## The Riemann Integral

The definition of the Riemann integral

Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. By a **partition** we mean a set of points

\[
a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b.
\]

These points are called **mesh points**. We will denote this partition by \( P \) and let \( \Delta x_i := x_i - x_{i-1} \). A partition \( Q \) is called a **refinement** of \( P \) if \( P \subseteq Q \). The graph of the the function \( f \) lies in a vertical strip in the \( xy \)-plane:

\[
\{(x, y) : a \leq x \leq b \}. 
\]

This strip consist of \( N \) **panels** \( \{(x, y) : x_{i-1} \leq x \leq x_i \} \). Loosely speaking, on this panel the graph of \( f \) varies between a minimum height \( m_i \) and a maximum height \( M_i \). More precisely

\[
m_i = \inf \{ f(x) | x_{i-1} \leq x \leq x_i \}, \quad M_i = \sup \{ f(x) | x_{i-1} \leq x \leq x_i \}.
\]

We can now define the **lower (Darboux) sum** \( L(P, f) \) and the **upper (Darboux) sum** \( U(P, f) \):

\[
L(P, f) := \sum_{i=1}^{N} m_i \Delta x_i, \quad U(P, f) := \sum_{i=1}^{N} M_i \Delta x_i.
\]

Note that

\[
m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \quad (1)
\]

Suppose we refine the partition \( P \) by adjoining one more mesh point \( z \). Suppose \( j \) is the integer such that \( x_{j-1} \leq z \leq x_j \) and let \( \tilde{P} \) denote the refined partition: \( \tilde{P} = P \cup \{z\} \). The extra point \( z \) divides the \( j^{th} \) panel of the original partition into a left panel and a right panel. Let

\[
m_j^L = \inf \{ f(x) | x_{j-1} \leq x \leq z \}, \quad M_j^L = \sup \{ f(x) | x_{j-1} \leq x \leq z \},
\]

\[
m_j^R = \inf \{ f(x) | z \leq x \leq x_j \}, \quad M_j^R = \sup \{ f(x) | z \leq x \leq x_j \}.
\]

When we compute the upper and lower sums for the refined partition we see that the only difference is that the \( j^{th} \) term is replaced by two new terms corresponding to the left and right panels:

\[
m_j \Delta x_j \quad \text{becomes} \quad m_j^L (z - x_{j-1}) + m_j^R (x_j - z),
\]

\[
M_j \Delta x_j \quad \text{becomes} \quad M_j^L (z - x_{j-1}) + M_j^R (x_j - z).
\]

Hence

\[
L(P, f) - L(\tilde{P}, f) = (m_j - m_j^L)(z - x_{j-1}) + (m_j - m_j^R)(x_j - z), \quad (2)
\]

\[
U(P, f) - U(\tilde{P}, f) = (M_j - M_j^L)(z - x_{j-1}) + (M_j - M_j^R)(x_j - z). \quad (3)
\]

Note that \( m_j \leq m_j^L \leq M_j \) and \( m_j \leq m_j^R \leq M_j \). This implies

\[
L(P, f) \leq L(\tilde{P}, f) \leq U(\tilde{P}, f) \leq U(P, f).
\]

Since we can think of a refinement \( Q \) of a partition \( P \) as being obtained by successively adding one point at a time we have:
**Lemma 1.** Let $Q$ be a refinement of $P$ then

$$L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f).$$

(4)

Note that equation (1) is actually a special case of equation (4).

**Lemma 2.** Let $P_1$ and $P_2$ be any two partitions of $[a,b]$ then $L(P_1,f) \leq U(P_2,f)$ and hence

$$\sup_P L(P,f) \leq \inf_P U(P,f)$$

$^1$ **Proof.** Let $Q := P_1 \cup P_2$, then it refines both $P_1$ and $P_2$. Therefore, by lemma 1:

$$L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f).$$

For later use we will also need another estimate:

**Lemma 3.** Let $Q$ be a refinement of $P$ obtained by adjoining K points, then

$$|L(P,f) - L(Q,f)| \leq K(M - m)\mu(P), \quad |U(P,f) - U(Q,f)| \leq K(M - m)\mu(P).$$

The proof is simply a consequence of the fact that neither suprema nor infima over any subinterval can differ by more than $M - m$. If we add an extra mesh point between $x_{i-1}$ and $x_i$ then the contribution from subinterval $[x_{i-1},x_i]$ is not changed by more than $(M_i - m_i)\Delta x_i \leq (M - m)\mu(P)$. So if we adjoin $K$ points the change in either the lower sum or the upper sum is no more in magnitude than $K(M - m)\mu(P)$.

