MAT 371 Exams 2012 - 2015

Spring 2012

A Exam 1

1. (15 pts) Give the precise definition of
   (a) \[ \lim_{n \to \infty} a_n = A. \]
   (b) A subsequence.
   (c) (10 pts) The Completeness Axiom.

2. (15 pts) Consider the sequence defined by: \( a_1 = 0 \) and \( a_{n+1} = 3 + \sqrt{11 + a_n} \).
   (a) Use induction to prove this sequence is bounded above by 14
   (b) Prove the sequence is increasing.
   (c) Why must the sequence be convergent?
   (d) Find the limit of the sequence.

3. (16 pts) Describe each of the following sets as either empty, \( \mathbb{R} \), or in interval notation:
   (a) \[ \bigcup_{n=1}^{\infty} [1/n, 1 - 1/n] \]
   (b) \[ \bigcap_{n=1}^{\infty} (0, 1/n) \]
   (c) \[ \bigcup_{n=-\infty}^{\infty} [n - 1, n] \]
   (d) \[ \bigcap_{n=1}^{\infty} (-1/n, 1 + 1/n) \]

4. (15 pts) Complete the following definitions:
   (a) The sequence \( \{a_n\}_{n=1}^{\infty} \) is said to be a Cauchy sequence if
   (b) An epsilon neighborhood of the real number \( x \) is
(c) $A$ is said to be an accumulation point of the set $S$ if

5. (10 pts)

(a) Give an example of a convergent sequence whose range has no accumulation points.
(b) Give an example of a sequence that does not converge but whose range has at least one accumulation point.

6. (15 pts) Use the definition of convergence directly (i.e. an $\epsilon - N$ proof) to show that

$$\lim_{n \to \infty} \frac{6n^3 + 5n + 2}{2n^3 + n^2} = 3.$$ 

7. (15 pts) Using only the definitions of convergence and of least upper bound prove that if $S$ is a nonempty set of numbers with least upper bound $B$, and $B \notin S$ then there exists a monotone increasing sequence $\{a_n\}_{n=1}^\infty$ such that $\forall n \in \mathbb{N}$ we have $a_n \in S$, and $\lim_{n \to \infty} a_n = B$.

B Exam 2

1. (10 pts) Define precisely:

(a) \[ \lim_{x \to x_0} f(x) = L. \]
(b) Continuity of a function $f$ at a point $x_0$.

2. (15 pts) Prove, using an $\epsilon - \delta$ argument that

$$\lim_{x \to 0} x^{1/3}\left[\sin(x) + 2\cos(x)\right] = 0.$$ 

3. (15 pts) Let us define the function $g(x) = 1$ when $x$ is irrational and $g(x) = 0$ if $x$ is rational. Show that $\lim_{x \to x_0} g(x)$ does not exist.

4. (15 pts) State the following theorems

(a) Intermediate Value Theorem.
(b) Extreme Value Theorem.
(c) Heine-Borel Theorem.

5. (15 pts) Carefully define the following terms:

(a) Compact set.
(b) Open set.
(c) Uniformly continuous
6. (10 pts) Consider the following functions with domain \([1, \infty)\). For each state whether it is uniformly continuous or not.

   (a) \(f(x) := \frac{x^4 - 9x^3 + 3}{x + 1}\)
   (b) \(g(x) := \sin^4(x)\)
   (c) \(u(x) := \sqrt{x}\).
   (d) \(v(x) := \sin(1/x)\).

7. (20 pts) Suppose that \(E\) is a sequentially compact set of real numbers.

   (a) Using only the definition of sequential compactness show that \(E\) is bounded above.
   (b) If \(M := \sup E\), prove that there exists a sequence \(\{x_n\}_{n=1}^\infty\) of members of \(E\) that converges to \(M\).

C Final Exam

1. (30 pts) Carefully state the following theorems:

   (a) The Fundamental Theorem of Calculus.
   (b) The Mean Value Theorem for Integrals.
   (c) The Mean Value Theorem (for derivatives).
   (d) The Heine-Borel Theorem.
   (e) The Inverse Function Theorem.
   (f) The Intermediate Value Theorem.

2. (5 pts) Define \(g : (0, \infty) \to \mathbb{R}\) by \(g(t) = t \ln(t)\). Use L'Hôpital's rule to show \(\lim_{t \to 0} g(t) = 0\). Hint: \(\ln(t)/t^{-1}\).

3. (20 pts) Evaluate the following:

   (a) \[\lim_{N \to \infty} \sum_{i=1}^{N} \left[ \frac{i^2}{N^3} + \frac{2i}{N^2} \right].\]
   (b) \(g'(3)\), where \(g\) is the inverse function of \(f : \mathbb{R} \to \mathbb{R}\) defined by \(f(x) = 1 + x^3 + x^5\).
   (c) The limit of the sequence \(\{a_n\}_{n=1}^\infty\) defined by: \(a_1 = 1, a_{n+1} = 1 + \sqrt{5 + a_n}\).
   (d) \[\frac{d}{dx} \int_{2x}^{x^2} \sqrt{1 + t^4} \, dt.\]

4. (15 pts) Give an example of each of the following, and if such an example does not exist simply state \(\not\exists\)

   (a) A sequence that does not converge, but whose range has an accumulation point.
(b) A bounded function on \([0, 1]\) that is not Riemann integrable.

(c) Two uniformly continuous functions \(f, g : \mathbb{R} \to \mathbb{R}\), whose composition is not uniformly continuous.

5. (30 pts) Answer True or False.

(a) If \(f : [a, b] \to \mathbb{R}\) and \(g : [a, b] \to \mathbb{R}\) are continuous then \(gf\) is continuous.

(b) If \(f : [a, b] \to \mathbb{R}\) and \(g : [a, b] \to \mathbb{R}\) are Riemann integrable then \(gf\) is Riemann integrable.

(c) If \(f : D \to \mathbb{R}\) and \(g : D \to \mathbb{R}\) are uniformly continuous then \(gf\) is uniformly continuous.

(d) If \(f : [a, b] \to [c, d]\) is Riemann integrable, and \(h : [a, b] \to \mathbb{R}\) is defined by \(h(x) := \sqrt{|f(x)|}\), then \(h\) is Riemann integrable on \([a, b]\).

(e) If \(S \subset \mathbb{R}\) is infinite then \(S\) has at least one accumulation point.

(f) If \(S \subset \mathbb{R}\) and \(S'\) is the set of all accumulation points of \(S\), then \(S'\) is closed.

(g) Every bounded sequence has a convergent subsequence.

(h) Suppose that \(f : [a, b] \to \mathbb{R}\) is a bounded function and that the function \(|f|\) is Riemann integrable on \([a, b]\). Then \(f\) is also Riemann integrable.

(i) Suppose \(f\) is uniformly continuous on \(\mathbb{R}\), and suppose that \(\{x_n\}_{n=1}^\infty\) is a Cauchy sequence. Then \(\{f(x_n)\}_{n=1}^\infty\) is a Cauchy sequence.

(j) \[\lim_{x \to 0} |x|^2 = 1.\]

6. (15 pts) Let \(S \subset \mathbb{R}\) and suppose that \(b\) is an accumulation point of \(S\). If \(b \notin S\) prove that there exists a sequence \(\{x_n\}_{n=1}^\infty\) of points in \(S\) such that \(\lim_{n \to \infty} x_n = b\).

7. (15 pts) Use the definition of convergence directly (i.e an \(\epsilon - \delta\) proof) to show that \[\lim_{x \to 2} (x^2 + 4x - 2) = 10.\]

8. (15 pts) Define the Riemann integral of a bounded function \(f : [a, b] \to \mathbb{R}\), starting with the definition of a partition.

9. (10 pts) State the definition of the derivative of a function at a point that does not use the limit of a ratio.

10. (25 pts) Carefully define the following.

(a) \[\lim_{x \to x_0} f(x) = L.\]

(b) Accumulation point of a set.

(c) Continuity of a function at a point.
(d) Compact set.
(e) The interior of a set $S \subset \mathbb{R}$.

11. (20 pts) Let $f : [a, b] \to \mathbb{R}$ be an increasing (bounded) function. Explain why each of the following statements is true.

(a) $f$ is continuous everywhere on $[a, b]$ except at countably many points.
(b) Let $F(x) := \int_a^x f(t) \, dt$. Then $F(x)$ is defined at all $x \in [a, b]$.
(c) $F$ is uniformly continuous.
(d) $F'(x) = f(x)$ on $[a, b]$ except at countably many points.

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Fall 2012

D Exam 1

1. (20 pts) Definitions.

   (a) Give the precise definition of
   \[ \lim_{n \to \infty} a_n = L. \]

   (b) Give the precise definition of \textit{cardinal equivalence} of two sets $A$ and $B$, i.e. what does $|A| = |B|$ mean?

   (c) Give one of the equivalent definitions of \textit{accumulation point}.

   (d) Define \textit{subsequence}.

2. (10 pts) True or false:

   (a) All bounded sequences are convergent.

   (b) All convergent sequences are bounded.

   (c) All sequences whose range have exactly one accumulation point are convergent.

   (d) All monotone sequences are convergent.

   (e) All Cauchy sequences are convergent.

   (f) All convergent sequences have monotone subsequences.

   (g) The sum of two divergent sequences is divergent.

3. (10 pts) Let \( \{a_n\}_{n=1}^\infty \) be a sequence whose range has at least two accumulation points. Prove directly, using only the definition of convergence, that this sequence diverges.

4. (15 pts) Use the definition of convergence directly to show that
   \[ \lim_{n \to \infty} \frac{n^2 + n}{n^2 + 4} = 1. \]
5. (10 pts)
   (a) Give an example of a divergent sequence whose range is finite.
   (b) Give an example of a convergent sequence whose range is finite.
   (c) Give an example of a divergent sequence whose range is bounded and infinite.
   (d) Give an example of a convergent sequence whose range is infinite.
   (e) Give an example of a sequence that has no convergent subsequence, but whose range is infinite.

6. (20 pts)
   (a) Suppose that $C > 1$. Prove that
   \[ \lim_{n \to \infty} C^{1/n} = 1. \]
   (b) Prove that
   \[ \lim_{n \to \infty} [3 + \sin(n)]^{1/n} = 1. \]

7. (15 pts) Prove that if $\{a_n\}_{n=1}^\infty$ is a sequence that converges to $A$ then the sequence $\{|a_n|\}_{n=1}^\infty$ converges to $|A|$.

E Exam 2

1. (35 pts) Give the precise definitions of each of the following:
   (a)\[ \lim_{x \to x_0} f(x) = L. \]
   (b) $f : D \to \mathbb{R}$ is continuous at $x_0$.
   (c) $f : D \to \mathbb{R}$ is differentiable at $x_0$.
   (d) $E$ is an open set.
   (e) The closure of a set.
   (f) $E$ is a compact set.
   (g) $E$ is a sequentially compact set.

2. (10 pts) Give an examples of a function defined on the real line such that:
   (a) $f$ is continuous, but has no derivative at $x = 0$
   (b) $f$ is continuous, bounded, but not uniformly continuous.

3. (10 pts) Using only the definition of limit, prove
   \[ \lim_{x \to 1} x^2 + 2x + 3 = 6. \]
4. (15 pts) Using only the epsilon-delta definition of continuity prove the following:

(a) If \( f : D \to \mathbb{R} \) is continuous at \( x_0 \) then there is a \( \delta_0 > 0 \) and an \( M > 0 \) such that \( |f(x)| \leq M \) for all \( x \in (x_0 - \delta_0, x_0 + \delta_0) \).

