C_c^\infty(U),

$$\int_V (u - \phi_n) \partial^\alpha \psi \, dx = \int_V u \partial^\alpha \psi \, dx - \int_V \phi_n \partial^\alpha \psi \, dx$$

$$= \int_V u \partial^\alpha \psi \, dx - (-1)^{[a]} \int_V \partial^\alpha \phi_n \psi \, dx$$

$$= \int_V u \partial^\alpha \psi \, dx - (-1)^{[a]} \int_V v \psi \, dx + (-1)^{[a]} \int_V (v - \partial^\alpha \phi_n) \psi \, dx$$

Then it follows that

$$\left| \int_V u \partial^\alpha \psi \, dx - (-1)^{[a]} \int_V v \psi \, dx \right| \leq C_1 \int_V |\phi_n - u|^p \, dx + C_2 \int_V |\partial^\alpha \phi_n - v|^p \, dx$$

which implies that every strong $L_p$-derivative is a weak derivative.
Weyl's Lemma

Weyl's lemma is a famous result that asserts that for $U$ a bounded open set in $\mathbb{R}^n$, if
$u = u(x)$ is harmonic in $U$, (i.e., $u \in C^2(U)$ and $\nabla^2 u(x) = 0$, $x \in U$) then $u(x)$ is infinitely differentiable in $U$.

To see why this result is true, recall that every harmonic function has the mean value property. That is,

$$\forall x \in U_\varepsilon, \ r < \varepsilon, \quad u(x) = \int_{\partial B_r(x)} u(y) dS(y) = \frac{1}{n\pi^{n-1}A_n} \int_{\partial B_r(x)} u(y) dS(y).$$

Then

$$J_\varepsilon u(x) = \int_{|x-y| \leq \varepsilon} S_\varepsilon(x-y) u(y) dy = \frac{1}{\varepsilon^n} \int_{|x-y| \leq \varepsilon} T\left(\frac{x-y}{\varepsilon}\right) u(y) dy$$

$$= \frac{1}{\varepsilon^n} \int_0^1 r T\left(\frac{r}{\varepsilon}\right) \int_{\partial B_1(r)} u(y) dS(y) dr = u(x) \int_0^1 \frac{n A_n \pi}{\varepsilon^n} T\left(\frac{r}{\varepsilon}\right) r^{n-1} dr$$

$$= u(x) \int_{B_1(0)} S_\varepsilon(y) dy = u(x).$$

But this says that $\forall \varepsilon > 0, \ \forall x \in U_\varepsilon, \ J_\varepsilon u(x) = u(x).$ Since $J_\varepsilon u(x)$ is infinitely differentiable on $U$, it follows that $u(x)$ is infinitely differentiable on $U$ although $u$ need not even be continuous on the closure, $\overline{U}$. 

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