Homework Set 5

1. Consider the problems

\[ y''(t) + a_1y'(t) + a_0y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1, \]

and

\[ u''(t) + a_1u'(t) + a_0u(t) = f(t), \quad u(0) = 0, \quad u'(0) = 0. \]

Let \( Y, U, \) and \( F \) denote the Laplace transforms of \( y, u, \) and \( f. \)

(a) Find expressions for \( Y \) and \( U \).

(b) Use a convolution theorem to derive Duhamel’s Principle for this problem.

2. Extend the data as odd functions and use the Fourier Transform to solve the problem

\[ u_t = ku_{xx} + f(x,t); \quad u(x,0) = \phi(x), \quad x > 0; \quad u(0,t) = 0, \quad t > 0. \]

3. You should use the table of transforms available at the MAT462 web site: Consider the diffusion problem on \( \mathbb{R} \):

\[ u_t = ku_{xx} + f(x,t), \quad u(x,0) = \phi(x). \]

Let’s use capital letters to denote the Laplace transform and \( \hat{\cdot} \) to denote Fourier transforms. So, given \( g(x,t) \), its Laplace and Fourier transforms are respectively \( G(x,s) \) and \( \hat{g}(\omega,t) \), and

\[ \mathcal{F}[G(x,s)] = \hat{G}(\omega,s) = \mathcal{L}[\hat{g}(\omega,t)]. \]

(a) Laplace transform and Fourier transform the PDE to obtain an expression for \( \hat{U}(\omega,s) \).

(b) Use the table and the inverse Fourier transform to obtain \( U(x,s) \) from your answer in part (a).

(c) Use the table and the inverse Laplace transform to obtain \( \hat{u}(\omega,t) \) from your answer in part (a).

(d) Finally, take the inverse transforms of \( U(x,s) \) and of \( \hat{u}(\omega,t) \) and show that both give the same expression for \( u(x,t) \).