Chapter 3
CONTINUITY

The first theorem of this chapter, Theorem 3.1, relates continuity, limits and sequences. You will note that we use these three equivalent conditions in providing variations on proofs such as that of Theorem 3.2. You may want to provide other proofs for some of these theorems and allow your students to read those in the text.

The notion of a compact set is presented as a natural concept, useful in considering uniform continuity. If your students go on to study more in the areas of topology or analysis, they will see further uses of compactness. Don't slight this topic. However, Project 3.1 gives the option of proving Theorem 3.3 without the need for compactness.


Suggestions for Solutions to Exercises

1. Apply Theorem 3.1 and the remarks about polynomials from page 75.

2. For \( x \neq -3 \), \( f(x) = 2x - 6 \), hence \( f \) has a limit at \(-3\) and that limit is \(-12\), \( f(-3) \). Apply 3.1.

3. Since \( f(x) = e^x \) is continuous on \( \mathbb{R} \), \( \{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0) \).

   if \( \{x_n\}_{n=1}^{\infty} \rightarrow x_0 \), \( \frac{n+1}{n} e^{n+1} = e^n = f \left( \frac{n+1}{n} \right) \) and

   \( \prod_{n=1}^{\infty} \frac{n+1}{n} \rightarrow 1 \), hence \( \prod_{n=1}^{\infty} \left( \frac{n+1}{n} \right)^n - e^1 = e \).

4. If \( x_0 \) is not an accumulation point of \( E \) and \( \{x_n\}_{n=1}^{\infty} \rightarrow x_0 \), \( x_n \in E \) for all \( n \), then there is \( N \) such that \( n \geq N \) implies that \( x_n = x_0 \). Hence \( f(x_n) = f(x_0) \) for \( n \geq N \) and so \( \{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0) \).

5. For \( x \neq 0 \), \( f(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x + 1}} = \frac{1 - \sqrt{x + 1}}{\sqrt{x} (\sqrt{x + 1})} = \frac{-x}{\sqrt{x} (1 + \sqrt{x + 1})} \). So \( f \) has a limit at 0, it is

\[ 32 \]
0. Define $f(0) = 0$.

6. Use 3.1 and Exercise 28 of Chapter 1.

7. There is a sequence $\{r_n\}_{n=1}^{\infty}$ of rational number converging to $\sqrt{2}$. By the continuity of $f$, $\{f(r_n)\}_{n=1}^{\infty}$ converges to $f(\sqrt{2})$. But $f(r_n) = r_n^2$ for each $n$ and $\{r_n^2\}_{n=1}^{\infty}$ converges to $(\sqrt{2})^2 = 2$. So $f(\sqrt{2}) = 2$.

8. If $x \in (a,b)$ is irrational, there is a sequence $\{r_n\}_{n=1}^{\infty}$ of rational numbers in $(a, b)$ converging to $x$. But then the continuity of $f$ at $x$ guarantees that $\{f(r_n)\}_{n=1}^{\infty}$ converges to $f(x)$. Noting that $f(r_n) = 0$ for all $n$, we see that $f(x) = 0$.

9. Use Theorem 2.6 and Theorem 3.1.

10. Choose $\epsilon > 0$. There is $\delta > 0$ such that $|x - x_0| < \delta$ and $x \in E$ imply that $|f(x) - f(x_0)| < \epsilon$. Then if $|x - x_0| < \delta$ and $x \in F \subset E$, $|g(x) - g(x_0)| = |f(x) - f(x_0)| < \epsilon$.

Define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(x) = x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = 6$ for $\frac{1}{2} < x \leq 1$. Then $f$ is not continuous at $\frac{1}{2}$ but $g: [0, \frac{1}{2}] \rightarrow \mathbb{R}$ defined by $g(x) = f(x)$ for all $x \in [0, \frac{1}{2}]$ is continuous at $\frac{1}{2}$.