We define the **lower Riemann integral** $\int_a^b f(x) \, dx$ and the **upper Riemann integral** $\overline{\int_a^b f(x)} \, dx$ as follows

$$\int_a^b f(x) \, dx = \sup_P L(P,f), \quad \overline{\int_a^b f(x)} \, dx = \inf_P U(P,f).$$

By lemma 2 the lower Riemann integral is less than or equal to the upper Riemann integral. We say that the function $f$ is Riemann integrable on $[a,b]$ if its lower and upper Riemann integrals have the same value. In that case we denote that common value by $\int_a^b f \, dx$, called the **Riemann integral** of $f$ on $[a,b]$. We now summarize

**Definition.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

$$M := \sup \{ f(x) | a \leq x \leq b \}, \quad m := \inf \{ f(x) | a \leq x \leq b \}.$$ 

Let $P$ denote the partition $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ and define $\Delta x_i := x_i - x_{i-1}$. The **mesh** or **norm** of this partition is defined as $|P| := \min \{ \Delta x_i | 1 \leq i \leq N \}$. We define

$$M_i := \sup \{ f(x) | x_{i-1} \leq x \leq x_i \}, \quad m_i := \inf \{ f(x) | x_{i-1} \leq x \leq x_i \}.$$ 

We define the **upper sum** and the **lower sum** respectively by

$$U(P,f) := \sum_{i=1}^N M_i \Delta x_i, \quad L(P,f) := \sum_{i=1}^N m_i \Delta x_i,$$

and the **upper Riemann integral** and the **lower Riemann integral** respectively by

$$\int_a^b f(x) \, dx := \inf_P U(P,f), \quad \overline{\int_a^b f(x)} \, dx := \sup_P U(P,f).$$

$^1$sup$_P L(P,f) = \sup \{ L(P,f) : P \text{ is a partition on } [a,b] \}$ and similarly for inf$_P U(P,f)$. 

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We say that $f$ is Riemann integrable on $[a, b]$ if the upper and lower Riemann integrals are equal. Their common value is then called Riemann integral and is denoted by
\[ \int_a^b f(x) \, dx. \]

We have the following important result:

**Riemann Lemma.** $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for any $\epsilon > 0$ there exists a partition $Q$ such that $U(Q, f) - L(Q, f) < \epsilon$.

**Proof.** Suppose that $f$ is Riemann integrable, then by the definitions of lower and upper Riemann integrals (which now have the same value) there exist partitions $P_1$ and $P_2$ such that
\[
L(P_1, f) > \int_a^b f(x) \, dx - \epsilon/2, \quad U(P_1, f) < \int_a^b f(x) \, dx + \epsilon/2.
\]

Let $Q = P_1 \cup P_2$, then
\[
U(Q, f) - L(Q, f) \leq U(P_2, f) - L(P_1, f) \leq \left( U(P_2, f) - \int_a^b f(x) \, dx \right) + \left( \int_a^b f(x) \, dx - L(P_1, f) \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Conversely, if for each $\epsilon > 0$ we can find a partition $Q$ such that $U(Q, f) - L(Q, f) < \epsilon$ then
\[
0 \leq \int_a^b f(x) \, dx - \int_a^b f(x) \, dx \leq U(Q, f) - L(Q, f) < \epsilon.
\]

Since $\epsilon$ is an arbitrarily small positive number, the upper and lower integrals must have the same value.

**Notation.** We use $C[a, b]$ to denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$ and $R[a, b]$ to denote the set of all functions that are Riemann integrable on $[a, b]$.

**Theorem.** $C[a, b] \subset R[a, b]$.

**Proof.** Let $f \in C[a, b]$. Then $f$ must, in fact, be uniformly continuous. Given any $\epsilon > 0$ we can find a $\delta > 0$ such that if $u, v \in [a, b]$ with $|u-v| < \delta$ then $|f(u) - f(v)| < \epsilon/(b-a)$. Let $P$ be any partition with mesh size $\mu(P) < \delta$. This means that on each panel $M_i - m_i < \epsilon/(b-a)$. Therefore
\[
U(P, f) - L(P, f) = \sum_{i=1}^{N} (M_i - m_i) \Delta x_i \leq \sum_{i} \left[ \frac{\epsilon}{b-a} \right] \Delta x_i = \epsilon.
\]

Integrability now follows from the Riemann lemma.

