Sustained Oscillations via Coherence Resonance in SIR
Rachel Kuske, Luis F. Gordillo, Priscilla Greenwood

Example: influenza.

- Population = $N = 2,000,000$
- Average life span = $1/\mu = 80$ years
- Reproductive number = $R_0 = 15$
- Time of infectivity = $1/\gamma = 15$ days
The SIR model

\[ \Delta S = (\mu (N-S) - \beta SI/N) \Delta t + \Delta Z_1 - \Delta Z_2 \]
\[ \Delta I = (\beta SI/N - (\gamma + \mu) I) \Delta t + \Delta Z_2 - \Delta Z_3 \]

\( \Delta Z_i \) are centered Poisson increments with variances

\[ \mu (N+S) \Delta t, \ \beta SI/N \Delta t, \ (\gamma + \mu) I \Delta t. \]
Deterministic and stochastic paths in the phase plane, 25 years

The deterministic system has equilibrium at

\[ S_{eq} = \frac{N}{R_0}, \quad I_{eq} = N \, \frac{\mu (R_0 - 1)}{\beta} \]
Stochastic Differential Equations:

\[
dS = (\mu(N-S) - \beta \frac{SI}{N})dt + \sqrt{\mu(N+S)}dW_1 - \sqrt{\beta \frac{SI}{N}}dW_2 \\
dI = (\beta \frac{SI}{N} - (\gamma + \mu)I)dt + \sqrt{\beta \frac{SI}{N}}dW_2 - \sqrt{(\gamma + \mu)I}dW_3
\]

Computations:

1. Change of variables:

\[
u = \frac{S - S_{eq}}{S_{eq}} \quad \nu = \frac{I - I_{eq}}{I_{eq}} \quad \text{time: } \Omega t, \quad \Omega = \sqrt{\beta \frac{\mu}{R_0}(R_0-1)}
\]

Now the stable equilibrium is at (0,0) and damped oscillations have frequency 1.
2. Linearize about \((u,v) = (0,0)\), and set \((u,v) = (0,0)\) in the noise coefficients

\[
d \begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} dt + G \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix}.
\]

The eigenvalues of \(M\) are

\[
\lambda = -\epsilon^2 \pm \sqrt{\epsilon^4 - 1},
\]

where

\[
\epsilon^2 = \frac{\mu R_0}{2 \Omega}.
\]

For \(\epsilon \ll 1\), oscillations are slowly decaying with frequency 1.
Conjecture: The Stochastic model is approximately

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(T) \begin{pmatrix} b \cos(t) \\ \sin(t) \end{pmatrix} + B(T) \begin{pmatrix} b \sin(t) \\ -\cos(t) \end{pmatrix}$$

where $b^2 = \frac{I_{eq}}{S_{eq}}$, $T = \epsilon^2 t$.

$A$ and $B$ are diffusion processes with drift coefficients depending on $(A,B)$ and constant diffusion coefficients.
Multiscale Analysis

Write \((du, dv)\) using Ito's formula and compare with

\[
d\begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} dt + G \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix}.
\]

2. Integrate each with respect to \(t\) over \([0, 2\pi]\).
3. Consider functions of \(T\) as constant in \(t\) over \([0, 2\pi]\).
4. Identify the drift and diffusion coefficients in the SDE of \(A(T)\) and \(B(T)\):

\[
d\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} dT + \frac{\delta}{\sqrt{2\epsilon}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}.
\]

\((A, B)\) are independent Ornstein-Uhlenbeck processes.
Parameter region for Ro and γ

- We need $\epsilon^2 \ll 1$ for separation of scales; $\delta^2/2\epsilon^2$, the stationary variance of A and B, of moderate size.

- Red: $\epsilon^2 = 0.1$
- Blue: $\delta^2/2\epsilon^2 = 1$
- Green: $\delta^2/2\epsilon^2 = 0.04$
- $N = 500,000, \mu = 1/55$
- $X = (25,15)$
Power Spectral Density of Stochastic Model and Multiscale Approximation

- Solid: Approximation
- Dot-dash: Stochastic Model
- $R_0 = 15$, $\gamma = 25$
- $N = 500,000$, $\mu = 1/55$
Histograms produced with 100 realizations during 200 years

The stochastic model

The multi-scale approximation