11. Choose $\epsilon > 0$. Let $\delta = \min\{1, \frac{\epsilon}{10}\}$. Then if $x \in \mathbb{R}$, $x$ is rational, and $|x - 2| < \delta$, then $|f(x) - 16| = |8x - 16| = 8|x - 2| < 8 \cdot \frac{\epsilon}{10} < \epsilon$; if $x$ is irrational and $|x - 2| < \delta$, then $|f(x) - 16| = |2x^2 + 8 - 16| = 2|x^2 - 4| = 2|x + 2| \cdot |x - 2| < 2(5)|x - 2| < 10 \cdot \frac{\epsilon}{10}$. If $\{r_n\}_{n=1}^{\infty}$ is a sequence of rational numbers converging to 1, $\{f(r_n)\}_{n=1}^{\infty} \rightarrow 8$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of irrational numbers converging to 1, then $\{f(a_n)\}_{n=1}^{\infty} \rightarrow 10$. So $f$ is not continuous at 1. In fact, 2 is the only point where $f$ is continuous.
12. Since $x_0$ is a zero of $q$ of multiplicity $m$, there is a polynomial $r$ such that $q(x) = r(x)(x - x_0)^m$ and $r(x_0) \neq 0$. If $x_0$ is not a zero of $p$, let $n = 0$. If $x_0$ is a zero of $p$, let $n$ denote the multiplicity of that zero. Then there is a polynomial $s$ such that $p(x) = s(x)(x - x_0)^n$ and $s(x_0) \neq 0$.

Thus, for all $x \neq x_0$, $p(x) = \frac{s(x)(x - x_0)^n}{r(x)(x - x_0)^m} = \frac{s(x)(x - x_0)^{n-m}}{r(x)}$. The function $\frac{s}{r}$ has a non-zero limit at $x_0$ in fact, it is $\frac{s(x_0)}{r(x_0)}$. Thus $\frac{p}{q}$ has a limit at $x_0$ iff $n - m \geq 0$, i.e., the multiplicity of $x_0$ as a zero of $p$ is greater than or equal to the multiplicity of $x_0$ as a zero of $q$.

13. If $f$ is continuous at $x_0$, then either $f$ has a limit at $x_0$ or $x_0$ is not an accumulation point of $D$. If $f$ has a limit at $x_0$, then Theorem 2.3 yields the desired result. If $x_0$ is not an accumulation point of $D$, there is a neighborhood $Q$ of $x_0$ such that $Q \cap D = \{x_0\}$ in which case we set $M = |f(x_0)|$.

14. Use Theorem 3.1 and Exercise 12 of Chapter 2.

15. $h = \frac{1}{2} [ |f - g| + (f + g)]$.

16. $h(x) = e^x \ln x = f(xg(x))$. Use 3.2 on $x \cdot g$ and 3.4 on $f(xg)$.

17. The function $g(x) = \sqrt{x}$ is continuous for $x \geq 0$ and $\{f(x)\} = gof(x)$ is continuous by Theorem 3.4.

18. See the solution to Exercise 17, Chapter 2.

19. It is clear that the uniform continuity of $f$ and $g$ imply the uniform continuity of $f + g$. Just note that

$$\left| (f + g)(x) - (f + g)(y) \right| = \left| f(x) + g(x) - (f(y) + g(y)) \right| \leq \left| f(x) - f(y) \right| + \left| g(x) - g(y) \right|.$$
When we try the same game with \( fg \), life gets more complicated, to wit: 
\[
\left| (fg)(x) - (fg)(y) \right| = \left| f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y) \right| \\
\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|.
\]

The role of \( f(x) \) and \( g(y) \) in this inequality makes us suspect that all is not well. Your suspicions are correct. Let \( f(x) = x \). \( f \) is uniformly continuous but
\[
[f(x)]^2 = x^2 \text{ is not, as noted following Theorem 3.5.}
\]
However, if both \( f \) and \( g \) are bounded and uniformly continuous on \( D \), then \( f \cdot g \) is uniformly continuous.