**Telescoping sum.** Note that
\[
\sum_{j=1}^{N} [\gamma_j - \gamma_{j-1}] = \gamma_N - \gamma_0.
\]

Such a sum is called a telescoping sum.

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then $f \in R[a, b]$.

**Proof.** We prove the case where $f$ is increasing; the case where $f$ is decreasing is handled similarly. Given any $\epsilon$ choose a positive integer $N$ such that $(f(b) - f(a))(b-a)/N < \epsilon$. Choose the partition $P$ with $x_i := a + i(b-a)/N$
so that $\Delta x_i = \Delta x := (b-a)/N$. From the fact that $f$ is increasing it follows that $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{N} (f(x_i) - f(x_{i-1})) \Delta x$$

which is a telescoping sum that is equal to $(f(x_N) - f(x_0)) \Delta x = (f(b) - f(a))(b-a)/N < \epsilon$. Integrability, once again, follows from the Riemann lemma.

### Riemann sums

Let $P := \{x_0, x_1, \cdots, x_N\}$ be a partition, $\Delta x_i := x_i - x_{i-1}$ and 

$$\mu(P) := \max_i \Delta x_i.$$ 

The number $\mu(P)$ is called the mesh size of the partition $P$. A set of points $\{t_1, t_2, \cdots, t_N\}$ is called a marking of the partition $P$ if for each $i$ we have $x_{i-1} \leq t_i \leq x_i$. We denote by $P^T$ the partition $P$ together with the marking $T$. We call $P^T$ a marked partition. Sometimes we will simplify the notation and denote $P^T$ by $\Pi$ and define $\mu(\Pi) := \mu(P)$. Given such a marked partition we define the corresponding Riemann sum as

$$S(\Pi, f) = S(P^T, f) := \sum_{i=1}^{N} f(t_i) \Delta x_i.$$ 

Clearly, since $m_i \leq f(t_i) \leq M_i$, we have

$$L(P, f) \leq S(P^T, f) \leq U(P, f).$$

**Lemma 4.** Let $\{P_k | k \in \mathbb{N}\}$ be a family of partitions for $[a, b]$ such that

$$\lim_{k \to \infty} \mu(P_k) = 0.$$ 

For each $k$ let $T_k$ be a marking for $P_k$. If $f$ is Riemann integrable on $[a, b]$ then

$$\lim_{k \to \infty} S(P_k^{T_k}, f) = \int_a^b f(x) \, dx.$$ 

We also have a converse to this lemma:

**Lemma 5.** Suppose $f : [a, b] \to \mathbb{R}$ is bounded and has the property that for any family of marked partitions $\{\Pi_k\}$ with $\mu(\Pi_k) \to 0$ as $k \to \infty$ the sequence $\{S(\Pi_k, f)\}_{k=1}^{\infty}$ converges. Then $f \in R[a, b]$, i.e. $f$ is Riemann integrable on $[a, b]$.

The above two lemmas can be combined into a single theorem:

**Riemann Integrability Theorem.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $L$ a real number. Then $f \in R[a, b]$ and its integral over $[a, b]$ is $L$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|L - S(P^T, f)| < \epsilon$ whenever $\mu(P) < \delta$. If $f$ is Riemann integrable then $L = \int_a^b f(x) \, dx$.

Some authors use the conclusion of this theorem as the definition of the Riemann integral. The proof is much like the proof of theorem 2.1 since it relates an $\epsilon - \delta$ statement to a statement about sequences.
The algebra of integrable functions

Riemann sums are real handy to use to prove various algebraic properties for the Riemann integral. Here we list the main properties as a theorem and indicate how the proofs go.

**Theorem.** Let \( f, g \in R[a, b] \) and let \( k \in \mathbb{R} \). Then

1. \( f + g \in R[a, b] \) and \( \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \).
2. \( kf \in R[a, b] \) and \( \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \).
3. If \( f(x) \leq g(x) \forall x \in [a, b] \) then \( \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \).
4. If \([p, q] \subset [a, b]\) then \( f \in R[p, q] \).
5. If \( a < c < b \) then \( \int_a^c f(x) \, dx = \int_c^b f(x) \, dx + \int_a^b f(x) \, dx \).
6. If \( f([a, b]) \subset [c, d] \) and \( \phi \in C[c, d] \) then \( \phi \circ f \in R[a, b] \).
7. \( |f| \in R[a, b] \) and \( \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx \).
8. \( fg \in R[a, b] \).

