Chapter 0
PRELIMINARIES

This chapter contains the basic material on sets, functions, relations and induction, as well as the system of real numbers. Your students may have seen a good share of this material before. You will need to make some choices about what material from this chapter you cover in detail. My experience has been that most students at this level have little appreciation of the structure of the system of real numbers as a complete ordered field. For that reason, I have found that a comprehensive coverage of Section 0.5 is very important.

Suggestions for Solutions to Exercises

1. a) \{1,2,3,4,5\}  
   b) \{-5,-4,-3,-2,-1,0,1,2\}  
   c) \{1,2,3,4,5\}  
   d) \{2,3,4\}

2. a) \(\frac{1}{2},1\)  
   b) [-1,7]

3. The proof is very similar to that of (v) of Theorem 0.2.

4. The proof is very similar to that of (i) of Theorem 0.3.

5. If \(x \in A \cap B\), then \(x \in A\) and \(x \in B\), in particular, \(x \in A\). Hence, \(A \cap B \subseteq A\). If \(x \in A\), then \(x \in A \cup B\), hence \(A \subseteq A \cup B\).

6. If \(x \in C \setminus B\), then \(x \in C\) and \(x \notin B\). Since \(A \subseteq B\) and \(x \notin B\), then \(x \notin A\). Therefore, \(x \notin C \setminus A\). Thus, \(C \setminus B \subseteq C \setminus A\). The converse is false as evidenced by the example \(C = \mathbb{J}\), \(A = \{-1,2,3\}\) and \(B = \{-3,2,3\}\). Here \(C \setminus A = C \setminus B\) but \(A \notin B\) and \(B \notin A\).

7. \(A \setminus (A \setminus B) = B\) if and only if \(B \subseteq A\). To show this, you may want to prove that \(A \setminus (A \setminus B) = A \cap B\).

8. Let \(x \in (A \setminus B) \cup (B \setminus A)\). Then \(x \in A \setminus B\) or \(x \in B \setminus A\). If \(x \in A \setminus B\), then \(x \in A\) and \(x \notin B\), hence, \(x \in A \cup B\) and \(x \notin A \cap B\). If \(x \in B \setminus A\), a similar argument shows that \(x \in A \cup B\) and \(x \notin A \cap B\). In either case, \(x \in (A \cup B) \setminus (A \cap B)\). Now assume \(x \in (A \cup B) \setminus (A \cap B)\). Then \(x \in A \cup B\) and \(x \notin A \cap B\). If \(x \in A\), then \(x \notin B\) since \(x \notin A \cap B\). But \(x \notin A\) or \(x \notin B\), hence, \(x \in A \setminus B\) or \(x \in B \setminus A\). Thus, \(x \in (A \setminus B) \cup (B \setminus A)\). We have shown that \((A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)\) and \((A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)\), hence \((A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)\).

9. The point of Russell's paradox is to show the student that using a rule to define a set can lead to logical difficulties. However, reassure your students that no such
Problems will arise in this book.

10. a) \{0\}   b) \mathbb{R}   c) [0,1]   d) (-1,3)

11. This proof is very similar to that of (i) of Theorem 0.4. One distinction is that if \( x \in S \setminus \bigcap_{\lambda \in A} A_\lambda \), then \( x \in S \setminus \bigcap_{\lambda \in A} A_\lambda \) and \( x \notin \bigcap_{\lambda \in A} A_\lambda \), hence \( x \notin A_\mu \) for some \( \mu \in A \). Therefore, \( x \notin S \setminus A_\mu \), etc., etc.

12. a) \( \mathbb{R} \setminus \{0\} \)   b) \( \mathbb{R} \setminus [1,2] \)

13. \( \text{im } f = \{ m \in J : m \text{ is odd} \} \). \( f \) is 1-1, but not onto \( J \). \( f \) has an inverse defined by

\[
\text{f}^{-1}(m) = \frac{m + 1}{2}
\]

for each \( m \in \text{im } f \).