20. Let's try the obvious thing, it usually works! Suppose \( f:A \to B \) and \( g:B \to C \) are uniformly continuous. Choose \( \varepsilon > 0 \). There is \( \delta_1 > 0 \) such that \( |u - v| < \delta_1 \) and \( u, \ v \in B \) imply that \( |g(u) - g(v)| < \varepsilon \). There is \( \delta_2 > 0 \) such that \( |x - y| < \delta_2 \) and \( x, \ y \in A \) imply that \( |f(x) - f(y)| < \delta_1 \). But then \( |x - y| < \delta_2 \) and \( x, \ y \in A \) imply that \( |f(x) - f(y)| < \delta_1 \), hence
\[
|(gof)(x) - (gof)(y)| = |g(f(x)) - g(f(y))| < \varepsilon.
\]
Nuff said?

21. First, notice that if \( 3.4 \leq t \), then \( 0.4 \leq t - 3 \), hence \( \frac{1}{t - 3} \leq 2.5 \). Choose \( \varepsilon > 0 \). Let \( \delta = \frac{\varepsilon}{15} \). If \( x, \ y \in [3.4, 5] \), \( |x - y| < \delta \), then
\[
\left| \frac{1}{x - 3} - \frac{1}{y - 3} \right| = \left| \frac{2(y - x)}{(x - 3)(y - 3)} \right| < (2.5)^2(2\delta) < \varepsilon.
\]

22. If \( x, y \in (2, 7) \), then \( x^2 + xy + y^2 - 1 \leq 146 \). Choose \( \varepsilon > 0 \). Let \( \delta = \frac{\varepsilon}{146} \). Then if \( x, y \in (2, 7) \) and \( |x - y| < \delta \), we have
\[
|f(x) - f(y)| = |(x - y)(x^2 + xy + y^2 - 1)| < 146\delta = \varepsilon.
\]

23. Suppose \( f: \mathbb{R} \to \mathbb{R} \) is continuous and periodic of period \( h \). The set \([-h, h]\) is compact, hence \( f \) is uniformly continuous on that set, i.e., given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( |x - y| < \delta \) and \( x, y \in [-h, h] \) imply that \( |f(x) - f(y)| < \varepsilon \). We may further impose the condition that \( \delta < \frac{h}{2} \). Suppose \( |x - y| < \delta \). There is an integer \( n \) such that \( x + nh \in [-\frac{h}{2}, \frac{h}{2}] \). Then \( y + nh \in [-h, h] \) since
\[
|x - y| < \delta < \frac{h}{2}.
\]
But then
\[
|f(x) - f(y)| = |f(x + nh) - f(y + nh)| < \varepsilon. \text{ Thus, } f \text{ is uniformly continuous.}
\]

24. Since \( A \) is bounded and not compact, \( A \) is not closed. Let a
be an accumulation point of $A$ which does not belong to $A$. Then define $f: A \to \mathbb{R}$ by $f(x) = \frac{1}{x - a}$ for all $x \in A$. The function $f$ is not uniformly continuous on $A$ since it doesn't have a limit at $a$. See Theorem 3.5. The set $A$ is not compact, but any function defined on $A$ is uniformly continuous.

25. Let $A = \{x: x < 0\}$ and $B = \{x: x > 0\}$. Define $f: A \cup B \to \mathbb{R}$ by $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Then $f$ is uniformly continuous on $A$ and uniformly continuous on $B$, and continuous on $A \cup B$, but not uniformly continuous on $A \cup B$. To see that $f$ is not uniformly continuous on $A \cup B$, note that $0$ is an accumulation point of $A \cup B$ and $f$ does not have a limit at $0$.