**Proof.** Let \( \Pi_k := P_k^2 \), \( k = 1, 2, \cdots \) be marked partitions such that \( \mu(\Pi_k) \to 0 \) as \( k \to \infty \). Then

\[
R(\Pi_k, f + g) = R(\Pi_k, f) + R(\Pi_k, g).
\]

the right hand side converges to \( \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \) as \( k \to \infty \). Therefore \( f + g \) is integrable and its integral is the sum of the integrals of \( f \) and \( g \). The proofs of (2) and (3) follow similarly. To prove (4) we modify the partitions so that \( p \) and \( q \) are always among the mesh points. Let \( U_*(P_k, f) \) and \( L_*(P_k, f) \) be the sum of only those terms in the upper and lower Riemann sums that correspond to panels that lie between \( p \) and \( q \). Then

\[
0 \leq U_*(P_k, f) - L_*(P_k, f) \leq U(P_k, f) - L(P_k, f) \to 0,
\]

and hence \( f \) is integrable on \([p, q]\). To prove (5) we arrange it so that \( c \in P_k \) for all \( k \). That is \( c = x_{m(k)} \in P_k \). Then

\[
R(\Pi_k, f) = R_L(\Pi_k, f) + R_R(\Pi_k, f),
\]

where \( R_L \) denotes the sum of terms \( 1, 2, \cdots, m(k) \) and \( R_R \) denotes the sum of the remaining terms. Clearly

\[
R_L(\Pi_k, f) \to \int_a^c f(x) \, dx, \quad R_R(\Pi_k, f) \to \int_c^b f(x) \, dx, \quad R(\Pi_k, f) \to \int_a^b f(x) \, dx.
\]

The proof of (6) is somewhat lengthy and so we will do that last. The first part of (7) follows immediately from (6) with \( \phi(s) := |s| \). The second part follows from the triangle inequality applied to the Riemann sums:

\[
|R(\Pi_k, f)| \leq R(\Pi_k, |f|).
\]

Taking limits we have the inequality in (7). We can also use (6) with \( \phi(s) := s^2 \) to deduce that \( f^2, g^2 \) and \((f + g)^2\) are all in \( R[a, b] \). Therefore

\[
f g = \frac{1}{2} \left[ (f + g)^2 - f^2 - g^2 \right] \in R[a, b].
\]

To prove (6), let \( \epsilon > 0 \) be given. Since \( \phi \) is uniformly continuous and bounded, there exists a \( \delta > 0 \) such that \( |\phi(u) - \phi(v)| < \epsilon/[2(b - a)] \) whenever \( u, v \in [c, d] \) and \(|u - v| < \delta\), and there exist numbers \( p \) and \( q \) such that \( \phi([c, d]) \subset [p, q] \). Since \( f \) is integrable we can find a partition \( P := \{x_0, x_1, \cdots, x_N\} \) such that

\[
U(P, f) - L(P, f) = \sum_{i=1}^N (M_i - m_i) \Delta x_i < \eta := \epsilon \delta/[2(q - p)],
\]
where $M_i$ (resp. $m_i$) is the supremum (resp. infimum) of $f$ on $[x_{i-1}, x_i]$. Let

$$J_k := \{ j \in \mathbb{N} : 1 \leq j \leq N, M_j - m_j < \delta \}, \quad K_k := \{ j \in \mathbb{N} : 1 \leq j \leq N, M_j - m_j \geq \delta \}.$$  

We note that

$$\eta > \sum_{i=1}^{N} (M_i - m_i) \Delta x_i \geq \sum_{i \in K_k} \delta \Delta x_i.$$  

This implies that

$$\sum_{i \in K_k} \Delta x_i < \eta/\delta = \epsilon/[2(q - p)].$$  

Let $M_i^*$ (resp. $m_i^*$) be the supremum (resp. infimum) of $\phi \circ f$ on $[x_{i-1}, x_i]$, then

$$U(P, \phi \circ f) - L(P, \phi \circ f) = \sum_{i=1}^{N} (M_i^* - m_i^*) \Delta x_i = \sum_{i \in J_k} (M_i^* - m_i^*) \Delta x_i + \sum_{i \in K_k} (M_i^* - m_i^*) \Delta x_i <$$

$$\sum_{i \in J_k} \epsilon \Delta x_i/[2(b - a)] + \sum_{i \in K_k} (q - p) \Delta x_i \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

**Warning.** If $|f|$ is Riemann integrable it does not follow that $f$ is Riemann integrable. Counterexample: $f(x) = 1$ if $x$ rational, $f(x) = -1$ if $x$ is irrational.

