1. Solve the Laplace equation in the annulus, \( \Omega := \{(r, \theta) \mid \rho < r < R\} \) with boundary conditions \( u(\rho, \theta) = g(\theta) \), and \( u(R, \theta) = f(\theta) \).

2. Consider the singular Sturm-Liouville problem on \([0, R]\) given by
\[
-(ry'')(r) + n^2 yr(r) = \lambda yr(r), \quad y(R) = 0, y(r) \text{ bounded as } r \to 0^+.
\]
Show that all eigenvalues are real and greater or equal to zero. Show that two eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the inner product \( \langle f, g \rangle = \int_0^R f(r)g(r) r \, dr \).

3. Let \( \beta_{nm} \) be the \( m^{th} \) positive root of \( J_n \), and let
\[
\Phi_{mn} := J_n(\beta_{mn} r/R)e^{in\theta}, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}.
\]
Use the previous problem to show that these functions are mutually orthogonal on \( L^2(\Omega) \).

4. Consider the following equation which is a model for a vibrating drumhead
\[
\ddot{u} + \gamma^2 \dot{u} - \Delta u = F(r, \theta, t), \quad (r, \theta) \in \{(r, \theta) \mid \rho < r < R\}
\]
with boundary condition \( u(R, \theta) = 0 \) and initial conditions \( u(r, \theta, 0) = 0, \quad u_t(r, \theta, 0) = g(r, \theta) \). For each of the following cases, just outline the method of proof - you need to convince me that you know what to do. If you wish, you can come to my office and explain it in person, so that you won’t have to write up the solution.

(a) \( F \equiv 0 \).
(b) \( F := F(r, \theta) \).
(c) The most general case: \( F \) depends on \( t \).