26. Suppose $E$ is closed and $(x_n)_{n=1}^\omega$ is a sequence of points of $E$ converging to $x_0$. If there is a subsequence $(x_{n_k})_{k=1}^\omega$ of $(x_n)_{n=1}^\omega$ such that $x_{n_k} \neq x_0$ for all $k$, then by Theorem 1.17, $x_0$ is an accumulation point of $E$, hence $x_0 \in E$ since $E$ is closed. If no such subsequence exists, then $x_0 = x_{n_0}$ for some $n_0 > 0$ and since $x_{n_0} \in E$, $x_0 \in E$.

Suppose now that $E$ contains all limits of sequences of members of $E$. Let $x_0$ be an accumulation point of $E$. Then by Theorem 1.17, there is a sequence $(x_n)_{n=1}^\omega$ of members of $E$ which converges to $x_0$, hence $x_0 \in E$.

Thus, $E$ contains all its accumulation points, i.e., is closed.

27. By Exercise 3 of Chapter 1, every open interval is a neighborhood of each of its members, i.e., is open. Suppose $x_0$ is an accumulation point of $[a, b]$. If $a \leq x_0 \leq b$, then $x_0 \in [a, b]$. If $x_0 < a$, then let $\epsilon = \frac{a - x_0}{2}$. Then $(x_0 - \epsilon, x_0 + \epsilon) \cap [a, b] = \emptyset$ contrary to $x_0$ being an accumulation point of $[a, b]$. Similarly, if $b < x_0$, let $\epsilon = \frac{x_0 - b}{2}$ and then $(x_0 - \epsilon, x_0 + \epsilon) \cap [a, b] = \emptyset$ leading to the same contradiction. Thus $[a, b]$ contains all of its accumulation points.

This is an example of a perfect set, a set $S$ which is closed and such that every $x_0 \in S$ is also an

28. If $F$ is a closed set which contains $D$, then $F$ must contain all the accumulation points of $D$ (since they are also accumulation points of $F$), hence $D = D \cup D' \subseteq F$. It only remains to show that $D$ is closed.

Let $x_0$ be an accumulation point of $D$. Let $Q$ be any neighborhood of $x_0$. Then $Q$ contains infinitely many points of $D$. If one of those points comes from $D'$, i.e., is an accumulation point of $D$, then there is a neighborhood $P$ of that point such that $P \subseteq Q$ and $P$ must contain infinitely many points of $D$. Thus, in any case, $Q$ contains infinitely many points of $D$. We may conclude $x_0 \in D' \subseteq D$ and $D$ is closed.

It should be noted that there is an alternative method of defining the closure of a set. First, one proves that the intersection of any family of closed sets is again a closed set. Then define the closure of a set $D$ to be the intersection of all closed sets which contain $D$. See John L. Kelly, "General Topology", New York, Van Nostrand, 1955, pp. 42-43 for more on this notion.

29. Suppose $D$ is bounded, i.e., there is $M > 0$ such that $|x| \leq M$ for all $x \in D$. But this means that $D \subseteq [-M,M]$ and $[-M,M]$ is a closed set, hence by Exercise 28, $D \subseteq [-M,M]$.

30. Let $A = \{x: f(x) \neq r_0\}$. If we can show $R \setminus A$ is closed then by Theorem 3.6, $A$ is open. $R \setminus A = \{x: f(x) = r_0\}$. If $x_0$ is an accumulation point of $R \setminus A$, then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of members of $R \setminus A$ converging to $x_0$. By the continuity of $f$ at $x_0$, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$, but $f(x_n) = r_0$ for all $n$, hence $f(x_0) = r_0$.

Thus, $x_0 \in R \setminus A$ and we have shown that $R \setminus A$ is closed.

A direct proof is also quite easy. If $x_0 \in A$,

$$
\epsilon = \frac{|f(x_0) - r_0|}{2}.
$$

There is $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$. But then it is easy to see that $f(x) \neq r_0$.