**Remark.** In general composition of two Riemann integrable functions is not integrable as can be seen from the following example. Let

$$f(x) = 0 \quad \forall x > 0 \text{ and } f(x) = 1 \quad \forall x \leq 0,$$

and let $\psi$ be Thomae’s function:

$$\psi(x) = 0 \quad \forall x \notin \mathbb{Q}, \quad \psi(p/q) = 1/q \quad \forall p \in \mathbb{Z}, \forall q \in \mathbb{N},$$

where we assume that $p$ and $q$ have no common factors in $\mathbb{N}$ except 1. The function $\psi$ is continuous at all irrationals. It is a fact (not proven here) that a bounded function that is continuous on $[a, b]$ except at countably many points is Riemann integrable, and hence $\psi$ is Riemann integrable on any bounded interval. The function $f$ is integrable on $[0, 1]$, but the function $f \circ \psi$ is the Dirichlet function ($f(x)$ is zero on all rationals and 1 on all irrationals). The Dirichlet function is not Riemann integrable on any interval $[a, b]$ with $a < b$, since all upper sums have the value $b - a$ while all lower sums are zero. This provides an example of a composition of two Riemann integrable functions that is not Riemann integrable. There are even examples where $f$ is Riemann integrable, $\psi$ is continuous, but $f \circ \psi$ fails to be Riemann integrable.

There is a beautiful characterization of Riemann integrable functions due to Lebesgue. We say that a set $S$ of real numbers has measure zero if for every $\epsilon > 0$ there exists a family of intervals $(a_1, b_1), (a_2, b_2), \cdots$ such that

$$S \subseteq \bigcup_j (a_j, b_j) \text{ and } \sum_j (b_j - a_j) \leq \epsilon.$$  

Note that any countable set $\{ \alpha_i \}$ has measure zero since for any $\epsilon > 0$ it can be covered by the intervals $(\alpha_i - \epsilon/2^{j+2}, \alpha_i + \epsilon/2^{j+2})$ where $\sum \epsilon/2^{j+1} \leq \epsilon.$

**Lebesgue’s characterization of Riemann integrable functions:** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff its points of discontinuity form a set of measure zero.\(^2\)

\(^2\)If a property holds everywhere except on a set of measure zero we say that it hold “almost everywhere”. So a bounded real-valued function on an interval $[a, b]$ is Riemann integrable iff it is continuous almost everywhere.
Using this characterization of Riemann integrability, the proof of the above theorem becomes very easy!

**Exercise R1.** Prove 1, 2, 6, 7, and 8 of the algebra if integrable functions theorem using Lebesgue’s characterization of Riemann integrability.

**Definition.** Suppose that $\alpha < \beta$ the we define

$$\int_{\alpha}^{\beta} f(x) \, dx := - \int_{\beta}^{\alpha} f(x) \, dx.$$  

**Theorem.** Let $f \in R[a, b]$ and $p, q, r \in [a, b]$ then

$$\int_{p}^{q} f(x) \, dx + \int_{q}^{r} f(x) \, dx = \int_{p}^{r} f(x) \, dx$$

irrespective of the relative sizes of $p, q,$ and $r$.

**The Fundamental Theorem of Calculus**

**Theorem.** Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable function and suppose $f'$ is Riemann integrable on $[a, b]$. Then

$$\int_{a}^{b} f'(x) \, dx = f(b) - f(a).$$

**Proof.** Let $P_n$, $n = 1, 2, \cdots$ be partitions of $[a, b]$ such that $\lim_{n \to \infty} \mu(P_n) = 0$. That is $P_n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \cdots\}$. Let $T_n := \{t_0^{(n)}, t_1^{(n)}, t_2^{(n)}, \cdots\}$ be a marking of the partition $P_n$ in such a way that

$$f'(t_i^{(n)}) = [f(x_i^{(n)}) - f(x_{i-1}^{(n)})] / [x_i^{(n)} - x_{i-1}^{(n)}].$$

This is possible by the Mean Value Theorem. Let $\Pi_n$ be the partition $P_n$ together with the marking $T_n$. Then

$$R(\Pi_n, f') = \sum_{i} f'(t_i^{(n)}) \Delta x_i = \sum_{i} [f(x_i^{(n)}) - f(x_{i-1}^{(n)})] = f(b) - f(a). \quad (5)$$

The last equality follows from the fact that the sum is a telescoping series. Now, letting $n$ tend to infinity we obtain the result we wanted.