14. \( \text{dom } f = \{ x : x \neq -2 \} \). \( f \) is 1-1 and as in example 0.8, the best way to find \( \text{im } f \) is to find the domain of the inverse. We find \( f^{-1}(x) = \frac{2x}{1 - x} \) and

\[
\text{dom } f^{-1} = \{ x : x \neq 1 \} = \text{im } f.
\]

15. \( f(x) = 7 \) for all \( x \in A \).

16. \( f(x) = x + 1 \) for each \( x \in A \). \( \text{im } f = \{ 2, 3, 4, 5, 6 \} = D \).

\( f^{-1} : D \to A \) is defined by \( f^{-1}(x) = x - 1 \) for each \( x \in D \).

17. \( f : A \to B \) defined by \( f(x) = x + 1 \) for each \( x \in A \); \( g : B \to C \) defined by \( g(2) = a, g(3) = b, g(4) = c, g(5) = d, g(6) = e, g(7) = e \).

18. Suppose \( a \in A \). There is \( b \in B \) such that \( (a, b) \in f \), i.e., \( f(a) = b \). Therefore, \( (b, a) \in f^{-1} \), hence

\[
f^{-1}(b) = a.
\]

Then \( (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a \).

The proof for \( (f \circ f^{-1})(b) = b \) is similar.

Proofs by induction consist of verifying a statement for one or more initial values and then showing that the truth of the statement for \( k \) or for all \( m \leq k \) implies the truth of the statement for \( k + 1 \). This last process is often called the induction step. For the exercises that involve induction, we will offer only the induction step.

19. Assuming \( 1 + 2 + \ldots + n = \frac{n(n + 1)}{2} \), we have
20. Assuming $1 + 3 + \ldots + (2n - 1) = n^2$, we have
\[
1 + 3 + \ldots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2
\]

21. Assuming $n^3 + 5n$ is divisible by 6, we have
\[
(n + 1)^3 + 5(n + 1) = n^3 + 3n^2 + 3n + 1 + 5n + 5
= (n^3 + 5n) + 3(n)(n + 1) + 6
\]
$n^3 + 5n$ is divisible by 6, as is 6, and since either $n$ or $n + 1$ is even, $3n(n + 1)$ is divisible by 6.

22. We will use the result of Example 0.11 here, that is, $2n + 1 < 2^n$ for all $n \geq 3$. Notice that $n^2 < 2^n$ is false for $n = 2, 3$ and 4. Now assume that $n \geq 5$ and $n^2 < 2^n$.
Then
\[
(n + 1)^2 = n^2 + 2n + 1 < 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1}
\]

23. Assume the hypotheses of 0.9 and assume that $P(n)$ is false for some $n \in \mathbb{J}$. There is a smallest $n_0 \in \mathbb{J}$ such that $P(n_0)$ is false, by the well-ordering principle, and by the hypothesis, $n_0 > m$. But then $P(n_0 - 1)$ is true and $P(k)$ is true for $1 \leq k \leq n_0 - 1$, hence $P(n_0)$ is true, a contradiction.

24. Assume $f(k) < 2^k$ for all $k \leq n$. Then $f(n + 1)$
\[
= f(n) + f(n - 1) + f(n - 2) < 2^n + 2^{n-1} + 2^{n-2}
= 2^n + 2^{n-1} + 2^{n-1} = 2^n + 2^n = 2^{n+1}
\]

25. Assume $f(k) < 2.4$ for some $k \geq 2$. Then
\[
f(k + 1) = \sqrt[3]{f(k)} < \sqrt[3]{2.4} = \sqrt[3]{5.4} < 2.4.
\]

26. Assume $f(k) = -5 \cdot 3^k + 5^{k-1} + 2^{k+3}$ for $1 \leq k \leq n$.
Then
\[
f(n + 1) = 8 \cdot f(n) - 15 \cdot f(n - 1) + 6 \cdot 2^{n+1}
= 8[-5 \cdot 3^n + 5^{n-1} + 2^{n+3}]
27. Assume the hypotheses of 0.10 and assume \( P(n) \) is false for some \( n > n_0 \). Let \( S = \{ n \in \mathbb{Z} : n \geq n_0 \} \) and \( T = \{ n \in S : P(n) \) is false\}. \( T \) is a non-empty subset of \( S \), hence by the modified well-ordering principle, \( T \) has a smallest member, call it \( k_0 \). Then \( P(k_0) \) is false, \( k_0 > n_0 \), so \( k_0 - 1 \geq n_0 \) and \( P(k_0 - 1) \) is true. But then \( P(k_0) \) is true by the hypothesis, a contradiction.