31. Let $x_0$ be an accumulation point of $T$. Since
$T \subseteq [a, b]$, $x_0$ is an accumulation point of $[a, b]$, hence $x_0 \in [a, b]$. This is important because we want to use the continuity of $f$ and $g$ at $x_0$, hence we must have $x_0 \in [a, b]$. There is a sequence $\{x_n\}_{n=1}^\infty$ of members of $T$ converging to $x_0$ and since $f - g$ is continuous at $x_0$, 
\[ \{(f - g)(x_n)\}_{n=1}^\infty \] converges to $(f - g)(x_0)$. But $x_n \in T$, hence $(f - g)(x_n) = 0$ for all $n$, hence $(f - g)(x_0) = 0$, i.e., $x_0 \in T$. Thus, $T$ is closed.

32. Let $A$ be any open set contained in $D$. Then each point of $A$ is an interior point of $A$, hence also an interior point of $B$. Hence $A \subseteq D^\circ$. It remains to show that $D^\circ$ is open.

Let $x_0 \in D^\circ$. Then there is an open interval 
\[ (x_0 - \varepsilon, x_0 + \varepsilon) \subset D. \] But the set $(x_0 - \varepsilon, x_0 + \varepsilon)$ is open (see Exercise 27, this chapter), hence 
\[ (x_0 - \varepsilon, x_0 + \varepsilon) \subset D^\circ. \] Thus, $D^\circ$ is open.

Along the same lines as in Exercise 28, one can define $D^\circ$ to be the union of all open sets which are contained in $D$.

33. $\{(\frac{1}{n}, n)\}_{n \in \mathbb{J}}$

34. $\{(1 + \frac{1}{n}, 2 - \frac{1}{n})\}_{n \in \mathbb{J}}$

35. If $E$ is compact, then $E$ is bounded. Thus, if $E$ is nonempty, both $\sup E$ and $\inf E$ are defined. By Exercise 22 of Chapter 1, $\sup E$ and $\inf E$ are either members of $E$ or accumulation points of $E$. Since $E$ is compact, it is closed, and hence $\sup E$ and $\inf E$ belong to $E$.

36. Suppose $E_1, \ldots, E_n$ are compact and let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $\bigcup_{i=1}^n E_i$. There are finite sets $A_1, \ldots, A_n$ contained in $A$ such that $E_i \subseteq \bigcup_{\alpha \in A_i} G_\alpha$ since each $E_i$ is compact. But $A_0 = \bigcup_{i=1}^n A_i$ is also finite and
\[ \bigcup_{i=1}^n E_i \subseteq \bigcup_{\alpha \in A_0} G_\alpha. \] Thus, $\bigcup_{i=1}^n E_i$ is compact.
37. For each $x \in [a,b]$, there are a number $M_x > 0$ and an open set $Q_x$ containing $x$ such that $y \in Q_x \cap [a,b]$ implies that $|f(y)| \leq M_x$. (By Theorem 2.3.) The set $[a,b]$ is compact and the collection $\{Q_x\}_{x \in [a,b]}$ is an open cover of $[a,b]$, hence there is a finite set $F \subseteq [a,b]$ such that $[a,b] \subseteq \bigcup_{x \in F} Q_x$. Let $M = \max \{M_x : x \in F\}$. Thus, if $y \in [a,b]$, then for some $x \in F$, $y \in Q_x$. Then $|f(y)| \leq M_x \leq M$.

38. Let $\{x : 0 \leq f(x) \leq 1\} = S$. $S$ is a subset of $D$, hence bounded. If $\{x_n\}_{n=1}^\infty \to x_0$ with $x_n \in S$ for all $n$, then $\{f(x_n)\}_{n=1}^\infty \to f(x_0)$. Since $0 \leq f(x_n) \leq 1$ for all $n$, $0 \leq f(x_0) \leq 1$, i.e., $x_0 \in S$. By Exercise 25, $S$ is closed.