(b) If \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) are continuous at \( x_0 \) then \( fg \) is continuous at \( x_0 \). Of course you are asked here to prove one item of the algebra of continuity theorem and so you are therefore not to use that theorem.

5. (20 pts) Say whether the following statements are true or false:

(a) If \( f : [a, b] \to \mathbb{R} \) is continuous, it is bounded.

(b) If \( f : [a, b] \to \mathbb{R} \) is continuous, \( \{x_n\}_{n=1}^\infty \) a divergent sequence in \([a, b]\), then the sequence \( \{f(x_n)\}_{n=1}^\infty \) has a convergent subsequence.

(c) If \( f : (a, b) \to \mathbb{R} \) is uniformly continuous, \( \{x_n\}_{n=1}^\infty \) a convergent sequence in \((a, b)\) then the sequence \( \{f(x_n)\}_{n=1}^\infty \) is a convergent sequence.

(d) If \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) are differentiable, then so is \( \max(f(x), g(x)) \).

(e) If \( f : (a, b) \to \mathbb{R} \) is differentiable, then it is continuous.

6. (12 points) Consider each of the following functions and decide whether or not it is uniformly continuous:

(a) \( f : [0, \infty) \to \mathbb{R} \), \( f(x) = \sin(\sqrt{x}) \).

(b) \( f : [0, \infty) \to \mathbb{R} \), \( f(x) = \sin(x^2 + 6x) \).

(c) \( f : [1, 2] \to \mathbb{R} \),

\[ f(x) = \frac{x \cos(1/x) + \sin(x^3)}{4 + 6x^2 + x^3 \sin(x)} \]

(d) \( f : \mathbb{R} \to \mathbb{R} \), \( f(x) = \frac{x^2}{1 + x^2} \).

\[ \text{F Final Exam} \]

1. (30 pts) Definitions.

(a) Give the precise definition of \( \lim_{x \to a} f(x) = L \).

(b) A set \( S \) is said to be compact if ....

(c) The function \( f : D \to \mathbb{R} \) is said to be continuous at \( x_0 \) if ....

(d) Define what is meant by an accumulation point.

(e) Two sets \( A \) and \( B \) are said to have the same cardinality if ....

(f) Define: Riemann sum
2. (15 pts) Give three equivalent definitions for the derivative of a function \( f : [a, b] \to \mathbb{R} \) at \( x_0 \).

3. (5 pts) Give an example of a countable set that has uncountably many accumulation points.

4. (30 pts) Carefully state \( \mathbf{6} \) of the following 7 theorems:
   a. The \textit{Mean Value Theorem} for derivatives.
   b. The \textit{Mean Value Theorem} for integrals.
   c. The \textit{Extreme Value Theorem}.
   d. The \textit{Inverse Function Theorem}.
   e. The \textit{Heine-Borel Theorem}.
   f. The \textit{Change-of-Variables Theorem}.
   g. The \textit{Riemann Lemma}.

5. (15 pts) Use the definition of limit directly (\( \varepsilon - \delta \) proof) to show:
   \[
   \lim_{x \to 0} [x^3 + x^2 + x + 1] = 1.
   \]

6. (5 pts) Find an open cover for \((0, 1)\) that has no finite subcover.

7. (15 pts) Suppose \( f(x) = e^x + \sin(x) - 2 \).
   a. Prove that \( f \) has a root on \([0, \pi]\)
   b. Let \( g := f^{-1} \), the inverse function. Find \( g'(e^n - 2) \).

8. (5 pts) Show that \( \lim_{t \to 0} t^2 \ln(1 + 1/t^2) = 0 \).

9. (26 pts) For each of the following statements read carefully and decide whether it is true (T) or false (F):
   a. Every bounded sequence has a subsequence that is Cauchy.
   b. If a function is differentiable at \( x_0 \), then it is continuous at \( x_0 \).
   c. If \( f : K \to \mathbb{R} \) is a continuous function and \( K \) is a compact set and \( f \) attains both positive and negative values, then \( f(c) = 0 \) for some \( c \in K \).
   d. Let \( F \) be a closed subset of \( \mathbb{R} \) and \( \{x_n\}_{n=1}^\infty \) a sequence in \( F \). Then there is a subsequence \( \{x_{n_i}\}_{i=1}^\infty \) that converges to a point in \( F \).
   e. Suppose that \( f : [a, b] \to [1, 2] \) is Riemann integrable and suppose \( g(x) := \ln(f(x)) \). Then \( g \) is integrable on \([a, b]\)
   f. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with period \( \pi \), i.e. \( f(x+\pi) = f(x) \forall x \in \mathbb{R} \). Then \( f \) is uniformly continuous.

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(g) If \( f : [a, b] \to \mathbb{R} \) achieves its maximum at \( c \), then at least one of the following must be true:
(i) \( c = a \), (ii) \( c = b \), (iii) \( f \) is not differentiable at \( c \), (iv) \( f'(c) = 0 \).
(h) If \( f : [a, b] \to [c, d] \) and \( g : [c, d] \to \mathbb{R} \) are both Riemann integrable, then \( g \circ f \) is Riemann integrable.
(i) Let \( f : \mathbb{R} \to \mathbb{R} \) be the continuous function, and let \( x_0 \in \mathbb{R} \) be a point where \( f \) satisfies \( |f(x) - f(x_0)| \leq |x - x_0| \). Then, if \( f(x_0) = 0 \) we have that \( f^2 \) is differentiable at \( x_0 \).
(j) The sequence \( \{2 + \sin(n)\}_{n=1}^{\infty} \) has a convergent subsequence.
(k) Let \( f : [a, b] \to \mathbb{R} \) be a function and \( c \in [a, b] \). Suppose that for each sequence \( \{x_n\}_{n=1}^{\infty} \) in \( [a, b] \) that converges to \( c \) we know that the sequence \( \{f(x_n)\}_{n=1}^{\infty} \) converges to some limit. Then \( f \) is continuous at \( c \).
(l) Let the sets \( S_1, S_2, \ldots \) be countable, the the union \( \bigcup_{n=1}^{\infty} S_n \) is countable.

10. (15 pts) Consider the function \( f : [0, 2] \to \mathbb{R} \) defined by \( f(x) = x^2 + 4x \). Let \( P := \{0, 1, 1.5, 2\} \) be a partition marked with the points \( \{1/2, 1, 2\} \). Compute each of the following:
(a) The upper sum.
(b) The lower sum.
(c) The Riemann sum.

11. (15 pts) Suppose that \( p > 0 \) and suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function that has period \( p \), i.e. \( \forall x \in \mathbb{R} \) we have \( f(x + p) = f(x) \). Suppose \( F(x) := \int_{x}^{x+p} f(s) \, ds \).
(a) Show that there is a constant \( k \) such that \( F(x) = k \) \( \forall x \in \mathbb{R} \)
(b) Show that for all \( a \in \mathbb{R} \) we have
\[
\int_{0}^{p} [f(x) - f(x + a)] \, dx = 0
\]
(c) Prove that for any real number \( a \) there is an \( c \in \mathbb{R} \) such that \( f(c + a) = f(c) \).

12. (14 pts) Using only the definitions of open prove that if \( \{U_\lambda\}_{\lambda \in \Lambda} \) is a family of open sets then
\[
\bigcup_{\lambda \in \Lambda} U_\lambda
\]
is an open set.

13. (10 pts) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable and that \( f \) has three roots at \( a < b < c \). Prove that there is a point \( z \in (a, c) \) where \( f''(z) = 0 \).
Spring 2013

G Exam 1

1. (20 pts) Definitions.
   (a) Give the precise definition of convergence of a sequence. In other words what does \( \lim_{n \to \infty} a_n = A \) mean?
   (b) Give the precise definition of the power set of a set.
   (c) Give three of the equivalent definitions of accumulation point.
   (d) Define the term Cauchy sequence.

2. (10 pts) True or false (T or F):
   (a) All sequences whose range has at least two accumulation points are divergent.
   (b) All monotone sequences are convergent.
   (c) The range of a Cauchy sequence has at least one accumulation point.
   (d) All bounded sequences have a monotone subsequence.
   (e) The product of two divergent sequences is divergent.

3. (15 pts) Use the definition of convergence directly to show that
   \[
   \lim_{n \to \infty} \frac{n^2 + 6n}{n^2 - 6n + 1} = 1.
   \]

4. (15 pts) Prove that \((0, 1)\) is cardinally equivalent to \([0, 1)\).

5. (15 pts) Describe each of the following sets as either empty, \(\mathbb{R}\), or in interval notation:
   (a) \[\bigcup_{n=3}^{\infty} (1/n, 1 - 1/n)\]
   (b) \[\bigcap_{n=1}^{\infty} [0, 1/n]\]
   (c) \[\bigcup_{n=1}^{\infty} [0, 1 - 1/n]\]
   (d) \[\bigcup_{n=1}^{\infty} [0, n + 1/n]\]
6. (10 pts) In each case below supply the example requested in the form \( a_n = \ldots \). If no such example exists, explain why.

(a) Give an example of a divergent sequence whose range is infinite.
(b) Give an example of a divergent sequence whose range is bounded and infinite.
(c) Give an example of a convergent sequence whose range is unbounded.
(d) Give an example of a convergent sequence whose range has two accumulation points.
(e) Give an example of a sequence that has no convergent subsequence.

7. (5 pts) State the Bolzano-Weierstrass theorem.

8. (10 pts) Prove that if a set \( S \) has an accumulation point \( A \), then \( A \) is also an accumulation point for at least one of the sets \( S \cap (-\infty, A] \) or \( S \cap [A, \infty) \).

H Exam 2

1. (20 pts) Give the precise definition (in complete sentences) of each of the following.

(a) Convergence at a point of a function, i.e. \( \lim_{x \to a} f(x) = L \).
(b) Open set.
(c) Sequentially compact set.
(d) Continuity of a function at a point.
(e) Uniform continuity.
(f) Interior of a set.

2. (20 pts) Give an example of each of the following. Note that the definition of a function requires that you give its domain.

(a) Two uniformly continuous functions whose product is not uniformly continuous.
(b) Two discontinuous functions whose product is continuous.
(c) An open cover of \([0, \infty)\) that has no finite subcover.
(d) A bounded function that does not have a maximum.

3. (15 pts) Using only the definition of closed, prove that the intersection of two closed sets is closed.

4. (15 pts) Using only the definition of limit (i.e. an “epsilon-delta” proof) prove that

\[
\lim_{x \to 1} x^4 - 1 = 0.
\]
5. (10 pts) Find the following limit and justify your answer (state any theorems you use):

\[
\lim_{x \to 1} \frac{x^2 + (x - 1) \sin(1/(x - 1))}{(x^2 - 1) \cos(x^2) + x}.
\]

6. (5 pts) Using only the definition of convergence of a function, prove that if \(\lim_{x \to a} f(x) = L\) then \(\lim_{x \to a} |f(x)| = |L|\).

7. (15 pts) Prove the theorem that says that if \(K\) is a sequentially compact set and \(f : K \to \mathbb{R}\) a continuous function that \(f(K)\) is sequentially compact.

I Final Exam

1. (25 pts) Definitions.

(a) Give the precise definition of \(\lim_{x \to a} f(x) = L\).