28. Assume the hypotheses of the theorem and assume \( P(n) \) is false for some \( n > n_0 \). As in Exercise 27, define \( S \) and \( T \). Then \( T \) is a non-empty subset of \( S \) and hence has a smallest member, call it \( k_0 \). Then \( P(k_0) \) is false, \( k_0 - 1 \geq n_0 \) and for \( n_0 \leq k \leq k_0 - 1 \), \( P(k) \) is true, hence \( P(k_0) \) is true by (b) of the theorem. This is a contradiction.

29. Assume \( f(k) = 5 \cdot 2^k + 2(-3)^k \) for \( 0 \leq k \leq n \). Then
\[
\begin{align*}
f(n + 1) &= 6 \cdot f(n - 1) - f(n) \\
&= 6[5 \cdot 2^{n-1} + 2(-3)^{n-1}] - [5 \cdot 2^n + 2(-3)^n] \\
&= 2^{n-1}[30 - 10] + (-3)^{n-1} [6 + 12] \\
&= 5 \cdot 2^{n+1} + 2(-3)^{n+1}
\end{align*}
\]
Just a reminder: Be sure to check \( f(0) \) and \( f(1) \).

30. Assume \( S \) is a countable set and \( T \subseteq S \). If \( S \) is finite, then \( T \) is also finite and hence countable. If \( S \) is countably infinite, there is a 1 - 1 function \( f:S \rightarrow J \) that is onto. Define \( g:T \rightarrow J \) by \( g(t) = f(t) \) for each \( t \in \mathcal{T} \). \( g \) is 1 - 1, hence \( T \) is equivalent to \( \text{im} g \) which is a subset of \( J \). Hence \( T \) is countable as it is a subset of \( J \), hence countable by 0.14 and \( T \) is equivalent to \( \text{im} g \).

31. \( f(n) = n \) if \( n \) is odd and \( f(n) = 1 - n \) if \( n \) is even.

32. The proof is by induction. \( P_0 \) is the set of all constant polynomials with integer coefficients, and of course that's just \( \mathbb{Z} \), hence it is countable. Some people don't view the zero polynomial as having degree zero, so include it in \( P_0 \) anyway and the countability of \( P_0 \) isn't affected one way.
or the other. Assume now that \( P_n \) is countable. Let 
\( A = \{ z : z \in \mathbb{Z}, z \neq 0 \} \) and define \( f : A \times P_n \to P_{n+1} \) as
follows: \( f(a, p) = ax^{n+1} + p \) for each \( a \in A \) and 
\( p \in P_n \).

This is a 1 - 1 function from \( A \times P_n \) onto \( P_{n+1} \) (there
are a few details to fill in), hence \( P_{n+1} \) is countable
since both \( A \) and \( P_n \) are countable as is \( A \times P_n \) by
Theorem 0.16.

33. The set of all polynomials with integer coefficients is the
union of the family \( \{ P_n \}_{n \in \mathbb{N}} \). This is a countable set by
Theorem 0.17.

34. If \( S \) is countable, then it is equivalent to a subset of
\( \mathbb{N} \), let \( f : S \to \mathbb{N} \) be the function that gives that
equivalence. Define an indexed family \( \{ B_n \}_{n \in \mathbb{N}} \) as
follows:

\[ B_n = A_s \text{ where } f(s) = n \]

Then \( \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{s \in S} A_s \) is countable by 0.17.