**Notation.**

$$f(x)|_a^b := f(b) - f(a).$$

The formula for the Fundamental Theorem of Calculus then reads

$$\int_{a}^{b} f'(x) \, dx = f(x)|_a^b.$$

**Remark.** You might suspect that if a function $f$ is differentiable on an interval $[a, b]$ then its derivative would have to be integrable. After all the derivative has an anti-derivative! Curiously, this is not true. The Volterra function
$V$ (not defined here but you can google it) is a function that is everywhere differentiable, $V'$ is a bounded function, but $V'$ is not integrable.

**Leibniz’s Rule**

**Lemma 6.** Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $g$ is continuous at $c \in [a, b]$. Let

$$G(x) := \int_a^x g(s) \, ds.$$  

Then $G$ is differentiable at $c$ and $G'(c) = g(c)$.

**Proof.** It suffices to show that

$$\lim_{t \to 0} \left\{ \frac{[G(c + t) - G(c)]}{t} - g(c) \right\} = 0.$$

But the left side can be written as

$$\lim_{t \to 0} \frac{1}{t} \int_0^t [g(c + s) - g(c)] \, ds = \lim_{t \to 0} \frac{1}{t} \int_c^{c+t} g(s) - g(c) \, ds.$$

Given any $\epsilon > 0$ we can find a $\delta > 0$ such that $|g(c + s) - g(c)| < \epsilon$ whenever $|s| < \delta$. Hence, if $|t| < \delta$ then

$$\left| \frac{1}{t} \int_0^t [g(c + s) - g(c)] \, ds \right| < \epsilon.$$

A more general result is

**Leibniz’s Rule** Let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let $\alpha : [c, d] \to [a, b]$, and $\beta : [c, d] \to [a, b]$ be continuous functions that are differentiable at $x_0 \in [c, d]$. Let

$$\Phi(x) := \int_{\alpha(x)}^{\beta(x)} g(s) \, ds.$$  

Then $\Phi$ is differentiable at $x_0$ and

$$\Phi'(x_0) = g(\beta(x_0))\beta'(x_0) - g(\alpha(x_0))\alpha'(x_0).$$

**Proof**³ Choose $x_\neq x_0$ and define

$$G(s) := \int_{x_\neq}^s g(t) \, dt,$$

then we can write

$$\Phi(x) = G(\beta(x)) - G(\alpha(x)).$$

The result then follows from the chain rule together with the above lemma.

³Actually Leibniz’s rule is a bit more general. Under the right hypotheses it is true that

$$\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} g(x, s) \, ds = \int_{\alpha(x)}^{\beta(x)} \frac{\partial g(x, s)}{\partial x} \, ds + g(x, \beta(x))\beta'(x) - g(x, \alpha(x))\alpha'(x).$$
A few more theorems on integrals.

**Mean Value Theorem for Integrals** Let \( f \in C[a, b] \) and \( g \in R[a, b] \) with \( g(x) \geq 0 \quad \forall x \in [a, b] \). Then there exists a \( c \in [a, b] \) such that

\[
\int_{a}^{b} f(x)g(x) \, dx = f(c) \int_{a}^{b} g(x) \, dx.
\]  

(6)

**Proof.** Let \( m := \min(f) \), and \( M := \max(f) \) and \( G := \int_{a}^{b} g(x) \, dx \). Then since \( g(x) \geq 0 \) we have \( mg(x) \leq f(x)g(x) \leq Mg(x) \) so that

\[
mG = m \int_{a}^{b} g(x) \, dx \leq \int_{a}^{b} f(x)g(x) \, dx \leq M \int_{a}^{b} g(x) \, dx = MG.
\]

(7)

If \( G = 0 \) then all integrals in (3) are zero and obviously (2) will be true for any choice of \( c \). If \( G > 0 \) then \( m \leq G^{-1} \int_{a}^{b} f(x)g(x) \, dx \leq M \) so that by the intermediate value theorem there exists a \( c \) such that \( f(c) = G^{-1} \int_{a}^{b} f(x)g(x) \) for some \( c \in [a, b] \).

**Change of Variables Theorem.** Let \( J \) be an interval, \( \phi : [a, b] \to J \) a differentiable function with \( \phi' \in R[a, b] \). Let \( f : J \to \mathbb{R} \) be continuous. Then

\[
\int_{a}^{b} f(\phi(t)) \phi'(t) \, dt = \int_{\phi(a)}^{\phi(b)} f(x) \, dx.
\]

**Proof.** Let

\[
F(x) := \int_{a}^{x} f(\phi(t)) \phi'(t) \, dt - \int_{\phi(a)}^{\phi(x)} f(s) \, ds.
\]

Clearly \( F(a) = 0 \) and using Leibniz’s rule we see that \( F'(x) = 0 \). This means \( F(x) = 0 \quad \forall x \in [a, b] \). In particular \( F(b) = 0 \), which is precisely what the theorem asserts.