(b) A set \(S\) is said to be compact if ....

(c) A point \(b\) is said to be an accumulation point of a set \(S\) of real numbers if ....

(d) A function \(f : D \subset \mathbb{R} \to \mathbb{R}\) is said to be continuous at \(x_0\) if ....

(e) A point \(a\) is said to be an isolated point of the set \(S\) if ...


3. (10 pts) Give three equivalent definitions for the derivative of a function \(f : D \to \mathbb{R}\) at \(x_0\).

4. (35 pts) Carefully state the following theorems:

(a) The Main Value Theorem for derivatives.

(b) The Fundamental Theorem of Calculus.

(c) Taylor’s Theorem.

(d) The Inverse Function Theorem.

(e) The Heine-Borel Theorem.

(f) The Riemann Lemma.

(g) The Bolzano-Weierstass Theorem.

5. (10 pts) Suppose \(f : [0, \infty) \to [1, \infty)\) is defined by \(f(x) = (x + 1)e^x + x^3\).

(a) Show \(f\) is one-to-one.

(b) Let \(g := f^{-1}\), the inverse function for \(f\). Find \(g'(1)\).

6. (10 pts) Show that the function \(f(x) := x \sin(1/x)\) is uniformly continuous on \([1, \infty)\).
7. (25 pts) For each of the following statements decide whether it is true (T) or false (F):

(a) Every monotone function on a bounded interval is Riemann integrable.

(b) If a function is differentiable on an interval $[a, b]$, then it is uniformly continuous on any subset of $[a, b]$.

(c) If $f : K \to \mathbb{R}$ is a continuous function and $K$ is a compact set then $f(K)$ is compact.

(d) Let $K$ be a subset of $\mathbb{R}$ and $\{G_a\}_{a \in A}$ an open cover for $K$. Suppose that there exists a finite subcover. Then $K$ is closed and bounded.

(e) If the range of the sequence $\{x_n\}_{n=1}^{\infty}$ has an accumulation point $b$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to $b$.

(f) If $f : [a, b] \to \mathbb{R}$ achieves its maximum at $c$, then at least one of the following must be true:
   (i) $c = a$, (ii) $c = b$, (iii) $f$ is not differentiable at $c$, (iv) $f'(c) = 0$.

(g) If $\int_1^b f(x) \, dx = 2$ where $f$ is an integrable function, then $f(c) = 1$ for some $c \in [a, b]$.

(h) The sequence $\{\sin(n)\}_{n=1}^{\infty}$ has a convergent subsequence.

(i) A set of real numbers is compact iff it is sequentially compact.

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The next part of the exam consists of short proofs. You will be graded on clarity of your argument. Use the top half of the page to do your scratch work and give your answer (i.e. the proof) on the bottom half. **Think before you try to answer.** All the proofs are short.

8. (15 pts) Suppose $f, f_1, f_2, \cdots$ are integrable functions on $[a, b]$ such that

$$\sup\{|f(x) - f_n(x)| : a \leq x \leq b\} \leq 1/n.$$  

Prove that

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$  

9. (15 pts) Use the definition of limit directly ($\varepsilon, \delta$ – proof) to prove:

$$\lim_{x \to 2} x^4 = 16.$$  

13
10. (15 pts) Show that if $f$ is a twice continuously differentiable function on $\mathbb{R}$ then

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

11. (15 pts) Suppose that $f$ and $g$ are Riemann integrable functions on $[a, b]$ and that there is a constant $c > 0$ such that $g(x) \geq c$ for all $x \in [a, b]$. Show that the function $f/g$ is Riemann integrable on $[a, b]$.

12. (15 pts) Suppose that $f : [a, b] \to \mathbb{R}$ is twice continuously differentiable, with $f(a) = 0$, $f(b) = 0$ and $f(c) > 0$, where $a < c < b$. Prove that there is a point $x_0 \in (a, b)$ where $f''(x_0) < 0$.

**Summer 2013**

**J Exam 1**

1. (25 pts) Give the precise definition of each of the following.

   (a) $\lim_{n \to \infty} a_n = A$.

   (b) Accumulation point of a set.

   (c) Countable.

   (d) Neighborhood of a point $x \in \mathbb{R}$.

   (e) $\lim_{x \to a} f(x) = L$.

2. (5 pts) State the Bolzano Weierstrass theorem

3. (15 pts) Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences that converge to the same limit, $L$. Define the shuffled sequence $\{y_n\}_{n=1}^\infty$ as follows. If $n$ is odd then $y_n = b_{(n+1)/2}$ and if $n$ even then $y_n = a_{n/2}$. Prove that the sequence $\{y_n\}_{n=1}^\infty$ also converges to $L$.

4. (20 pts) Define the sequence $\{a_n\}_{n=1}^\infty$ as follows: $a_1 := 1$, $a_{n+1} := [7 + a_n]^{1/3}$.

   (a) Use mathematical induction to show $a_n \leq 20 \ \forall n \in \mathbb{N}$.

   (b) Use mathematical induction to show $\{a\}_{n=1}^\infty$ is an increasing sequence.

   (c) Bounded monotone sequences converge to a real number. On what crucial property of the real numbers does this depend?

   (d) Find an equation for the limit, $A$, of the sequence $\{a_n\}_{n=1}^\infty$.

5. (20 pts) In each of the following cases give an example of a sequence with the stated properties. If no such sequence exists, explain why.
(a) A convergent sequence whose range is infinite, but does not have any accumulation points.
(b) A sequence that does not converge but whose range has at least one accumulation point.
(c) A sequence that converges and whose range has at least two accumulation points.
(d) A sequence whose range is infinite but has no accumulation point.

6. (15 pts) Use the definition of convergence directly (i.e an $\epsilon - N$ proof) to show that
$$\lim_{n \to \infty} \frac{6n^3 + 30}{n^3 + n^2 + n - 1} = 6.$$  

K Exam 2

1. (25 pts) Carefully complete the following definitions:
   (a) $f : E \to \mathbb{R}$ is uniformly continuous on $E$ if
   (b) $K$ is a sequentially compact set if
   (c) $U$ is a open set if
   (d) $f : D \to \mathbb{R}$ is continuous at $x_0$ if
   (e) If $f : D \to \mathbb{R}$ then
$$\lim_{x \to x_0} f(x) = L$$
   means that

2. (15 pts) Using only the epsilon-delta definition of limit, prove that
$$\lim_{x \to 2} (x^2 - 4)(x^4 + 2) = 0.$$  

3. (10 pts) Carefully state the following theorems:
   (a) The Intermediate Value theorem.
   (b) The Extreme Value Theorem.

4. (10 pts) For each of the following functions determine if it is uniformly continuous (“Yes”) or not (“No”). Careful: +2 points for each right answer, -1 point for each wrong answer, 0 points for no answer.
   (a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin^2(2x + 7) \cos(3x - 4)$.
   (b) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x^2 - 10) \cos(3x^2 + 7)$.
   (c) $f : [-\pi, \pi] \to \mathbb{R}$ with $f(x) = x^2 \sin(x^4)$.
   (d) $f : [1, \infty) \to \mathbb{R}$ with $f(x) = \sin(1/x).$
(e) $f : (0,1) \to \mathbb{R}$ with $f(x) = 1/x$.

5. (15 pts) For each of the following statements say whether it is true or false and give your reason. If the statement is false, construct a counter-example.

(a) Let $f : E \to \mathbb{R}$ be a uniformly continuous function whose range has a (finite) supremum. Then $f$ attains a maximum somewhere on $E$.

(b) Let $E$ be a bounded set and let $g : E \to \mathbb{R}$ be a uniformly continuous function. Then the range of $g$ is a bounded set.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then it attains a maximum and a minimum.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is a bounded uniformly continuous function and $h : \mathbb{R} \to \mathbb{R}$ is continuous then $h \circ f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

(e) Finite sets are sequentially compact.

6. (10 pts) Show that if $f : [a,b] \to [a,b]$ is a continuous function, then the function $g(x) := f(x) - x$ has a root on $[a,b]$.

7. (15 pts) Let $S \subset \mathbb{R}$. Prove that the closure of its complement is the complement of its interior, i.e. $\mathbb{R} \setminus S = \mathbb{R} \setminus S^o$, or equivalently, $\mathbb{R} \setminus (\mathbb{R} \setminus S) = S^o$.

L Final Exam

1. (35 pts) Carefully state the the following theorems:

(a) The Fundamental Theorem of Calculus.

(b) The Mean Value Theorem for Integrals.

(c) The Mean Value Theorem (for derivatives).

(d) The Heine-Borel Theorem.

(e) The Inverse Function Theorem.

(f) L'Hôpital's Rule.

(g) Taylor's Theorem.

2. (15 pts) Define the Riemann integral of a bounded function $f : [a,b] \to \mathbb{R}$, starting with the definition of a partition.
3. (25 pts) Carefully define the following.
   (a) \[ \lim_{n \to \infty} a_n = A. \]
   (b) Compact set.
   (c) Equivalence relation on a set \( S \)
   (d) \( f : D \to \mathbb{R} \) is continuous at \( x_0 \).
   (e) \( a^p \) for \( a > 0 \) and \( p \in \mathbb{R} \)

4. (10 pts) Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous and define
   \[ F(x) := \int_a^x f(t) \, dt. \]
   Prove that \( F \) is uniformly continuous.

5. (10 pts) Use the definition of convergence directly (i.e. an \( \epsilon - \delta \) proof) to show that
   \[ \lim_{x \to 1} (x - 1)(2x^2 + x + 3) = 0. \]

6. (10 pts) Evaluate each of the following:
   (a) \[ \frac{d}{dx} \int_{\ln(x)}^{e^x} \sin(\sqrt{t}) \, dt. \]
   (b) \[ \lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{j+n}. \]

7. (10 pts) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined as \( f(x) = \sin(x^2)/x \) if \( x \neq 0 \) and \( f(0) = 0 \). Prove that \( f \) is uniformly continuous. Note: you may use the inequality \( |\sin(\theta)| \leq |\theta| \) that we proved in class.

8. (15 pts) Consider the function \( f(x) := 1 + x^2 \) on the interval \([0, 6]\). Let \( P \) be the partition \( \{0, 2, 3, 5, 6\} \) and let \( T \) be the marking \( \{0, 2, 4, 6\} \). Compute the upper, lower, and Riemann sums:
   (a) \( U(P, f) = \)
   (b) \( L(P, f) = \)
   (c) \( S(P^T, f) = \)

9. (15 pts) Define the sequence \( \{a_n\}_{n=1}^{\infty} \) recursively as follows: \( a_1 = 1, \ a_{n+1} = \ln(2 + a_n) \).
   (a) Show that the sequence is monotone.
(b) Use mathematical induction how that the sequence is bounded above by \( e^3 - 2 \) 
(recall that \( 2 - e < 3 \)).

(c) Explain why the sequence is convergent.

10. (10 pts) prove that if \( a < b \) then \((a, b)\) is a neighborhood for each of its members.

11. (10 pts) Define \( f(x) := \int_{1/2}^x \ln(s) \, ds \) and let \( g := f^{-1} \), the inverse function. Find \( g'(0) \).