35. For each \( p \in P_n \), \( B(p) \) is finite since a polynomial of
degree \( n \) has at most \( n \) real zeros. \( P_n \) is countable,
hence by Exercise 34, \( \bigcup_{p \in P_n} B(p) \) is countable.

36. Let \( Q_n = \bigcup_{p \in P_n} B(p) \) as in Exercise 35. \( Q_n \)
is countable and \( \bigcup_{n \in \mathbb{N}} Q_n \) is the set of all algebraic numbers.

37. If \( f : A \to P(A) \) define \( C = \{ x : x \in A \text{ and } x \notin f(x) \} \).

Furthermore, suppose that there is \( c \in A \) such that \( f(c) = C \).
Is \( c \in C? \) If \( c \in C \), then \( c \notin f(c) = C \) which is a
contradiction. So it must be that \( c \notin C \), in which case
\( c \notin f(c) = C \), again a contradiction. So it must be the
case that there is no such \( c \in A \), that is, \( C \notin \text{im } f \).

38. \( f(x) = \frac{x - a}{b} - \frac{a}{b} \) is a 1 - 1 function from \( [a,b] \) onto
\( [0,1] \). Theorem 0.11 then shows that \( [a,b] \) is equivalent
to \( [c,d] \).

39. \( x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} = y \)
40. \[ 0 \leq (\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \text{ since } x \geq 0 \text{ and } y \geq 0. \]
Thus, \[ \sqrt{xy} \leq \frac{x + y}{2}. \]

41. Since \( 0 < a < b \), then \( a^2 = a \cdot a < ab < b \cdot b = b^2 \).
Since \( 0 < a < b \), then \( 0 < \sqrt{a} \) and \( 0 < \sqrt{b} \). Since \( a \neq b \), then \( \sqrt{a} \neq \sqrt{b} \) and so either \( \sqrt{a} < \sqrt{b} \) or \( \sqrt{a} > \sqrt{b} \).
However, if \( \sqrt{a} > \sqrt{b} \), then by the first part of this exercise, \( a > b \). Thus, \( 0 < \sqrt{a} < \sqrt{b} \).

42. Since \( \frac{x}{y} < \frac{a}{b} \) and \( x, y, a, \) and \( b \) are positive, then \( xb < ay \).
So \( xy + xb < xy + ay \) or \( x(y + b) < y(x + a) \)
and hence \( \frac{x}{y} < \frac{x + a}{y + b} \). The manipulations with the inequalities were justified by the fact that \( x, y, a, \) and \( b \) are positive real numbers. The remaining inequality is established in a similar way.

43. Assume \( r \) is a rational number and \( r^2 < 2 \). We may as well assume \( r \geq 0 \) since all we need to establish is that there is a \( x \in A \) such that \( r < x \) and if \( r < 0 \), then \( 0 \) serves the purpose. Now if \( r^2 < 2 \) and \( r > 0 \), choose a rational number \( \delta \) such that \( 0 < \delta < 1 \) and \( \delta < \frac{2 - r^2}{2r + 1} \).
We will show that \( (r + \delta)^2 < 2 \).
\[
(r + \delta)^2 = r^2 + 2r\delta + \delta^2 < r^2 + 2r\delta + \delta \\
= r^2 + (2r + 1)\delta < r^2 + (2 - r^2) = 2.
\]
Note: Since \( 0 < \delta < 1 \), then \( \delta^2 < \delta \).

44. Suppose \( A \) is nonvoid and \( x = \sup A \). If \( \epsilon > 0 \), then \( x - \epsilon < x \), and hence \( x - \epsilon \) is not an upper bound for \( A \), hence there is a \( a \in A \) such that \( x - \epsilon < a \), but \( x \) is an upper bound for \( A \), hence
\[
x - \epsilon < a \leq x.
\]

45. This proof is very similar to that of Exercise 44. If \( \epsilon > 0 \), then \( y < y + \epsilon \), so \( y + \epsilon \) is not a lower bound for \( A \), hence there is a \( a \in A \) such that \( a < y + \epsilon \). As \( y \) is a lower bound for \( A \),
\[
y \leq a < y + \epsilon.
\]