MAT371, Thomae’s function

We will use the convention that whenever we use $p/q$ to denote a rational number then $q$ is a positive integer and $p$ an integer, and such that $|p|$ and $q$ have no common integral factors other than 1. Thomae’s function is defined as

$$T(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0 \\ 1/q & \text{if } x = p/q \text{ where } q \in \mathbb{N}, \text{ and } p \in \mathbb{Z} \end{cases}$$

This function is obviously discontinuous at all nonzero rational numbers since $f(p/q) = 1/q > 0$ while there exists a sequence $\{x_n\}_{n=1}^\infty$ of irrational numbers that converges to $p/q$ but $\lim_{n \to \infty} f(x_n) = 0 \neq f(p/q)$. Also, $T$ is continuous at all irrationals. To see this, suppose that $x_0$ is irrational. Given $\epsilon > 0$ let $N$ be a positive integer such that $1/N < \epsilon$. Let $S$ be the (finite) set of all rational numbers $p/q \in [x_0 - 1, x_0 + 1]$ such that $q \leq N$. Let $\delta := \min\{|x - y| : y \in S\}$. If $|x - x_0| < \delta$ then either $x$ is irrational or $x = p/q$ with $1/q \leq 1/N < \epsilon$. In either case $|f(x) - f(x_0)| < \epsilon$. We have

1. Thomae’s function is continuous except at countably many points, namely at the nonzero rational numbers.

2. Thomae’s function is Riemann integrable on any interval.

The second property follows from a more general result (see below), but can be proved directly: Let $T$ denote Thomae’s function with domain $[a, b]$. Given any $\epsilon > 0$ let $N$ be an integer such that $1/N < \epsilon/[2(b - a)]$. Let $S$ be the set of rational numbers $p/q$ in $[a, b]$ with $q \leq N$. Let $K$ be the number of elements in $S$ and let $P$ be a partition with mesh size $\mu(P) < \epsilon/(4K)$. We see that the lower Riemann sum $L(P, f) = 0$. The partition $P$ has at most $2K$ panels which contain members of $S$. These panels contribute at most $2K \times \epsilon/(4K) = \epsilon/2$ to the upper sum. On the other panels $0 \leq f(x) \leq 1/N$ and so these panels contribute at most $(b - a) \times (1/N) < \epsilon/2$ to the upper sum. Therefore the upper sum $U(P, f) < \epsilon$.

**Example.** Let

$$g(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

This function is integrable. But $g \circ T$ is the Dirichlet function, which is not integrable. So here we have an example that the composition of two Riemann integrable functions is not necessarily Riemann integrable.

There is a precise description of the set of Riemann integrable functions. From it, one may conclude that any bounded function that is discontinuous at at most countably many points of an interval $[a, b]$ is Riemann integrable. To state that result we need

**Definition.** A set $S$ of real numbers is said to have measure zero if for any $\epsilon > 0$ there exist countably many open intervals $(a_i, b_i)$, $b_i > a_i$, such that $S \subset \bigcup_i (a_i, b_i)$ and such that $\sum_i (b_i - a_i) < \epsilon$.

Note that any countable set $S = \{x_1, x_2, \ldots\}$ has measure zero (use the intervals $(x_i - \epsilon/2^{i+2}, x_i + \epsilon/2^{i+2}$ and note that $\sum_i \epsilon/2^{i+1} = \epsilon/2 < \epsilon$.

**Theorem.** A bounded function on a closed interval is Riemann integrable iff the set on which it is discontinuous has measure zero.

The proof of this result is well beyond the scope of this course.