12. (10 pts) Define \( g(s) := \sin(s)/s \) if \( s \neq 0 \) and \( g(0) = 1 \). Note that \( g \) is a continuous function. Define

\[
f(x) := \int_0^x g(s) \, ds.
\]

Find each of the following:

(a) \( f'(x) \) for \( x \neq 0 \).
(b) \( f'(0) \).
(c) \( f''(x) \) for \( x \neq 0 \).
(d) \( f''(0) \).
(e) Is \( f'' \) continuous at the origin?

13. (5 pts) Let \( a + bx + cx^2 + dx^3 + ex^4 + R_5(x) \) be a Taylor expansion for \( \ln(1 + x) \). Evaluate \( a, b, c, d, \) and \( e \).

14. (10 pts) Let \( S \) be a bounded set and let \( b \) be its least upper bound. If \( b \notin S \), prove that \( b \) must be an accumulation point of \( S \).

15. (10 pts) Let \( a < b < c \) and \( f : [a, c] \to \mathbb{R} \) a bounded function that, for every \( \delta > 0 \) is integrable on both \([a, b - \delta]\) and \([b + \delta, c]\). Use the Riemann Lemma to prove that \( f \) is integrable on \([a, c]\).

**Fall 2013**

**M Exam 1**

1. (20 pts) Definitions.

(a) Give the precise definition of convergence of a sequence. In other words what does \( \lim_{n \to \infty} a_n = A \) mean?

(b) Give the precise definition of countably infinite.

(c) Give three of the equivalent statements of \( x_0 \) is an accumulation point of the set \( S \).

(d) Define the term equivalence relation on a set.
2. (20 pts) True or false (T or F). Careful: 4 points for each correct answer -2 points for each wrong answer!

(a) Every convergent sequence has a monotone subsequence.
(b) If the range of a sequence has two accumulation points then the sequence is divergent.
(c) Every Cauchy sequence has a unique limit.
(d) If \( \{a_n\}_{n=1}^{\infty} \) is a sequence such that \( \lim_{n \to \infty} |a_n| \) exists, then \( \{a_n\}_{n=1}^{\infty} \) has a convergent subsequence.
(e) The quotient of two convergent sequences can be divergent.

3. (15 pts) Use the definition of convergence directly to show that

\[
\lim_{n \to \infty} \frac{n^3 + n^2}{n^3 - 20n + 1} = 1.
\]

4. (10 pts) State the following:

(a) The Bolzano-Weierstrass Theorem.
(b) The Completeness Axiom.

5. (10 pts) Evaluate each of the following:

(a)

\[
\bigcap_{n=1}^{\infty} \left( \frac{n-1}{n}, \frac{n+1}{n} \right) =
\]

(b)

\[
\bigcup_{n=1}^{\infty} \left( \frac{1}{n}, \frac{n^2+1}{n} \right) =
\]

6. (10 pts) In each case below supply the example requested in the form \( a_n = \ldots \). If no such example exists, explain why.

(a) Give an example of a divergent sequence whose range is finite.
(b) Give an example of a Cauchy sequence whose range is unbounded.
(c) Give an example of a convergent sequence whose range has two accumulation points.
(d) Give an example of a convergent sequence whose range has no accumulation points.
(e) Give an example of a bounded sequence that has no convergent subsequence.

7. (15 pts) Suppose that \( \lim_{n \to \infty} a_n = A \). Using only the definition of convergence, prove that

\[
\lim_{n \to \infty} \frac{a_n + a_{2n}}{2} = A.
\]
Exam 2

1. (20 pts) Carefully complete the following definitions:
   (a) \( f : E \to \mathbb{R} \) is uniformly continuous if
   (b) \( E \) is a sequentially compact set of real numbers if
   (c) \( E \) is an open set if
   (d) The interior of a set \( E \) is

2. (10 pts) State the following theorems:
   (a) State the Extreme Value Theorem
   (b) Intermediate Value Theorem theorem.

3. (10 pts) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function and suppose that there exists a sequence \( \{x_n\}_{n=1}^\infty \) of points such that
   \[
   \lim_{n \to \infty} |f(x_n) - x_n| = 0.
   \]
   Prove that this function has a fixed point \( z \), i.e. where \( f(z) = z \).

4. (10 pts) Let \( f : D \to \mathbb{R} \) be a function and \( x_0 \) an accumulation point of \( D \). If for each sequence \( \{x_n\}_{n=1}^\infty \) in \( D \setminus \{x_0\} \) that converges to \( x_0 \) we have that the sequence \( \{f(x_n)\}_{n=1}^\infty \) is a Cauchy sequence, then show that \( f \) has a limit at \( x_0 \).

5. (20 pts) For each of the following statements say whether it is true or false and give your reason.
   (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by
       \[
       f(x) = \begin{cases} 
       0 & \text{if } x \leq 0 \\
       x \sin(1/x) & \text{if } x > 0 
       \end{cases}
       \]
       then \( f \) is not continuous at the origin.
   (b) The function \( f(x) = x^4 + x - 4 - x^2 \) has at least two roots on the interval \([-2, 2]\).
   (c) If \( f : E \to \mathbb{R} \) is uniformly continuous and \( E \) is a bounded, but not compact, then the range of \( f \) can be unbounded.
   (d) If \( h : [0, 1] \to \mathbb{R} \) is continuous and is not a constant function, then the range of \( h \) contains a nonempty open interval.

6. (15 pts) Check each of the following functions and see if it is uniformly continuous or not. Support your answer with a careful argument.
   (a) \( f : [1, \infty) \to \mathbb{R}, \quad f(x) = x^{-1} \).
   (b) \( f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \cos(x^2) \).
(c) $f : [-1, 1] \to \mathbb{R}$, where

$$f(x) = \frac{1 + x^3 \tan(x)}{2 + x^2 \cos(x) + x^4}.$$ 

7. (15 pts)

(a) Find the value of

$$\lim_{x \to 3} \frac{x^3 - 27}{x - 3}.$$ 

(b) Use an $\epsilon - \delta$ argument to show prove that your answer in part (a) is correct.

O Final Exam

1. (35 pts) Complete the following definitions:

(a) Two sets $A$ and $B$ are cardinally equivalent (have the same cardinality) if ______.

(b) A function $f : D \to \mathbb{R}$ is said to be continuous at $x_0$ if ______.

(c) The sequence $\{a_n\}_{n=1}^\infty$ is said to converge to $A$, i.e.

$$\lim_{n \to \infty} a_n = A$$

if ______.

(d) A point $a$ is said to be an accumulation point of a set $S \subset \mathbb{R}$ if ______.

(e) A sequence $\{a_n\}_{n=1}^\infty$ is said to be a Cauchy sequence if ______.

(f) A set $U \subset \mathbb{R}$ is said to be open if ______.

(g) A set $K \subset \mathbb{R}$ is said to be sequentially compact if ______.

2. (30 pts) Carefully state the following theorems:

(a) The Fundamental Theorem of Calculus.

(b) The Mean Value Theorem for derivatives.

(c) The Riemann Lemma.

(d) The Inverse Function Theorem.

(e) The Chain Rule.

(f) The Extreme Value Theorem.

3. (10 pts) Evaluate each of the following:

(a)

$$\frac{d}{dx} \int_{1+\sin(x)}^{x^2} \ln(\sqrt{t}) \, dt$$
(b) \[ \lim_{n \to \infty} \frac{\ln(n^2) + 2e^n}{e^n + n^6 \sin(n^3)} \]

4. (15 pts) Give three equivalent definitions for \textit{differentiable}. Be sure to include all of the hypotheses.

5. (15 pts) In each case below give an example as asked for. If no such example exists, explain why.
   
   (a) \( f : (0, 1) \to \mathbb{R} \) is continuous but not uniformly continuous.
   
   (b) \( \{a_n\}_{n=1}^\infty \) is a Cauchy sequence that does not have a limit.
   
   (c) \( \{a_n\}_{n=1}^\infty \) is a convergent sequence whose range has exactly 2 accumulation points.
   
   (d) \( f : \mathbb{R} \to \mathbb{R} \) is continuous and bounded, but that does not have a maximum.
   
   (e) A bounded function on \([0, 1]\) that is not Riemann integrable.

6. (10 pts) Suppose \( \{x_n\}_{n=1}^\infty \) is a sequence of real numbers converging to a number \( b \). Suppose \( a < b \). Using only the definition of convergence, prove that there exists an integer \( N \) such that \( x_n > a \) for all \( n \geq N \)

7. (10 pts) Show that \( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e \).

8. (45 pts) \textbf{Circle} the correct answer \textit{True} or \textit{False}. A correct answer gets 3 points, a wrong answer -1 point.

   (a) If \( f : \mathbb{R} \to [-1, 1] \) and \( g : [-1, 1] \to \mathbb{R} \) are continuous at the origin then \( g \circ f \) is continuous at the origin.  \textit{True} or \textit{False}

   (b) If \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to [1, 2] \) are Riemann integrable then \( f/g \) is Riemann integrable. \textit{True} or \textit{False}

   (c) If \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to (0, \infty) \) are continuous then \( f/g \) is uniformly continuous. \textit{True} or \textit{False}

   (d) Let \( a < b < c \) and suppose \( f \) is continuous on \([a, c] \setminus \{b\} \). The \( f \) is Riemann integrable on \([a, c] \).
   \textit{True} or \textit{False}

   (e) If \( f : [a, b] \to [1, 2] \) and \( g : [a, b] \to [1, 2] \) are uniformly continuous then \( fg \) is uniformly continuous. \textit{True} or \textit{False}

   (f) If \( f : [0, 1] \to [0, 1] \) is a monotone bijection, then \( f'(x) \geq 0 \) for all \( x \in [0, 1] \).
   \textit{True} or \textit{False}

   (g) If \( S \subset \mathbb{R} \) is not open, then \( S \) contains an accumulation point of \( \mathbb{R} \setminus S \). \textit{True} or \textit{False}

   (h) If \( S \subset \mathbb{R} \) is a bounded set then the closure \( \overline{S} \) is compact. \textit{True} or \textit{False}
(i) For every number \( a > 1 \) we have
\[
\int_{p}^{q} a^x \, dx = \frac{a^q - a^p}{\ln(a)}.
\]

**True or False**

(j) Every nonempty bounded set has a supremum. **True or False**

(k) Let \( E := \{1/n \mid n \in \mathbb{N} \} \) and let \( f : E \to \mathbb{R} \) be a continuous function where \( E \) is the closure of \( E \). Then \( \lim_{n \to \infty} f(1/n) \) exists. **True or False**

(l) If \( f : [a, b] \to \mathbb{R} \) is a continuous function, then there exists a point \( c \in (a, b) \) where \( f(c) = [f(a) + f(b)]/2 \). **True or False**

(m) If \( h : (-1, 1) \to \mathbb{R} \) is uniformly continuous then its range is bounded. **True or False**

(n) If \( A \) and \( B \) are sets and \( f : A \to B \) is a surjection, then there exists an injection \( g : B \to A \). **True or False**

(o) If \( \{x_n\}_{n=1}^{\infty} \) is a sequence of real numbers that converges to \( A \in \mathbb{R} \), and if for all \( n \) we have \( x_n \neq A \), then \( A \) is an accumulation point of the range of the sequence. **True or False**

(p) Suppose that \( f : [a, b] \to [0, 1] \) a twice continuously differentiable function that has at least two roots: \( f(\alpha) = 0, f(\beta) = 0 \). Then there is a point \( x \) where \( f''(x) = 0 \). **True or False**

9. (10 pts) Suppose that \( f, g : [a, b] \to \mathbb{R} \) are two continuous functions and suppose that if \( \{x_n\}_{n=1}^{\infty} \) is a sequence of points in \( [a, b] \) such that \( |f(x_n) - g(x_n)| < 1/n \) for all \( n \in \mathbb{N} \). Prove that there is a point \( x \in [a, b] \) where \( f(x) = g(x) \).