39. Choose $\epsilon > 0$. There is $M > 0$ such that if $|x| > M$, then $|f(x)| < \frac{\epsilon}{2}$. $f$ is uniformly continuous on $[-2M, 2M]$, hence there is $\delta > 0$, $\delta < M$, such that if $x, y \in [-2M, 2M]$, $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Now consider $|x - y| < \delta$, where either $x$ or $y$ fails to be in $[-2M, 2M]$. Then $|x| \geq M$ and $|y| \geq M$, so $|f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

40. $f(x) = \sin x^2$ is such a function. Note that $\{(x + 2n\pi - \sqrt{y + 2n\pi})\}_{n=1}^\infty$ converges to 0 for any $x$ and $y$. Choose $0 < \alpha < 1$. There are $x, y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $|\sin x - \sin y| = \alpha$. Now choose any $\epsilon$ such that $0 < \epsilon < \alpha$ and any $\delta > 0$. There is large enough $n$ such that $|\sqrt{x + 2n\pi} - \sqrt{y + 2n\pi}| < \delta$ and $|\sin (\sqrt{x + 2n\pi})^2 - \sin (\sqrt{y + 2n\pi})^2| = \alpha > \epsilon$.

41. Define $f(x) = xe^x - 1$. $f(0) = -1$, $f(1) = e - 1 > 0$.

42. Define $g(x) = x^3 - 6x^2 + 2.826$. $g(0) = 2.826 > 0$ $g(1) = -2.174 < 0$.

43. $-f:[a,b] \to \mathbb{R}$ is continuous and $(-f)(a) \leq -y \leq (-f)(b)$. Hence by Theorem 3.18, there is $c \in [a,b]$ such that $(-f)(c) = -y$, i.e., $f(c) = y$.

44. Define $g:[a,b] \to \mathbb{R}$ by $g(x) = f(x) - x$. Since $f:[a,b] \to [a,b], g(a) = f(a) - a \geq 0$ and
\[ g(b) = f(b) - b \leq 0. \] Thus by Theorem 3.13, there is \( x \in [a, b] \) such that \( g(x) = f(x) - x = 0. \)

45. The proof of Theorem 3.19 uses only the intermediate value property, hence can be mimicked to show that \( f \) is monotone. If \( f \) is monotone, then discontinuities can occur only when \( U(x_0) \neq L(x_0) \), see Section 2.4. Suppose \( f \) is increasing and \( U(x_0) > L(x_0) \). Then if \( L(x_0) < s < U(x_0) \), there are \( x < x_0 < y \) such that \( f(x) \leq L(x_0) \) \( < s \leq U(x_0) \) \( \leq f(y) \). By the intermediate value property, there must be \( x < z < y \) such that \( f(z) = s \). This is contrary to \( L(x_0) < s < U(x_0) \).

The endpoints may be handled similarly.

46. Suppose \( f: \mathbb{R} \to \mathbb{R} \) is continuous and the equation \( f(x) = 0 \) has exactly two solutions, \( x_1 \) and \( x_2 \) with \( x_1 < x_2 \). On the interval \( [x_1, x_2] \), \( f(t) \leq 0 \) for all \( t \), or \( f(t) \geq 0 \) for all \( t \), let us assume the latter. Then \( f(t) < 0 \) for \( t < x_1 \) and \( f(t) < 0 \) for \( t > x_2 \). \( f \) has a maximum value on \( [x_1, x_2] \) at exactly two points, \( a \) and \( b \), assume \( a < b \). Now choose any \( t \) such that \( a < t < b \). Then \( f(x_1) = 0 \leq f(t) < f(a) \) and \( f(x_2) = 0 < f(t) < f(b) \). So the equation \( f(x) = f(t) \) has three solutions, \( t, c_1 \) and \( c_2 \) where \( x_1 < c_1 < a \), and \( b < c_2 < x_2 \) by the intermediate value property.

47. Refer to Project 2.1 at the end of Chapter 2. The condition given guarantees that \( f \) has a limit at every point and the limit at \( x \) is \( f(x) \). Therefore, \( f \) is continuous. It remains to prove that \( f \) is uniformly continuous.