**Integration by parts.** Let \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) be differentiable functions with \( f, g \in R[a, b] \). Then

\[
\int_{a}^{b} f(x)g'(x) \, dx = -\int_{a}^{b} f'(x)g(x) \, dx + [f(b)g(b) - f(a)g(a)].
\]

The proof is an immediate consequence of applying the Fundamental Theorem of Calculus to the function \( f(x)g(x) \).

**Taylor’s Theorem.** Let \( J \) be an interval, \( a \in J \) and \( f \in C^{n+1}(J) \), i.e. \( f \) is \( n + 1 \) times continuously differentiable on \( J \). Then for all \( x \in J \) we have

\[
f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_{n+1}(x),
\]

(8)

where

\[
R_{n+1}(x) = \frac{1}{n!} \int_{a}^{x} f^{(n+1)}(t)(x-t)^n \, dt.
\]

Moreover, for each \( x \in J \) there is a number \( c \) between \( x \) and \( a \) such that

\[
R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.
\]

**Note.** For \( n = 0 \) equation (8) reduces to the Fundamental Theorem of Calculus and to the Mean Value Theorem, depending on the form of the remainder that is chosen:

\[
f(x) = f(a) + \int_{a}^{x} f'(t) \, dt, \quad f(x) = f(a) + f'(c)(x-a).
\]
The proof of the theorem for arbitrary \( n \) is effected by repeated integration by parts on the integral or, more simply, by mathematical induction.

**The logarithm, exponential and power functions.**

**Definition** We define the natural logarithm function as

\[
\ln(x) := \int_1^x \frac{1}{t} \, dt.
\]

Using this definition we can derive all the properties of the natural logarithm.

**Theorem** Let \( a > 0, b > 0 \).

1. \( \ln: (0, \infty) \to (-\infty, \infty) \) is a bijection. It is strictly increasing: if \( a > b \) then \( \ln(a) > \ln(b) \).
2. \( \ln(ab) = \ln(a) + \ln(b) \).
3. If \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \) then \( \ln(a^{m/n}) = \frac{m}{n} \ln(a) \).
4. \( \ln(1) = 0, \quad \ln(a/b) = \ln(a) - \ln(b) \).
5. \( [\ln(x)]' = 1/x \).

**Exercise R2.** Prove this theorem.

We can define the exponential function as the inverse function for the natural logarithm function:

\[
\exp := [\ln]^{-1} : (-\infty, \infty) \to (0, \infty).
\]

Now it is not difficult to prove all the standard properties of the exponential function.

**Theorem** Let \( y, z \in \mathbb{R} \), then

1. \( \exp : (-\infty, \infty) \to (0, \infty) \)

is a bijection. It is strictly increasing.
2. \( \exp(y + z) = \exp(y) \exp(z) \).
3. If \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \) then \( \exp\left(\frac{m}{n} y\right) = [\exp(y)]^{m/n} \).
4. \( \exp(0) = 1 \).
5. \[ \exp(x)' = \exp(x). \]

**Exercise R3.** Prove this theorem.

**Definition** We define \( e \) to be the number \( \exp(1) \).

This definition implies \( \ln(e) = 1 \).

Let \( a \) be a positive real number and let \( p \in \mathbb{R} \). Temporarily let us use the notation

\[ E(a, p) := \exp(p \ln(a)). \]

If \( p = m/n \) where \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \) then

\[ E(a, p) = \exp\left(\frac{m}{n} \ln(a)\right) = [\exp(\ln(a))]^{m/n} = a^{m/n} = a^p. \]

This suggests that we define \( a^p \) as \( E(a, p) \):

**Definition.** Let \( a > 0 \), and \( p \in \mathbb{R} \) then we define

\[ a^p := \exp(p \ln(a)). \]

**Exercise R4:** Let \( a > 0 \). Find the derivative of \( f(x) := a^x \) and \( \int_a^q a^x \, dx \).

**Exercise R5:** Let \( a > 1 \) and let \( \lambda(x) := a^x/[K + a^x] \) (a logistic function). Show that \( \lambda \) is a bijection between \( \mathbb{R} \) and \((0, 1)\). Find its inverse function and find an antiderivative \( \Lambda \), i.e. a function such that \( \Lambda' = \lambda \).