10. (20 pts) Use the definition of convergence, and nothing else, to prove the following statement:

If \( f, g : D \to \mathbb{R} \) are functions, \( x_0 \) an accumulation point of \( D \), \( \lim_{x \to x_0} f(x) = A \) and \( \lim_{x \to x_0} g(x) = B \), then \( \lim_{x \to x_0} f(x)g(x) = AB \)

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**Spring 2014**

**P Exam 1**

1. (20 pts) Definitions.

   (a) Give the precise definition of
   \[
   \lim_{n \to \infty} a_n = L.
   \]

   (b) Give the precise definition of an accumulation point of a set \( S \) of real numbers.
(c) Define what is meant by a \textit{neighborhood} of a point

(d) State the Bolzano-Weierstrass Theorem.

2. (8 pts) Complete the following definitions:

(a) Two sets $A$ and $B$ are cardinally equivalent (have the same cardinality) if

(b) A subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$ is

3. (15 pts) Using \textbf{only the definition of convergence}, prove that if $\{b_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{a_n\}_{n=1}^{\infty}$ converges to zero then

$$\lim_{n \to \infty} a_n b_n = 0.$$ 

4. (5 pts) Consider the following sequence

$$a_n = \begin{cases} n &\text{if } n \text{ is odd}, \\ (-1)^k &\text{if } n = 2^k \text{ for some integer } k > 0, \\ 1/n &\text{if } n \text{ is even, but not a power of } 2 \end{cases}$$

(a) List all the limit points of this sequence.

(b) List all the accumulation points of the range of this sequence.

5. (15 pts) The sequence $a_n := 3^{1/n}$ is obviously bounded below by 1 and above by 3.

(a) Prove that this sequence is monotone.

(b) Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 1.

(c) Prove that the sequence $\{[2 + \sin(n)]^{1/n}\}_{n=1}^{\infty}$ converges and find its limit.

6. (15 pts) Use the definition of convergence directly (i.e., \textit{an $\epsilon - N$ proof}) to show that

$$\lim_{n \to \infty} \frac{n^3 + 8n^2}{n^3 - 17n} = 1.$$ 

7. (12 pts) Write each of the following sets in interval notation:

(a)

$$\bigcap_{n \in \mathbb{N}} \left[0, \frac{1}{n}\right].$$

(b)

$$\bigcup_{n \in \mathbb{N}} \left[0, \frac{1}{n}\right].$$
(c) \[ \bigcap_{n \in \mathbb{N}} \left( -2 + \frac{1}{n}, 2 - \frac{1}{n} \right). \]

(d) \[ \bigcup_{n \in \mathbb{N}} \left( -2 + \frac{1}{n}, 2 - \frac{1}{n} \right). \]

8. (10 pts) Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence that converges to \( A \) where \( A > 0 \). Prove that there exists a number \( M \) and a positive integer \( N_0 \) such that
\[
\frac{1}{M} \leq a_n \leq M, \quad \forall n \geq N_0.
\]

Q Exam 2

1. (20 pts) Carefully state the complete definitions of:
   (a) \[ \lim_{x \to x_0} f(x) = L, \]
   (b) The interior of a set.
   (c) An open set.
   (d) \( f : D \to \mathbb{R} \) is continuous at \( x_0 \) if ...

2. 10 pts State the following two theorems:
   (a) The Intermediate value Theorem.
   (b) Theorem 2.1 (relating the existence of the limit of a function to the convergence of certain sequences).

3. (20 pts) For each of the following statements say whether it is true or false and give your reason. If the statement is false, construct a counter-example.
   (a) Let \( E := \{1/2^n \mid n \in \mathbb{N}\} \) and let \( f : E \to \mathbb{R} \) be a continuous function where \( E \) is the closure of \( E \). Then \( \lim_{n \to \infty} f(1/2^n) \) exists
   (b) Let \( E := \{1/2^n \mid n \in \mathbb{N}\} \) and let \( g : E \to \mathbb{R} \) be a continuous function. Then the range of \( g \) is a bounded set.
   (c) If \( f : \mathbb{R} \to \mathbb{R} \) is a bounded uniformly continuous function and \( g : \mathbb{R} \to \mathbb{R} \) is continuous then \( g \circ f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous.
   (d) If \( h : [a, b] \to \mathbb{R} \) is continuous then its range is closed.
4. (10 pts) Prove, using only the definition of convergence, that

\[ \lim_{x \to 1} x^4 - x^2 + x - 1 = 0. \]

Note that the above expression can be factored as \((1 + x^2 + x^3)(x - 1)\).

5. (10 pts) Suppose \( f : K \to \mathbb{R} \) is a continuous function where \( K \) is sequentially compact. Let \( L \) be a real number. Suppose that there is a sequence \( \{x_n\}_{n=1}^\infty \) of points in \( K \) such that \( \lim_{n \to \infty} f(x_n) = L \). Prove there exists a point \( x \in K \) such that \( f(x) = L \).

6. (15 pts) Prove that if \( D \) is a set of real numbers and \( D' \) the set of all accumulation points of \( D \), then \( D \cup D' \) is closed.

7. (15 pts) Check each of the following functions and see if it is uniformly continuous or not. Explain your reason.

(a) \( f : [1, \infty) \to \mathbb{R}, \ f(x) = 1/x. \)

(b) \( f : [0, \infty) \to \mathbb{R} \ f(x) = \cos^2(\sqrt{x}). \)

(c) \( f : [-1, 1] \to \mathbb{R}, \text{ where} \)

\[ f(x) = \frac{1 + x^3}{2 + x \cos(x^2)}. \]

Spring 2014

**Final Exam**

1. (30 pts) Definitions.

   (a) Let \( f : D \to \mathbb{R} \) be a function. Give the precise definition of

   \[ \lim_{x \to x_0} f(x) = L. \]

   (b) A set \( S \) is said to be sequentially compact if ....

   (c) A set \( S \) is said to be compact if ....

   (d) The function \( f : D \to \mathbb{R} \) is said to be continuous at \( x_0 \) if ....

   (e) The interior of a set \( D \) is ....

   (f) Two sets \( A \) and \( B \) are cardinally equivalent (have the same cardinality) if ....

2. (15 pts) Use the definition of convergence directly (i.e an \( \epsilon - \delta \) proof) to show that

\[ \lim_{x \to 2} x^4 = 16. \]

3. (25 pts) Carefully state the the following theorems:
(a) The **Fundamental Theorem of Calculus.**

(b) The **Mean Value Theorem for derivatives.**

(c) The **Heine-Borel Theorem.**

(d) The **Inverse Function Theorem.**

(e) The **Chain Rule.**

4. (15 pts) Show that if $f : (a, b) \to \mathbb{R}$ has a bounded derivative ($|f'(x)| \leq M \quad \forall x$) then $f$ is uniformly continuous.

5. (15 pts) Show that the following equation has a real solution on the interval $[0, 2]$: $f(x) := \int_0^x e^{t^2} \, dt = 5$.

6. (20 pts) Look at the function $f : [-1/2, 2] \to \mathbb{R}$ defined by $f(x) = \frac{\ln(1 + x) - x}{x^2}$ if $x \neq 0$, with $f(0) = -1/2$.

   (a) Show that $f$ is continuous at zero.

   (b) Find $f'(0)$.

   (c) Is $f$ uniformly continuous? Explain.

7. (20 pts) Evaluate each of the following:

   (a) $\frac{d}{dx} \int_{\sin(x)}^{1/x} e^{t^2} \, dt$

   (b) Find $g'(5)$ where $g = f^{-1}$, the inverse function of $f$, and $f : (-1, \infty) \to \mathbb{R}$ is defined by $f(x) = 4x + x^4$.

   (c) The Riemann sum $S(P^T, f)$ if $f : [0, 2] \to \mathbb{R}$ is defined by $f(x) = x^2$, the partition is $P := \{0, 1/2, 1, 2\}$ and the marking is $T := \{1/4, 1/2, 2\}$. **Don’t leave the problem incomplete. Do the addition.**

8. (15 pts) Define the Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$, starting with the definition of a partition.

9. (5 pts) Find an open cover for $(0, 1)$ that has no finite subcover.

10. (10 pts) Prove that the function $y = x^3$ is not uniformly continuous on $\mathbb{R}$.

11. (10 pts) Say whether the statement is true or false:
(a) Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence whose range has an accumulation point, then this sequence has a subsequence that converges.

(b) The closure of a countable set is also countable.

(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function that is differentiable at the origin, and suppose \( f(0) \neq 0 \). Then \( |f| \) is also differentiable at the origin.

(d) Let \( f : [a, b] \to \mathbb{R} \) be a bounded function that is continuous everywhere in its domain except at the points \( a_1, a_2, \ldots, a_N \), then \( f \) is integrable on \([a, b]\).

(e) Let \( A \) and \( B \) be two disjoint compact sets of real numbers and let \( \delta := \inf\{|x - y| \mid x \in A, y \in B\} \). Then there exists a point \( x_0 \in A \) and a point \( y_0 \in B \) such that \( |x_0 - y_0| = \delta \).

12. (20 pts) Let \( f : D \to \mathbb{R} \) be a function and \( x_0 \) an accumulation point of \( D \). Suppose

\[
\lim_{x \to x_0} f(x) = L > 0.
\]

Using only the definition of convergence of a function, prove that

\[
\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{L}.
\]

Spring 2015

S Exam 1

1. (20 pts) Definitions.

(a) Give the precise definition of \( \lim_{n \to \infty} a_n = L \).

(b) Give the precise definition of an accumulation point of a set \( S \) of real numbers.

(c) Define what is meant by a neighborhood of a point.

(d) State the Bolzano-Weierstrass Theorem.
2. (8 pts) Complete the following definitions:
   (a) Two sets $A$ and $B$ are cardinally equivalent (have the same cardinality) if
   (b) A subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$ is

3. (15 pts) Using only the definition of convergence, prove that if $\{b_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{a_n\}_{n=1}^{\infty}$ converges to zero then
   \[ \lim_{n \to \infty} a_n b_n = 0. \]

4. (5 pts) Consider the following sequence
   \[ a_n = \begin{cases} 
   n & \text{if } n \text{ is odd,} \\
   (-1)^k & \text{if } n = 2^k \text{ for some integer } k > 0, \\
   0 & \text{if } n \text{ is even, but not a power of 2} 
   \end{cases} \]
   (a) List all the limit points of this sequence.
   (b) List all of the accumulation points of the range of this sequence.

5. (15 pts) The sequence $a_n := 2^{1/n}$ is obviously bounded below by 1 and above by 2.
   (a) Prove that this sequence is monotone.
   (b) Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 1.
   (c) Prove that the sequence $\{[2 + \sin(n)]^{1/n}\}_{n=1}^{\infty}$ converges and find its limit.

6. (15 pts) Use the definition of convergence directly (i.e. an $\epsilon - N$ proof) to show that
   \[ \lim_{n \to \infty} \frac{n^3 + 8n^2}{n^3 - 17n} = 1. \]

7. (12 pts) Write each of the following sets in interval notation:
   (a) \[ \bigcap_{n \in \mathbb{N}} \left[ 0, \frac{1}{n} \right). \]
   (b) \[ \bigcup_{n \in \mathbb{N}} \left[ 0, \frac{1}{n} \right). \]
   (c) \[ \bigcap_{n \in \mathbb{N}} \left( -2 + \frac{1}{n}, 2 - \frac{1}{n} \right). \]
(d) \[ \bigcup_{n \in \mathbb{N}} \left( -2 + \frac{1}{n}, 2 - \frac{1}{n} \right). \]

8. (10 pts) Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence that converges to \( A \) where \( A > 0 \). Prove that there exists a number \( M \) and a positive integer \( N_0 \) such that

\[ \frac{1}{M} \leq a_n \leq M, \quad \forall n \geq N_0. \]

T

Exam 2

1. (20 pts) Carefully state the complete definitions of:

(a) \[ \lim_{x \to x_0} f(x) = L, \]

(b) The interior of a set.

(c) An open set.

(d) \( f : D \to \mathbb{R} \) is continuous at \( x_0 \) if ...

2. 10 pts State the following two theorems:

(a) The Intermediate value Theorem.

(b) Theorem 2.1 (relating the existence of the limit of a function to the convergence of certain sequences).

3. (20 pts) For each of the following statements say whether it is true or false and give your reason. If the statement is false, construct a counter-example.

(a) Let \( E := \{1/2^n \mid n \in \mathbb{N}\} \) and let \( f : E \to \mathbb{R} \) be a continuous function where \( E \) is the closure of \( E \). Then \( \lim_{n \to \infty} f(1/2^n) \) exists

(b) Let \( E := \{1/2^n \mid n \in \mathbb{N}\} \) and let \( g : E \to \mathbb{R} \) be a continuous function. Then the range of \( g \) is a bounded set.

(c) If \( f : \mathbb{R} \to \mathbb{R} \) is a bounded uniformly continuous function and \( g : \mathbb{R} \to \mathbb{R} \) is continuous then \( g \circ f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous.

(d) If \( h : [a, b] \to \mathbb{R} \) is continuous then its range is closed.

4. (10 pts) Prove, using only the definition of convergence, that

\[ \lim_{x \to 1} x^4 - x^2 + x - 1 = 0. \]

Note that the above expression can be factored as \((1 + x^2 + x^3)(x - 1)\).

5. (10 pts) Suppose \( f : K \to \mathbb{R} \) is a continuous function where \( K \) is sequentially compact. Let \( L \) be a real number. Suppose that there is a sequence \( \{x_n\}_{n=1}^{\infty} \) of points in \( K \) such that \( \lim_{n \to \infty} f(x_n) = L \). Prove there exists a point \( x \in K \) such that \( f(x) = L \).
6. (15 pts) Prove that if \( D \) is a set of real numbers and \( D' \) the set of all accumulation points of \( D \), then \( D \cup D' \) is closed.

7. (15 pts) Check each of the following functions and see if it is uniformly continuous or not. Explain your reason.

(a) \[ f : [1, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}. \]

(b) \[ f : [0, \infty) \to \mathbb{R} \quad f(x) = \cos^2(\sqrt{x}). \]

(c) \[ f : [-1, 1] \to \mathbb{R}, \text{ where} \]

\[ f(x) = \frac{1 + x^3}{2 + x \cos(x^2)}. \]

U Final Exam

1. (30 pts) Definitions.

(a) Let \( f : D \to \mathbb{R} \) be a function. Give the precise definition of

\[ \lim_{x \to x_0} f(x) = L. \]

(b) A set \( S \) is said to be sequentially compact if ....

(c) A set \( S \) is said to be compact if ....

(d) The function \( f : D \to \mathbb{R} \) is said to be continuous at \( x_0 \) if ....

(e) The interior of a set \( D \) is ....

(f) Two sets \( A \) and \( B \) are cardinally equivalent (have the same cardinality) if ....

2. (15 pts) Use the definition of convergence directly (i.e an \( \epsilon - \delta \) proof) to show that

\[ \lim_{x \to 2} (x^4) = 16. \]

3. (25 pts) Carefully state the the following theorems:

(a) The Fundamental Theorem of Calculus.

(b) The Mean Value Theorem for derivatives.

(c) The Heine-Borel Theorem.
(d) The Inverse Function Theorem.

(e) The Chain Rule.

4. (15 pts) Show that if \( f : (a, b) \to \mathbb{R} \) has a bounded derivative \(|f'(x)| \leq M \quad \forall x\) then \( f \) is uniformly continuous.

5. (15 pts) Show that the following equation has a real solution on the interval \([0, 2]\):

\[
f(x) := \int_0^x e^t \, dt = 5.
\]

6. (20 pts) Look at the function \( f : [-1/2, 2] \to \mathbb{R} \) defined by

\[
f(x) = \frac{\ln(1 + x) - x}{x^2} \text{ if } x \neq 0,
\]

with \( f(0) = -1/2 \).

(a) Show that \( f \) is continuous at zero.

(b) Find \( f'(0) \).

(c) Is \( f \) uniformly continuous? Explain.

7. (20 pts) Evaluate each of the following:

(a) \[
\frac{d}{dx} \int_{\sin(x)}^{1/x} e^{(t^2)} \, dt
\]

(b) Find \( g'(5) \) where \( g = f^{-1} \), the inverse function of \( f \), and \( f : (-1, \infty) \to \mathbb{R} \) is defined by \( f(x) = 4x + x^4 \).

(c) The Riemann sum \( S(P, f) \) if \( f : [0, 2] \to \mathbb{R} \) is defined by \( f(x) = x^2 \), the partition is \( P := \{0, 1/2, 1, 2\} \) and the marking is \( T := \{1/4, 1/2, 2\} \). **Don’t leave the problem incomplete. Do the addition.**

8. (15 pts) Define the Riemann integral of a bounded function \( f : [a, b] \to \mathbb{R} \), starting with the definition of a partition.

9. (5 pts) Find an open cover for \((0, 1)\) that has no finite subcover.

10. (10 pts) Prove that the function \( y = x^3 \) is not uniformly continuous on \( \mathbb{R} \).

11. (10 pts) Say whether the statement is true or false:

(a) Let \( \{x_n\}_{n=1}^\infty \) be a sequence whose range has an accumulation point, then this sequence has a subsequence that converges.
(b) The closure of a countable set is also countable.

c Let \( f : \mathbb{R} \to \mathbb{R} \) be a function that is differentiable at the origin, and suppose \( f(0) \neq 0 \). Then \( |f| \) is also differentiable at the origin.

d Let \( f : [a, b] \to \mathbb{R} \) be a bounded function that is continuous everywhere in its domain except at the points \( a_1, a_2, \ldots, a_N \), then \( f \) is integrable on \([a, b]\).

e Let \( A \) and \( B \) be two disjoint compact sets of real numbers and let \( \delta := \inf\{|x - y| \mid x \in A, y \in B\} \). Then there exists a point \( x_0 \in A \) and a point \( y_0 \in B \) such that \( |x_0 - y_0| = \delta \).

12. (20 pts) Let \( f : D \to \mathbb{R} \) be a function and \( x_0 \) an accumulation point of \( D \). Suppose

\[
\lim_{x \to x_0} f(x) = L > 0.
\]

Using only the definition of convergence of a function, prove that

\[
\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{L}.
\]

Fall 2015

V Exam 1

1. (25 pts) Definitions.

   (a) Give the precise definition of

   \[
   \lim_{n \to \infty} a_n = L.
   \]

   (b) Give the precise definition of an accumulation point of a set \( S \) of real numbers.

   (c) State the Squeeze Theorem.

   (d) State the Cantor-Bernstein Theorem.

   (e) State the Bolzano-Weierstrass Theorem

2. (10 pts) Complete the following definitions:

   (a) Two sets \( A \) and \( B \) are cardinally equivalent (have the same cardinality) if

   (b) The sequence \( \{x_n\}_{n=1}^\infty \) is Cauchy if

3. (15 pts) Using only the definition of convergence prove that if \( \{a_n\}_{n=1}^\infty \) is a sequence that converges to \( A \) then \( \lim_{n \to \infty} a_n^2 = A^2 \)
4. (20 pts) Define the sequence \( \{a_n\}_{n=1}^{\infty} \) as follows:

\[
    a_1 = 0, \quad \text{for } n \geq 1 \text{ we have } a_{n+1} = \sqrt{2 + a_n^2}.
\]

(a) Use induction to prove that \( a_n \leq 3 \) for all \( n \).

(b) Use induction to prove that this sequence is increasing.

(c) State the theorem that guarantees that this sequence is convergent,

(d) Find \( \lim_{n \to \infty} a_n \).

5. (10 pts) Prove that there are integers \( n_1 < n_2 < \cdots \) and a number \( L \) such that

\[
    \lim_{k \to \infty} \sin(n_k) = L.
\]

6. (10 pts) Evaluate and explain what theorem(s) you are using:

\[
    \lim_{n \to \infty} \frac{3n^3 + 2n^2 \sin(n^3) - 7n + 10}{n^3 + 120n^2 \cos(n^4)}.
\]

7. (10 pts) Let \( \{a_n\}_{n=1}^{\infty} \) be a bounded sequence \( M := \sup \{a_n \mid n \in \mathbb{N}\} \). Suppose that \( a_n < M \forall n \). Prove, using only the definition of accumulation point that \( M \) is an accumulation point of the range of the sequence, i.e. an accumulation point of the set \( \{a_n \mid n \in \mathbb{N}\} \).

\[\mathbf{W} \quad \text{Exam 2}\]

1. (15 pts) Carefully define each of the following (be sure the definitions are complete):

   (a) A compact set.

   (b) Continuity of a function at a point.

   (c) The limit of a function at a point.

2. (10 pts) State the following theorems:

   (a) The Intermediate Value Theorem.

   (b) The Extreme Value Theorem.

3. (10 pts) Suppose that \( f : D \to \mathbb{R} \) is a continuous function on \( D := [0, 1] \cup [2, 3] \). Also suppose that there exists a sequence \( \{x_n\}_{n=1}^{\infty} \) of points in \( D \) such that \( |f(x_n)| < 1/n \). Prove that there is a point \( x \in D \) such that \( f(x) = 0 \).

4. (15 pts) Prove that the product of two bounded, uniformly continuous functions is uniformly continuous.
5. (25 pts) For each of the following functions say whether it is true or false and give your reason.

(a) Let \( f : [-1, 1] \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
x \sin(1/x) & \text{if } x \neq 0
\end{cases}
\]

then \( f \) is uniformly continuous on its domain.

(b) The function \( f(x) = x^3 - 5x + 3 \) has at least two roots on the interval \([0, 2]\).

(c) If \( f : \mathbb{R} \to \mathbb{R} \) is given by \( f(x) = \sin(\sqrt{x}) \cos(1 + \pi x) \) then \( f \) is uniformly continuous on its domain.

(d) If \( h : [a, b] \to \mathbb{R} \) is continuous and is not a constant function, then the range of \( h \) is a closed interval.

(e) If \( f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous then \( f(B) \) is bounded whenever \( B \) is a bounded subset of real numbers.

6. (15 pts)

(a) Find the value of

\[
\lim_{x \to 1} \frac{x^4 - 1}{x - 1}.
\]

(b) Use an \( \epsilon - \delta \) argument to show prove that your answer in part (a) is correct.

7. (10 pts) Prove the closure \( \overline{D} \) of a set \( D \) is a closed set.

\[ \chi \] Final Exam

1. (30 pts) Give the precise definitions of:

(a) \( \lim_{n \to \infty} x_n = L \).

(b) \( \lim_{x \to x_0} f(x) = L \).

(c) The (general) Riemann sum.

(d) Sequentially compact set.

(e) Accumulation point.

(f) Continuity of a function at a point.
2. (15 pts) Use the definition of convergence directly to show that
\[
\lim_{n \to \infty} \frac{n^3 + 1}{n^3 - 25n + 1} = 1.
\]

3. (10 pts) Prove that \( \mathbb{N} \), the natural numbers, and \( \mathbb{Z} \), the integers, have the same cardinality. Be complete.

4. (40 pts) State the following theorems:
   (a) The Extreme Value Theorem.
   (b) Bolzano-Weierstrass theorem.
   (c) Mean Value theorem for integrals.
   (d) Intermediate Value Theorem.
   (e) Heine Borel Theorem.
   (f) The Chain Rule.
   (g) The Fundamental Theorem of Calculus.
   (h) The Mean Value Theorem (for derivatives).

5. (15 pts) Evaluate each of the following:
   (a) \[
   \frac{d}{dx} \int_{\sin(x)}^{0} \cos(t^3) \, dt
   \]
   (b) \[
   \lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{n} \cos \left( \frac{j}{n} \right) \sin \left( \frac{j}{n} \right)
   \]
   (c) \[
   \lim_{n \to \infty} \frac{n^3 + 2n \ln(n) + n^2 \sin(n^4)}{2n^3 + 6n^2 \cos(n) + n^3 \exp(-n)}
   \]

6. (25 pts) Define
\[
f(x) = \begin{cases} 
1 & \text{if } x = 0 \\
\frac{\sin(x)}{x} & \text{if } x \neq 0
\end{cases}
\]
   (a) Show that \( f \) is differentiable at the origin and find \( f'(0) \).
   (b) Find \( f'(x) \) for \( x \neq 0 \).
   (c) Show that \( f \) is uniformly continuous on \([0, 2]\).
   (d) Show that \( f \) is uniformly continuous on \([1, \infty)\).
   (e) \( f \) is uniformly continuous on all of \([0, \infty)\), true or false?

7. (15 pts) Let \( f(x) = x^2 \exp(x^2) \) with domain \((0, \infty)\).
(a) Prove that $f$ is an injective function.
(b) Prove that $f : (0, \infty) \to (0, \infty)$ is a surjection.
(c) Let $g$ be the inverse function for $f$. Find $g'(e)$

8. (10 pts) Suppose that $f$ is a continuously differentiable function on $\mathbb{R}$ and $f(0) = 0$ and $f'(0) = \pi$. Find $F''(0)$ where

$$F(x) := \frac{1}{x} \int_0^x f(s) \, ds \text{ if } x \neq 0, \text{ and } F(0) = 0.$$ 

9. (10 pts) Let $S$ be a bounded set whose least upper bound $b$ is not a member of $S$. Using only the definitions of convergence and of least upper bound, prove that there is a sequence $\{x_n\}_{n=1}^\infty$ of points $x_n \in S$ that converges to $b$.

10. (10 pts) Define the function

$$f : \mathbb{R} \to [0, \infty), \quad f(0) = 0, \quad f(x) = \exp(-1/x^2) \text{ if } x \neq 0.$$ 

(a) Using the definition of continuity, show that $f$ is continuous at 0
(b) It can be shown that $f$ is also differentiable at 0. To find the derivative, what limit must be evaluated? You are not asked to evaluate it.

11. (5 pts). Suppose that $f : [0, \infty) \to \mathbb{R}$ is a differentiable function that satisfies $f(0) = a$ and $f'(x) \leq b$ for all $x$ where $a$ and $b$ are some constants. Prove that $f(x) \leq a + bx \forall x > 0$.

12. (15 pts) Suppose that $g : D \to \mathbb{R}$ is a uniformly continuous function that satisfies $g(x) \geq c$ for all $x \in D$ where $c$ is a constant, $c > 0$. Prove that $1/g$ is uniformly continuous.

Y Make-up Final Exam (300 points)

1. (21 pts) Consider the function $f(x) := 36 - (x-2)^2 (= 32 + 4x - x^2)$ on the interval $[0, 8]$. Let $P$ be the partition $\{0, 2, 4, 6, 8\}$ and let $T$ be the marking $\{1, 3, 5, 7\}$. Compute the upper, lower, and Riemann sums. For your convenience, the table below gives you the function values at the integers from zero to eight.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>32</td>
<td>35</td>
<td>36</td>
<td>35</td>
<td>32</td>
<td>27</td>
<td>20</td>
<td>11</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) $U(P, f) =$
(b) $L(P, f) =$
(c) $S(P^T, f) =$
(d) $\int_0^8 f(x) \, dx.$
(e) Explain why the values from (a)-(d) are in the order they are.
2. (15 pts).
(a) Suppose that \( \lim_{n \to \infty} a_n = A \) and suppose that \( a_i \neq a_j \) if \( i \neq j \). Does this imply that \( A \) is an accumulation point of the range?
(b) If your answer in (a) was “yes” then prove it. If your answer was “no” then give a counterexample.

3. (20 pts) Answer True or False.
(a) If \( f : [a, b] \to [c, d] \) and \( g : [c, d] \to \mathbb{R} \) are continuous then \( g \circ f \) is continuous.
(b) If \( f : [a, b] \to [c, d] \) and \( g : [c, d] \to \mathbb{R} \) are Riemann integrable then \( g \circ f \) is Riemann integrable.
(c) If \( f : D \to E \) and \( g : E \to \mathbb{R} \) are uniformly continuous then \( g \circ f \) is uniformly continuous.
(d) If \( f : [a, b] \to [c, d] \) and \( g : [c, d] \to \mathbb{R} \) are differentiable then \( g \circ f \) is differentiable.
(e) If \( S \subset \mathbb{R} \) is uncountable then \( S \) has at least one accumulation point.
(f) If \( S \subset \mathbb{R} \) is not closed, then \( S \) is open.
(g) If \( S \subset \mathbb{R} \) is a bounded set then the closure \( \overline{S} \) is compact.
(h) If \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) are functions, \( c \in [a, b] \) such that \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} f(x)g(x) \) both exist, then \( \lim_{x \to c} g(x) \) exists.
(i) Let \( a < b < c \) and suppose \( f \) is uniformly continuous on \( [a, c] \setminus \{b\} \). Then \( \lim_{x \to b} f(x) \) exists.
(j) For every number \( a > 1 \) we have
\[
\frac{d}{dx} a^x = \ln(a) a^x.
\]

4. (18 pts) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(0) = 0, f(x) = x^4 \sin(1/x) \) for \( x > 0 \). Define
\[
F(x) := \int_0^{\sqrt{x}} f(t) \, dt, \quad x \geq 0.
\]
(a) Find \( F'(x) \) for \( x > 0 \).
(b) Find \( F'(0) \).
(c) Is \( F' \) continuous at 0?
(d) Find \( F''(x) \) for \( x > 0 \).
(e) Find \( F''(0) \).
(f) Is \( F'' \) continuous at 0?

5. (20 pts) Define a sequence \( \{a_n\}_{n=1}^{\infty} \) as follows:
\[
a_1 = 0, \quad a_{n+1} = \left[ \sqrt{6 + \sqrt{a_n}} \right]^3.
\]

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(a) Use mathematical induction to show that \( a_n \leq 1000 \) for all \( n \in \mathbb{N} \).

(b) Use mathematical induction to show that this is an increasing sequence.

(c) How do you know that this is a convergent sequence?

(d) Find the limit of the sequence.

6. (15 pts) Let \( E \) be a set and let \( E' \) be the set of all accumulation points of \( E \). Prove that \( E' \) is closed.

7. (20 pts) Let \( N \) be some fixed positive integer.

(a) Use induction to prove that:

\[ N^N! \geq N^n \quad \forall n \geq N. \]

Note that the “base case” now is \( n = N \).

(b) Prove that

\[ \lim_{n \to \infty} [n!]^{-1/n} = 0. \]

8. (15 pts) Suppose that \( f \) is a continuously differentiable function on \( \mathbb{R} \) and \( f(0) = 0 \). Find \( F'(0) \) where

\[ F(x) := \frac{1}{x} \int_0^x f(s) \, ds \text{ if } x \neq 0, \text{ and } F(0) = 0. \]

9. (15 pts) Let \( f : [0, \infty) \to \mathbb{R} \) and \( g : [0, \infty) \to \mathbb{R} \) be two continuous functions such that for all \( x \) we have \( |f(x)| \leq g(x) \). Moreover assume that there is a number \( M \) such that for all \( b > 0 \)

\[ \int_0^b g(x) \, dx \leq M. \]

(a) Define the sequence \( b_n := \int_0^n g(x) \, dx \). Explain why the sequence \( \{b_n\}_{n=1}^\infty \) must converge.

(b) Explain why the sequence \( \{b_n\}_{n=1}^\infty \) must be a Cauchy sequence.

(c) Define the sequence \( a_n := \int_0^n f(x) \, dx \). Prove that the sequence \( \{a_n\}_{n=1}^\infty \) must converge.

10. (20 pts) Let \( A \subset \mathbb{R} \), let \( A^o \) denote its interior, \( \overline{A} \) its closure. Then its boundary \( \partial A \) is defined as the set \( \partial A := \overline{A} \setminus A^o \). Let \( a \) be a point in the boundary: \( a \in \partial A \).

(a) Using the above definition of \( \partial A \) prove that every epsilon neighborhood of \( a \) contains a point of \( A \).

(b) Using the above definition of \( \partial A \) prove that every epsilon neighborhood of \( a \) contains a point of \( \mathbb{R} \setminus A \).

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11. (15 pts)
   (a) State the Bolzano Theorem
   (b) Show that the following equation has a real solution on the interval $[0, 1]$:
   
   $$-x^2 + \int_0^x e^{[e^t]} \, dt = 1.$$ 

12. (15 pts) Let the sequence $\{a_n\}$ be defined inductively by: $a_1 = 0, \ a_{n+1} = 2 - 1/(1 + a_n)$.
   (a) Use induction to show that $0 < a_n < 2$ for all $n > 1$.
   (b) Use induction to show this sequence is monotone.
   (c) Find the limit of this sequence.

13. (15 pts) Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence that converges to $A$. Consider the sequence $\{b_n\}_{n=1}^\infty$ of moving averages defined by $b_n := (a_n + a_{n+1} + a_{n+2})/3$. Using only the definition of convergence prove that this sequence also converges to $A$.

14. (15 pts) Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence and $\lim_{n \to \infty} a_n = L$, prove that $\lim_{n \to \infty} \max(a_n, 0) = \max(L, 0)$. Make your arguments precise and clear.

15. (15 pts) Recall that $\sinh(x) := (e^x - e^{-x})/2$ and $\cosh(x) := (e^x + e^{-x})/2$.
   (a) Show $\cosh(x)^2 = 1 + \sinh(x)^2$.
   (b) Show that
   
   $$\frac{d}{dx} \sinh(x) = \cosh(x).$$

   (c) Let $f(x) := \sinh(x)$ and let $g$ be the inverse function of $f$. Using the inverse function theorem (i.e. without computing $g$) obtain $g'(y)$ explicitly, i.e. a formula in $y$.

16. (15 pts) Suppose that $K$ is a positive constant and that $f : [0, \infty) \to (0, \infty)$ is differentiable and $0 \leq f'(x) < K f(x)$ $\forall x \in (0, \infty)$. Let $g(x) := \ln(f(x))$.
   (a) Prove that $g$ is uniformly continuous.
   (b) Prove that $f(x) < f(0) \exp(Kx) \quad \forall x \in [0, \infty)$.

17. (15 pts) Let $S \subset \mathbb{R}$, then its characteristic function $\chi_S$ is defined as

   $$\chi_S(x) = \begin{cases} 
   1 & \text{if } x \in S \\
   0 & \text{if } x \notin S 
   \end{cases}$$

   (a) Prove that $\chi_S$ is continuous on the interior, $S^o$.
   (b) Prove that $\chi_S$ is continuous on the exterior, $\mathbb{R} \setminus \overline{S}$.
   (c) Prove that $\chi_S$ is discontinuous on the boundary, $\overline{S} \setminus S^o$.
18. (16 pts) Let \( f, g : D \to \mathbb{R} \) be uniformly continuous and suppose that \( h : E \to D \) is uniformly continuous. Prove two of the following:

(a) \( f + g : D \to \mathbb{R} \) is uniformly continuous.
(b) \( f \circ h : E \to \mathbb{R} \) is uniformly continuous.
(c) \( f \cdot g : D \to \mathbb{R} \) is uniformly continuous provided both \( f \) and \( g \) are bounded functions.

Spring 2016

Z1 Exam 1

1. (15 pts) Definitions.

(a) Give the precise definition of
\[
\lim_{n \to \infty} a_n = A.
\]

(b) Give one of the equivalent definitions of an accumulation point.

(c) Give the precise definition of a subsequence.

2. (10 pts) State the following:

(a) The Bolzano-Weierstrass Theorem.

(b) The Completeness Axiom for the real numbers.

3. (10 pts) Use the Cantor-Bernstein Theorem to prove that \( \mathbb{N} \times \mathbb{N} \) is countable.

4. (10 pts) Suppose that \( \{a_n\}_{n=1}^\infty \) is an increasing sequence, \( \lim_{n \to \infty} a_n = A \) and \( a_n < A \forall n \).

Prove that \( A \) is an accumulation point of the range.

5. (15 pts) Use the definition of convergence directly (i.e., an \( \epsilon - N \) proof) to show that
\[
\lim_{n \to \infty} \frac{2n^2 - 30n + 1}{n^2 - 6n + 1} = 2.
\]

6. (6 pts) In each of the following two statements supply the missing hypotheses:

(a) If \( \{a_n\}_{n=1}^\infty \) is a sequence of real numbers that converges to \( A \) and \( \{b_n\}_{n=1}^\infty \) is a sequence of real numbers that converges to \( B \) then \( \lim_{n \to \infty} a_n/b_n = A/B \) provided ______.

If \( \{a_n\}_{n=1}^\infty \) is a sequence of real numbers that converges to \( A \) and \( p, q \in \mathbb{N} \) then \( \lim_{n \to \infty} a_n^{p/q} = A^{p/q} \) provided ______.

7. (10 pts) Using only the definition of convergence, prove that if the sequence \( \{a_n\}_{n=1}^\infty \) that converges to \( A \), then \( \{|a_n|\}_{n=1}^\infty \) converges to \( |A| \).

8. (15 pts) Consider the sequence defined inductively by \( a_1 = 0, a_{n+1} = [16 + a_n]^{2/3} \).

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(a) Prove that \( a_n \leq 11 \) \( \forall n \in \mathbb{N} \).

(b) Prove that \( a_{n+1} \geq a_n \) \( \forall n \in \mathbb{N} \).

(c) Explain why the sequence \( \{a_n\}_{n=1}^{\infty} \) must converge.

(d) Let \( A := \lim_{n \to \infty} a_n \). Find the polynomial equation that needs to be solved to find \( A \).

9. (10 pts) Let

\[
[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset ... 
\]

be a nested family of intervals and define

\[ a = \sup \{a_1, a_2, \ldots\}, \quad b = \inf \{b_1, b_2, \ldots\}. \]

Prove that

\[
\bigcap_{n=1}^{\infty} [a_n, b_n] = [a, b].
\]

Z2 Exam 2

1. (20 pts) Carefully define the following:

(a) : The limit of \( f : D \to \mathbb{R} \) at \( x_0 \) is \( L \).

(b) \( f : E \to \mathbb{R} \) is continuous at \( x_0 \).

(c) A sequentially compact set.

(d) An open set.

2. (10 pts) State two theorems that deal with continuous functions on compact domains in which the domain does not need to be an interval.

3. (15 pts) Using only the definition of convergence prove

\[
\lim_{x \to 2} x^3 = 8.
\]

4. (25 pts) For each of the following statements say whether it is true or false. If it is true give the reason. If the statement is false, construct a counter-example.

(a) Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded uniformly continuous function and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function. Then \( f \circ g : \mathbb{R} \to \mathbb{R} \) is uniformly continuous.

(b) Let \( g : K \to \mathbb{R} \) be a continuous function where \( K \) is sequentially compact. Suppose that \( f \) has a positive maximum and a negative minimum. There is a point \( x_0 \in K \) where \( f(x_0) = 0 \).

(c) If \( h : (0, 1) \to (0, \infty) \) is a continuous function, then the function \( 1/h : (0, 1) \to \mathbb{R} \) is also continuous.
(d) If \( f : (0, 1) \to \mathbb{R} \) is uniformly continuous, then \( \lim_{n \to \infty} f(1/n) \) exists.

(e) If \( f : (a, b) \to \mathbb{R} \) are has the property that whenever \( \{x_n\}_{n=1}^{\infty} \) is a sequence in \( (a, b) \) that converges to \( a \) then \( \{f(x_n)\}_{n=1}^{\infty} \) is convergent. Then \( \lim_{x \to a} f(x) \) exists.

5. (15 pts) Prove that there is a point \( x_0 > 0 \) where \( 2 \sin(x_0) = x_0 \).

6. (15 pts)

(a) It is known that
\[
\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.
\]
Prove there exists a number \( \delta > 0 \) such that \( \sin(\theta)/\theta > 1/2 \) whenever \( 0 < \theta < \delta \).

(b) Prove there exists a positive integer \( N \) such that \( 2n\pi \sin(2n\pi + 1/n) > \pi \) whenever \( n \geq N \).

(c) Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x \sin(x) \). Prove that \( f \) is not uniformly continuous.

Z3 Final Exam

1. (30 pts) Look at the function \( f : [0, \infty) \to \mathbb{R} \) that is defined to be \( f(x) = x \cos(\ln(x)) \), \( x > 0 \) and \( f(0) = 0 \).

(a) Is \( f \) continuous? Explain briefly.

(b) Is it bounded? Explain.

(c) Is it differentiable on \((0, \infty)\)? Explain.

(d) Is it differentiable at 0? Explain.

(e) Is it uniformly continuous on \([0, \infty)\)? Explain.

(f) Is it integrable on \([0, 1]\)? Explain.

2. (25 pts) Definitions:

(a) Give the precise definition of convergence:
\[
\lim_{x \to a} f(x) = L.
\]

(b) A set \( K \) of real numbers is said to be compact if ....

(c) A point \( x_0 \) is said to be an accumulation point of the set \( S \) of real numbers if ....

(d) The function \( f : D \to \mathbb{R} \) is said to be continuous at \( x_0 \) if ....
(e) Give the complete definition of a Riemann sum.

3. (25 pts) State the following Theorems:
   
   (a) The Bolzano-Weierstrass Theorem.
   
   (b) The Riemann Lemma.
   
   (c) The Inverse Function Theorem.
   
   (d) The Mean Value Theorem for derivatives.
   
   (e) The Intermediate Value Theorem.

4. (15 pts) Use the definition of convergence directly (i.e. an \( \epsilon - N \) proof) to show that

\[
\lim_{n \to \infty} \frac{2n^2 + 4n}{n^2 - 3n + 1} = 2.
\]

5. (15 pts) Show that the following equation has a real solution on the interval \([e, 5]\):

\[
\int_e^x [\ln(t)]^2 \, dt = 2.
\]

Be sure to say what theorem(s) you are using.

6. (20 pts) Evaluate each of the following:

(a)

\[
\frac{d}{dx} \int_{\cos(x)}^{\ln(x)} \sin(1 + t^3) \, dt
\]

(b)

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{n+j}.
\]

Hint: \(\frac{1}{n+j} = \frac{1}{1+j/n} \cdot \frac{1}{n}\)

(c)

\[
\lim_{n \to \infty} n^{1/n}
\]

(d)

\[
\lim_{x \to 0} \frac{\sin(x) - x}{x^3}.
\]

7. (15 pts)

(a) Give a definition for the derivative of a function \( f \) at a point \( x_0 \).

(b) Prove the following: Suppose \( f : D \to \mathbb{R} \) is differentiable at \( x_0 \in D^o \) and that \( f'(x_0) \) exists and is positive, then there exists a number \( x_1 > x_0 \) such that \( f(x) > f(x_0) \) for all \( x \in (x_0, x_1) \).
8. (10 pts) On $[0, 1]$ define the function

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - s^3}} ds.$$ 

Let $g$ denote its inverse function: $g := f^{-1}$. Show that

$$g'(u)^3 + g(u)^3 = 1.$$ 

9. (15 pts) Let $\{a_n\}_{n=1}^\infty$ be a sequence and define

$$s_n := \sum_{i=1}^n a_i, \quad t_n := \sum_{i=1}^n |a_i|.$$ 

Prove that if $\{t_n\}_{n=1}^\infty$ is Cauchy, then $\{s_n\}_{n=1}^\infty$ is also Cauchy.

10. (10 pts) Determine if each of the following is true or false:

   (a) If $f, g : [a, b] \to \mathbb{R}$ are increasing functions then $f - g$ is Riemann integrable on $[a, b]$.

   (b) If $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ are Riemann integrable then $g \circ f : [a, b] \to \mathbb{R}$ is Riemann integrable.

   (c) Let $D \subset \mathbb{R}$. If $f : D \to \mathbb{R}$ is uniformly continuous and $f(x) > 0$ for all $x \in D$ then $1/f$ is also uniformly continuous on $D$.

   (d) If $f : [a, b] \to [c, d]$ is continuous and bijective, then it is monotone.

   (e) If $\{a_n\}_{n=1}^\infty$ is a bounded sequence, then it has a convergent monotone subsequence.

11. (10 pts) Let $D, E \subset \mathbb{R}$ and let $f : D \to E$ and $g : E \to \mathbb{R}$ be two uniformly continuous functions. prove that $g \circ f : D \to \mathbb{R}$ is uniformly continuous.

12. (10 pts) Suppose $f \in C^2[a, b]$, and suppose that $f(a) = f(b) = 0$ while $f(x) > 0$ for all $x \in (a, b)$.

   (a) Prove that there must exist a point $\gamma \in (a, b)$ where $f''(\gamma) < 0$. Mention the names of the theorem(s) you use.

   (b) Prove that there exists an interval $(\alpha, \beta) \subset [a, b]$ on which $f''(x) < 0$. 

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