Suppose \( f \) is continuous at zero. Thus given \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( |x| < \delta \), then \( |f(x)| < \epsilon \). Then if \( |x - y| < \delta \), then \( |f(x) - f(y)| = |f(x - y)| < \epsilon \). Thus \( f \) is uniformly continuous.

Again, suppose that \( f \) is continuous at zero, hence continuous everywhere. Let \( a = f(1) \). For each integer \( n \), \( f(n) = na \), moreover for each rational number \( r \), \( f(r) = ra \). To see this holds for rational numbers, note that \( p(f(r/p)) = f(1) = a \), hence \( f(r/p) = (r/p)a \). If \( x \) is a real number, there is a sequence \( \{r_n\}_{n=1}^{\infty} \) of rational numbers converging to \( x \), hence by the continuity of \( f \), \( \{f(r_n)\}_{n=1}^{\infty} \) converges to \( f(x) \). But \( f(r_n) = ar_n \), hence \( \{f(r_n)\}_{n=1}^{\infty} \) converges to \( ax \) and \( f(x) = ax \).
48. Exercise 25 in Chapter 2 states that \( g \) is continuous at \( x_0 \) if \( f \) has a limit at \( x_0 \) and that limit is \( f(x_0) \), i.e., if \( f \) is continuous at \( x_0 \). But \( f \) was assumed continuous on \([a,b] \), hence so is \( g \).

49. Since \( g(x_0) \neq 0 \), there is a neighborhood \( Q \) of \( x_0 \) such that if \( t \in Q \cap D \), then \( g(t) \neq 0 \). Since \( x_0 \) is an accumulation point of \( D \), there is \( x \neq x_0, x \in D \cap Q \) and hence \( g(x) \neq 0 \), that is \( x \in D_0 \). Therefore, \( x_0 \) is an accumulation point of \( D_0 \).

50. If \( x_0 \) is not an accumulation point of \( E \), then it is not an accumulation point of \( D \) and vice versa. In this case, both \( f \) and \( g \) are continuous at \( x_0 \). If \( x_0 \) is an accumulation point of \( E \), then it is an accumulation point of \( D \) and vice versa, and we may invoke Exercise 27 of Chapter 2. Actually, a direct proof is very easy also.

PROJECT 3.1

1. Assume \( f \) is not uniformly continuous. Then for some \( \epsilon > 0 \), no \( \delta \) can be found that satisfies the definition of uniform continuity. Thus for each \( n \in J \), there are \( a_n \) and \( b_n \) in \( A \) such that \( |a_n - b_n| < \frac{1}{n} \) (\( \frac{1}{n} \) doesn't work for \( \delta \)) and \( |f(a_n) - f(b_n)| \geq \epsilon \).

2. The sequence \( \{a_n\}_{n=1}^{\infty} \) is bounded, hence it has a convergent subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \). The sequence \( \{b_{n_k}\}_{k=1}^{\infty} \) has a convergent subsequence and the corresponding subsequence of \( \{a_{n_k}\}_{k=1}^{\infty} \) also converges. To avoid subscripts on subscripts, assume \( \{a_{n_k}\}_{k=1}^{\infty} \) and \( \{b_{n_k}\}_{k=1}^{\infty} \) both converge, call the limits \( a \) and \( b \).

3. Since \( A \) is closed, both \( a \) and \( b \) belong to \( A \). Also
\[
|a_{n_k} - b_{n_k}| < \frac{1}{n_k} \text{ for each } k \in J, \text{ hence } a = b.
\]
On the other hand, \( |f(a_{n_k}) - f(b_{n_k})| \geq \epsilon > 0 \), hence by the continuity of \( f \), \( 0 = |f(a) - f(b)| \geq \epsilon > 0 \). This is a contradiction.

Proof of Corollary 3.11: