Nested Intervals Theorem and Applications

Theorem [Nested Intervals Theorem]. Consider a nested family of intervals $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, where for each n we have $a_n \le a_{n+1} \le b_{n+1} \le b_n$, then $a := \sup_n a_n$ and $b := \inf_n b_n$ both exist and

$$\bigcap_{n=1}^{\infty} I_n = [a, b] \neq \emptyset.$$

Moreover, if $\lim_{n\to\infty}[b_n - a_n] = 0$ then $a = b$ and the intersection will consist of only the point $\{a\}$.

Proof. We see that for all $m, n \in \mathbb{N}$ we have $a_n \le b_m$ and therefore $\sup\{a_n \ : \ n \in \mathbb{N}\} \le b_m \quad m \in \mathbb{N}$. Therefore $a := \sup\{a_n \ : \ n \in \mathbb{N}\} \le b := \inf\{b_n \ : \ n \in \mathbb{N}\}$. This means $[a, b] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ so that

$$\bigcap_{n=1}^{\infty} I_n \supset [a, b] \neq \emptyset.$$

If $x < a$ then $x < a_n$ for some n and if $x > b$ then $x > b_n$ for some n. In either case $x \notin [a_n, b_n]$ for some n and hence is not in the intersection of all $[a_n, b_n]$. Therefore

$$\bigcap_{n=1}^{\infty} I_n = [a, b] \neq \emptyset.$$

Finally, $b - a \le b_n - a_n$ so that if $b_n - a_n$ is arbitrarily small then $b - a = 0$ and $[a, b] = \{a\} = \{b\}$.

Bolzano-Weierstrass Theorem. Every bounded infinite set has at least one accumulation point.

Proof. Let S be a bounded infinite set. Let a_1 be a lower bound and b_1 an upper bound. So $[a_1, b_1]$ contains infinitely many points of S. Now we inductively construct the following family of nested intervals. Given the interval $[a_n, b_n]$ that contains infinitely many elements of S. Let $m_n = (a_n + b_n)/2$, the midpoint of $[a_n, b_n]$. Now one or both of the intervals $[a_n, m_n]$ and $[m_n, b_n]$ must contain infinitely many points of S, so let $[a_{n+1}, b_{n+1}]$ be one of these two subintervals that contains infinitely many points of S. Note that $b_n - a_n = (b_1 - a_1)/2^n$. We may apply the nested intervals theorem to conclude that the intersection of this family of nested intervals consists of a single point p:

$$\bigcap_{n=1}^{\infty} I_n = \{p\}.$$

We claim that p is an accumulation point of S. To see this let $(p - \epsilon, p + \epsilon)$ be any epsilon neighborhood of p. Choose $N \in \mathbb{N}$ such that $b_N - a_N < \epsilon$. Then

$$a_N = (a_N - b_N) + (b_N - p) + p > -\epsilon + p \text{ and } b_N = (b_N - a_N) + (a_N - p) + p < \epsilon + p,$$

so that $[a_N, b_N] \subset (p - \epsilon, p + \epsilon)$ and hence every epsilon neighborhood of p contains infinitely many points of S.

Bolzano Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(a)f(b) \le 0$. Then there exists a point $c \in [a, b]$ such that $f(c) = 0$.

Proof. Let $a_1 = a$ and $b_1 = b$. Now we inductively construct the following family of nested intervals. Given the interval $[a_n, b_n]$ such that $f(a_n)f(b_n) \le 0$, Let $m_n = (a_n + b_n)/2$, the midpoint of $[a_n, b_n]$. Now either $f(a_n)f(m_n) \le 0$ or $f(m_n)f(b_n) \le 0$, for if both were positive then $f(a_n)f(b_n) > 0$. Choose one of these intervals, call it $[a_{n+1}, b_{n+1}]$, such that $f(a_{n+1})f(b_{n+1} \le 0$. We may apply the nested intervals theorem to conclude that the intersection of this family of nested intervals, $[a_n, b_n]$, consists of a single point c, and moreover $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$ so that by continuity $f(c)f(c) = \lim_{n\to\infty} f(a_n)f(b_n) \le 0$, i.e. $f(c) = 0$,