Green’s functions for the Dirichlet problem

The Green’s function for the Dirichlet problem in the region $\Omega$ is the function $G : \Omega \times \Omega \to \mathbb{R}$ such that if $-\Delta u = f$ in $\Omega$ with boundary condition $u = 0$ on $\partial \Omega$ then

$$u(x) = \int_{\Omega} G(x, y) f(y) \, dV_y.$$  

A Green’s function $G(x, y)$ is a weak solution of $-\Delta_y G(x, y) = \delta(x - y)$. Note that if $-\Delta_y K(x, y) = \delta(x - y)$ and $g$ is a function such that $\Delta_y g(x, y) = 0$ then $-\Delta_y [K(x, y) - g(x, y)] = \delta(x - y)$. This fact is used to create Green’s functions for various regions beginning with a radially symmetric function $w(r)$ that satisfies $-\Delta w(r) = \delta(0)$ and letting $K(x, y) := w(|x - y|)$, then finding the appropriate harmonic function $g(x, y)$ to subtract in order to satisfy the required boundary conditions. In 3 dimensions $w(r) = [4\pi r]^{-1}$, while in 2 dimensions $w(r) = \ln(1/r)/2\pi$. In 3 dimensions the Green’s function is therefore characterized by

1. $\Delta_y [G(x, y) - \frac{1}{4\pi |x - y|}] = 0$ for $y \in \Omega$,
2. $G(x, y) = 0$ for $y \in \partial \Omega$.

In 2 dimensions the Green’s function is characterized by

1. $\Delta_y [G(x, y) - \frac{1}{2\pi} \ln \frac{1}{|x - y|}] = 0$ for $y \in \Omega$,
2. $G(x, y) = 0$ for $y \in \partial \Omega$.

3-dimensional regions

• Whole space: $\Omega := \mathbb{R}^3$
  $$G(x, y) = \frac{1}{4\pi |x - y|}$$

• Upper half space: $\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. Let $x^* := (x_1, x_2, -x_3)$
  $$G(x, y) = \frac{1}{4\pi |x - y|} - \frac{1}{4\pi |x^* - y|}$$

• Ball of radius $a$: $\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < a^2\}, a > 0$. Let $q = a/|x|$ and $\hat{x} := q^2 x$.
  $$G(x, y) = \frac{1}{4\pi |x - y|} - \frac{q}{4\pi |x - y|} = \frac{1}{4\pi |x - y|} - \frac{1}{4\pi q |x - q^{-1} y|}$$

To prove that $G(x, y) = 0$ when $y \in \partial \Omega$ we let $r := |x|$, $u$ the unit vector $r^{-1} x$, and similarly write $y = av$ where $v$ is a unit vector. Note that $q = a/r$. Let $\gamma$ denote the angle between $u$ and $v$. Then $|x - y|^2 = |ru - av|^2 = r^2 - 2ra \cos(\gamma) + a^2$ while $|q \hat{x} - q^{-1}y|^2 = |au - rv|^2 = a^2 - 2ar \cos(\gamma) + r^2$. Hence $G(x, y) = 0$ if $y \in \partial \Omega$.  

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2-dimensional regions

- Whole plane: $\Omega := \mathbb{R}^2$
  \[ G(x,y) = -\frac{1}{2\pi} \ln(|x-y|) \]
- Upper half plane: $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$. Let $x^* := (x_1, -x_2)$
  \[ G(x,y) = -\frac{1}{2\pi} \ln(|x-y|) + \frac{1}{2\pi} \ln(|x^*-y|) \]
- Disk of radius $a$: $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x|^2 = x_1^2 + x_2^2 < a^2\}$, $a > 0$. Let $q = a/|x|$ and $\hat{x} := q^2x$.
  \[ G(x,y) = -\frac{1}{2\pi} \ln(|x-y|) + \frac{1}{2\pi} \ln(|\hat{x}-y|/q) \]

Note: One may also use the eigenfunction method to get this Green function. The eigenfunctions for the problem $-\Delta \Phi = \lambda \Phi$ with homogeneous boundary conditions are $\Phi_{nm}(r, \theta) := J_{nm}(\beta_{nm}r/a)e^{in\theta}$ where $\beta_{nm}$ is the $m$th root of $J_n(x)$. The corresponding eigenvalue is $\lambda_{nm} := [\beta_{nm}/a]^2$. Expanding $u$ and $f$ in terms of these eigenfunctions we see that

\[ u(r, \theta) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} U_{nm}\Phi_{nm}(r, \theta), \quad f(r, \theta) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} F_{nm}\Phi_{nm}(r, \theta) \]

So that $U_{nm} = F_{nm}/\lambda_{nm}$, where

\[ F_{nm} = \int_{0}^{\pi} \int_{0}^{a} f(\rho, \phi)\overline{\Phi_{nm}(\rho, \phi)}\rho d\rho d\phi/N_{nm}, \quad N_{nm} := \int_{0}^{\pi} \int_{0}^{a} \Phi_{nm}(\rho, \phi)\overline{\Phi_{nm}(\rho, \phi)}\rho d\rho d\phi. \]

This gives us the same Green’s function, in polar coordinates and as an unpleasant looking double sum:

\[ G(r, \theta; \rho, \phi) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{\Phi_{nm}(r, \theta)\Phi_{nm}(\rho, \phi)}{N_{nm}\lambda_{nm}} \]

- First quadrant: $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0 \text{ and } x_2 > 0\}$. Let $x^* := (x_1, -x_2)$
  \[ G(x,y) = -\frac{1}{2\pi} \ln(|x-y|) + \frac{1}{2\pi} \ln(|x^*-y|) + \frac{1}{2\pi} \ln(|x+y|) + \frac{1}{2\pi} \ln(|x^*+y|) \]
- Upper half disk of radius $a$: $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x|^2 = x_1^2 + x_2^2 < a^2, \ x_2 > 0\}$, $a > 0$. Let $q = a/|x|$ and $\hat{x} := q^2x$, and $\hat{x}^* := q^2x^*$
  \[ G(x,y) = -\frac{1}{2\pi} \ln(|x-y|) + \frac{1}{2\pi} \ln(|\hat{x}-y|/q) + \frac{1}{2\pi} \ln(|x^*-y|) - \frac{1}{2\pi} \ln(|\hat{x}^*-y|/q) \]
- Quarter disk of radius $a$: this is what you need for problem 4.