1. Separating variables, $X(x)Y(y)$, leads to the Sturm-Liouville problem $-X''(x) = \mu X(x), \quad X(0) = X(1) = 0$, which has solutions $X_n(x) = \sin(n\pi x)$ for $n = 1, 2, \cdots$ with $\mu_n = n^2\pi^2$. The associated equation for $Y$ is $Y''(y) = n^2\pi^2 Y(y)$. Since we also want $Y'(0) = 0$, we have $Y_n(y) = \sin(n\pi x)$. Therefore

$$u(x, y) = \sum_{n=1}^{\infty} U_n \sin(n\pi x) \sinh(n\pi y).$$

Setting $y = 1$ and using the last BC we see that

$$U_n = \frac{2}{\sinh(n\pi)} \int_0^1 f(x) \sin(n\pi x) \, dx.$$

2. The existence-uniqueness theorem requires is to investigate the matrix

$$
\begin{pmatrix}
    x^2 & dx/ds \\
    u + y & dy/ds
\end{pmatrix}
$$

along the initial curve. Evaluating this matrix along the initial curve yields

$$
\begin{pmatrix}
    s^2 & 1 \\
    1 + 2s & 2
\end{pmatrix}
$$

which is singular only if $2s^2 - 2s - 1 = 0$, that is if $s = 1/2 \pm \sqrt{3}/2$. There is therefore a unique solution in the neighborhood of each point on the initial curve except the points $(1/2 \pm \sqrt{3}/2, 1 \pm \sqrt{3})$.

3. By contradiction: Suppose $u$ were negative someplace in $\Omega$, then it would assume a negative minimum at some point $(x_0, y_0) \in \Omega$. At this point $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$, $u_{xx}(x_0, y_0) \geq 0$, and $-u(x_0, y_0)^3 > 0$. But then the equation would give us the contradiction $0 > 0$.

4. (a) Let $v$ be a steady-state solution: $0 = \Delta v + f$ in $\Omega$ and $\partial v/\partial n = g$ on $\partial \Omega$, then

$$\iint_{\Omega} f \, dx \, dy = -\iint_{\Omega} \nabla \cdot \nabla u \, dx \, dy = -\int_{\partial \Omega} \frac{\partial v}{\partial n} \, dS = -\int_{\partial \Omega} g \, dS.$$

The required condition is $\iint_{\Omega} f \, dx \, dy + \int_{\partial \Omega} g \, dS = 0$.

(b) Separating variables leads to

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}.$$

This leads to two eigenvalue problems: $-X''(x) = \mu X(x), \quad X'(0) = X'(\pi) = 0$, and $-Y''(y) = \nu Y(y), \quad Y'(0) = Y'(\pi) = 0$. This gives us $X_n = \cos(nx)$, with $n = 0, 1, \cdots$ and $Y_m = \cos(my)$, with $m = 0, 1, \cdots$. The corresponding ODE for $T$ is then $T''_n = -(n^2 + m^2)T_n$. Hence the proposed solution is

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{nm} \cos(nx) \cos(my) \exp[-(n^2 + m^2)t].$$

The initial condition yields

$$c_{nm} = \frac{\int_0^\pi \int_0^\pi u_0(x, y) \cos(nx) \cos(my) \, dx \, dy}{\int_0^\pi \int_0^\pi \cos^2(nx) \cos^2(my) \, dx \, dy}.$$
(c) Clearly
\[ \lim_{t \to \infty} u(x, y, t) = c_{00} = \pi^{-2} \int_0^\pi \int_0^\pi u_0(x, y) \, dx \, dy. \]

5. Separating variables leads to \( X''(x)/X(x) = -Y''(y)/Y(y) = \text{const.} \) Since the BC at \( y = 1 \) is nonhomogeneous, the relevant eigenvalue problem (i.e. Sturm-Liouville problem) must be \(-X''(x) = \lambda X(x)\). In order to get bounded solutions we need \( \lambda \geq 0 \). This leads to \( X = \exp(i\omega x) \) with \( \omega \in \mathbb{R} \).

Therefore \( Y'' = \omega^2 Y \). Taking into account that we want \( Y(0) = 0 \) we see that \( Y(y) = \sinh(\omega y) \), so that the proposed solution becomes
\[ u(x, y) = \int_{-\infty}^\infty U(\omega) \sinh(\omega y) e^{i\omega} \, d\omega. \]

Imposing the boundary value problem we see that \( U(\omega) \sinh(\omega) \) must be the Fourier transform of \( f \) and so
\[ U(\omega) = \frac{\int_{-\infty}^\infty f(x) e^{-i\omega x} \, dx}{2\pi \sinh(\omega)}. \]

6. (a) \[
\frac{dE}{dt} = \int_\Omega \frac{1}{2} [2u_t u_{tt} + 2\nabla u \cdot \nabla u_t] \, dx \, dy = \int_\Omega u_t [F + \Delta u + \nabla u \cdot \nabla u_t] \, dx \, dy.
\]

Integrating by parts (or using Green’s first identity)
\[
\frac{dE}{dt} = \int_\Omega F u_t \, dx \, dy + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} \, dS.
\]

(b) Suppose \( u_1 \) and \( u_2 \) are two solutions to the nonhomogeneous problem, the \( u := u_1 - u_2 \) is a solution to the homogeneous problem: \( u_{tt} = \Delta u \) in \( \Omega \), \( u(x, y, 0) \equiv 0 \), \( \partial u/\partial n + au = 0 \) on \( \partial\Omega \).

Using the energy integral for \( E \) we see that
\[
\frac{dE}{dt} = -\int_{\partial\Omega} au u_t \, dS = -\frac{d}{dt} \frac{1}{2} \int_{\partial\Omega} u^2 \, dS \Rightarrow E + \frac{1}{2} \int_{\partial\Omega} u^2 \, dS = c = \text{const.}
\]

Considering the initial condition, \( c = 0 \), and hence both \( E \equiv 0 \) and \( \int_{\partial\Omega} au^2 \, dS = 0 \). The fact that \( E(t) = 0 \, \forall t \geq 0 \) implies that \( u \) is a constant function. Then the boundary integral is
\[
u^2 \int_{\partial\Omega} a \, dS = 0. \]

Hence \( u \equiv 0 \).
Take-home part

1. This problem proceeds the same way as the search for the eigenfunctions on the disk that was done in class. But here the Sturm-Liouville for \( \Theta \) has boundary conditions \( \Theta(0) = \Theta(\pi/2) = 0 \). Hence \( \Theta_n(\theta) = \sin(2n\theta) \). The Sturm-Liouville for \( R \) is exactly the same, except that we only need the Bessel functions of even order. Letting \( \beta_k \) denote the \( m \)th root of the Bessel function \( J_k \) we have:

\[
\Phi_{nm} = J_{2n}(\beta_{2n, m} r) \sin(2n\theta).
\]

2. Separation of variables leads to the equation 
\[-\Theta''(\theta) = \mu \Theta(\theta), \quad \Theta(0) = \Theta(\pi/2) = 0\]
and the equation 
\[r (r R'(r))' = \mu^2 R(r), \quad R \text{ bounded}.\]
This means, as in problem 1, that \( \Theta_n(\theta) = \sin(2n\theta) \). For \( R \) we get the Euler equation 
\[r^2 R'' + r R' - (2n)^2 R = 0.\]
The general solution for this last equation is 
\[R(r) = Ar^{2n} + Br^{-2n}.\]
Clearly \( B \) must be zero. Calling the solution \( u_1 \) (for later purposes)
\[u_1 = \sum_{n=1}^{\infty} c_n r^{2n} \sin(2n\theta).\]
Imposing the BC yields
\[c_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin(2n\theta) \, d\theta.\]

3. Separating variables as in the previous problem. But now we have an inhomogeneous BC at \( \theta = \pi/2 \) and so the (singular) Sturm-Liouville problem that must be considered is the equation for \( R \):
\[-(r R'(r))' = \mu r R, \quad R(1) = 0, \quad R \text{ bounded}.\]
If \( \mu < 0 \), say \( \mu = -\omega^2 \) then \( R = A r^{\omega r} + Br^{-\omega r} \). To keep the solution bounded, \( B = 0 \). But then \( R(1) = 0 \) would imply \( A = 0 \). If \( \mu = 0 \) then \( R = A + B \ln(r) \). Again this leads to \( A = 0, B = 0 \). Hence \( \mu > 0 \). Let \( \sqrt{\mu} = \omega > 0 \). Then \( R = Ar^{i\omega} + Br^{-i\omega} = (A + iB) \cos(\omega \ln(r)) + i(A - B) \sin(\omega \ln(r)) \), or equivalently \( R = C \sin(\omega \ln(r)) + D \cos(\omega \ln(r)) \). The BC at \( r = 1 \) implies \( D = 0 \). Hence \( R = C \sin(\omega \ln(r)) \). The equation for \( \Theta \) becomes \( \Theta''(\theta) = \omega^2 \Theta(\theta), \quad \Theta(0) = 0 \). Therefore \( \Theta = \sinh(\omega r) \) and the solution (which we will call \( u_2 \)) is
\[u_2(r, \theta) = \int_0^\infty U(\omega) \sinh(\omega r) \sin(\omega \ln(r)) \, d\omega.\]
The BC at \( \theta = \pi/2 \) reads
\[g(r) = \int_0^\infty U(\omega) \sinh(\omega \pi/2) \sin(\omega \ln(r)) \, d\omega.\]
Letting \( s = \ln(r) \)
\[g(e^s) = \int_0^\infty U(\omega) \sinh(\omega \pi/2) \sin(\omega s) \, d\omega.\]
Therefore, using the Fourier sine transform
\[U(\omega) = \frac{2}{\pi \sinh(\omega \pi/2)} \int_0^\infty g(e^s) \sin(\omega s) \, ds\]
4. Proceeding as for the half disk: If \( \mathbf{x} = (x_1, x_2) \) let 
\[
r = \sqrt{x_1^2 + x_2^2}, \quad \mathbf{x} = x/r^2, \quad \mathbf{x}^* := (x_1, -x_2) \text{ and } \mathbf{x}^* = x^*/r^2.
\]
Note that \( q = |x|/r^2 = 1/r \). Then the method of images provides
\[
G(x, y) = \left[-\frac{1}{2\pi} \ln(|x - y|) + \frac{1}{2\pi} \ln(|\mathbf{x} - y|/q) + \frac{1}{2\pi} \ln(|\mathbf{x}^* - y|) - \frac{1}{2\pi} \ln(|\mathbf{x}^* - y|/q)\right] + \frac{1}{2\pi} \ln(|x + y|) + \frac{1}{2\pi} \ln(|\mathbf{x} + y|/q) + \frac{1}{2\pi} \ln(|\mathbf{x}^* + y|) - \frac{1}{2\pi} \ln(|\mathbf{x}^* + y|/q)
\]
and the solution (which we will call \( u_3 \)) is given by
\[
u_3(x) = \int_{\Omega} G(x, y) H(y) \, dx \, dy,
\]
where \( H(x) = h(r, \theta), \ x_1 = r \cos(\theta), \ x_2 = r \sin(\theta) \). This is good enough as an answer, but we can clean this up a little:
\[
G(x, y) = \frac{1}{2\pi} \ln \left[ \frac{|\mathbf{x} - y||\mathbf{x}^* - y||\mathbf{x} + y||\mathbf{x}^* + y|}{|x - y||x^* - y||x + y||x^* + y|} \right].
\]
One can now square the fraction inside the logarithm, make up for it with a factor \( 1/2 \) in front of the logarithm and use the cosine law to express the factors of the form \( |u \pm y|^2 = \rho^2 + \sigma^2 \mp 2\rho\sigma \cos(\phi - \alpha) \) where \( y = (\rho \cos(\phi), \rho \sin(\phi)) \) and \( u = (\sigma \cos(\alpha), \sigma \sin(\alpha)) \). Put this together with the fact that in polar coordinates \( x = (r, \theta), \ x^* = (r, -\theta), \ x = (1/r, \theta), \ x^* = (1/r, -\theta) \), and we can get \( G \) in polar coordinates: \( G(x, y) = G(r, \theta; \rho, \phi) \) where
\[
G(r, \theta; \rho, \phi) = \frac{1}{4\pi} \ln \left( \frac{1 + r^2 \rho^2 - 2r \rho \cos(\theta - \phi)}{|r^2 + \rho^2 - 2r \rho \cos(\theta - \phi)|^2} \right) \quad \text{and we can write}
\]
\[
u_3(r, \theta) = \int_0^{\pi/2} \int_0^1 G(r, \theta; \rho, \phi) h(\rho, \phi) \rho \, d\rho \, d\phi.
\]
5. Let \( v := u_1 + u_2 + u_3 \) then \( v \) is a steady state solution. Let \( u := \phi - v \), then \( u_t - \Delta u = 0 \) in \( \Omega, u = 0 \) on \( \partial \Omega \), and \( u(r, \theta, 0) = \phi_0(r, \theta) - v(r, \theta) \). Separating variables leads to
\[
T'(t)\Phi(r, \theta) - T(t)\Delta \Phi(r, \theta) = 0,
\]
From which we get (see problem 1): \( \Phi(r, \theta) = \Phi_{nm}(r, \theta) = J_{2n}(\beta_{2n}m r) \sin(2n\theta) \) with \( \Delta \Phi_{nm} = -\beta_{2n}^2 \Phi_{nm} \). Hence \( T(t) = \exp(-\beta_{2n}^2 t) \) and
\[
u(r, \theta, t) = \sum_{n=1}^\infty c_{nm} \Phi_{nm}(r, \theta) \exp(-\beta_{2n}^2 m t).
\]
Using the orthogonality:
\[
c_{nm} = \frac{\int_0^1 \int_0^{\pi/2} \phi_0(r, \theta) - v(r, \theta) \Phi_{nm}(r, \theta) r \, d\theta \, dr}{\int_0^1 \int_0^{\pi/2} \Phi_{nm}(r, \theta)^2 r \, d\theta \, dr}.
\]
Therefore \( \phi = u + v = u + u_1 + u_2 + u_3 \).