**Exercise R6:** Find

\[ \lim_{x \to 0} [1 + x]^{1/x}. \]

**Exercise R7:** Let \( f : [a, b] \to [m, M] \) be a Riemann integrable function and let \( \phi : [m, M] \to \mathbb{R} \) be a continuously differentiable function such that \( \phi'(t) \geq 0 \quad \forall t \) (i.e. \( \phi \) is monotone increasing). Using only the Riemann lemma, show that the composition \( \phi \circ f \) is Riemann integrable.
Remaining proofs

**Proof of Lemma 4:** Suppose that \( m \leq f(x) \leq M \) for all \( x \in [a, b] \). We assume \( M > m \); the case \( M = m \) would make the proof trivial. By the Riemann lemma, given any \( \epsilon > 0 \) we can find a partition \( Q \) such that \( U(Q, f) - L(Q, f) < \epsilon/2 \). Suppose that \( Q \) consists of \( K \) mesh points. Let \( k_0 \) be an integer such that

\[
\mu(P_k) < \frac{\epsilon}{4K(M-m)} \quad \forall k \geq k_0.
\]

Let \( \tilde{P}_k \) be the the partition of \( P_k \) obtained by adjoining all the mesh points of \( Q \). By our earlier lemma

\[
|L(\tilde{P}_k, f) - L(P_k, f)| \leq K(M-m)\mu(P_k) < \frac{\epsilon}{4}, \quad |U(\tilde{P}_k, f) - U(P_k, f)| \leq K(M-m)\mu(P_k) < \frac{\epsilon}{4}.
\]

Moreover, since

\[
L(P_k, f) \leq \int_a^b f(x) \, dx \leq U(P_k, f), \quad L(P_k, f) \leq S(P_k^{T_k}, f) \leq U(P_k, f)
\]

we have for \( k \geq k_0 \):

\[
\left| \int_a^b f(x) \, dx - S(P_k^{T_k}, f) \right| \leq U(P_k, f) - L(P_k, f)
\]

\[
\leq U(\tilde{P}_k, f) - L(\tilde{P}_k, f) + 2K(M-m)\mu(P_k) \leq U(Q, f) - L(Q, f) + \epsilon/2 < \epsilon.
\]

**Proof of Lemma 5:** Let \( P = \{x_0, x_1, \ldots, x_N\} \) be any partition of \([a, b]\) and let \( \delta \) be an arbitrary positive number. We can choose a marking \( T = \{t_1, t_2, \ldots, t_N\} \) so that \( |f(t_i) - M_i| < \delta/N \). Then we have \( |S(P^T, f) - U(P, f)| < \delta \). Similarly, it is possible to mark the partition such that \( |S(P^T, f) - L(P, f)| < \delta \). We can construct partitions \( \tilde{P}_k \) such that \( \mu(P_k) \rightarrow 0 \) as \( k \rightarrow \infty \) and construct marked partitions as follows. \( \Pi_{2k-1} \) is the partition \( P_k \) marked such that \( S(\Pi_{2k-1}, f) - L(P_k, f) < 1/k \) and \( \Pi_{2k} \) is the partition \( P_k \) marked such that \( U(P_k, f) - S(\Pi_{2k}, f) < 1/k \). Since \( |U(P_k, f) - L(P_k, f)| \leq |S(\Pi_{2k}, f) - S(\Pi_{2k-1}, f)| + 2/k \), and the sequence \( \{S(\Pi_n, f)\}_{n=1}^{\infty} \) is convergent we see that \( \lim_{k \to \infty} (U(P_k, f) - L(P_k, f)) = 0 \) and therefore, by the Riemann lemma, \( f \in R[a, b] \).

**Proof of the Riemann Integrability Theorem:** Suppose that for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |L - S(P^T, f)| < \epsilon \) whenever \( \mu(P) < \delta \). Then clearly the hypothesis of Lemma 5 is satisfied and \( f \) is integrable and (by Lemma 4) \( L = \int_a^b f \, dx \). Conversely, suppose that \( f \) is integrable but there exists some value \( \epsilon > 0 \) such that for arbitrarily small \( \delta \), say for \( 0 < \delta_k < 1/k \), there exists a marked partition \( \Pi_k \) with \( \mu(\Pi_k) < \delta_k \) such that

\[
|L - S(\Pi_k, f)| \geq \epsilon,
\]

where \( L = \int_a^b f \, dx \). But this contradicts lemma 4.