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Periodic Solutions of Differential Delay Equations with Threshold-Type Delays

H. L. SMITH AND Y. KUANG

Abstract. Periodic solutions are shown to exist for the differential delay equation $x'(t) = -\nu x(t) - e^{-\eta \tau} f(x(t - \tau))$ where $\tau$ is determined implicitly by the solution $x(t)$ through the threshold relation $\int_{t-\tau}^{t} k(x(t), x(s)) \, ds = m$. These periodic solutions have two simple zeros in the period interval. If $k$ is independent of the second variable then $\tau = \sigma(x(t))$ while if $k$ is independent of the first variable then the delay is of threshold-type. The latter arise in many models in the biological sciences. The proof consists of modifying the classical proof for the case of constant delays.

0. Introduction

In this paper the existence of periodic solutions for the state-dependent delay equation with threshold-type delay of the form

$$x'(t) = -\nu x(t) - e^{-\eta \tau} f(x(t - \tau))$$

$$\int_{t-\tau}^{t} k(x(t), x(s)) \, ds = m$$

is established. In (0.1), $\nu$ and $\eta$ are nonnegative constants, $m$ is a positive constant and $k(x, y)$ is a positive-valued locally Lipschitz function on $\mathbb{R}^2$. The delay $\tau$ is determined implicitly by the integral relation in (0.1). The function $f$ is assumed to be locally Lipschitz and satisfy

$$xf(x) > 0, \quad x \neq 0.$$ 

Thus (0.1) is of negative feedback type.

In the special case that $k$ is independent of $y$, that is, $k(x, y) = (\sigma(x))^{-1}m$ where $\sigma : \mathbb{R} \to (0, \infty)$ is locally Lipschitz, then the delay $\tau$ is given by

$$\tau = \sigma(x(t)).$$

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If \( k(x, y) = k(y) \), then the delay \( \tau \) is determined implicitly by

\[
(0.3) \quad \int_{t-\tau}^{t} k(x(s)) \, ds = m.
\]

Delays of the form (0.3) have appeared frequently in applications (see below) and are called threshold delays. By allowing \( k \) to depend on \( x(t) \) in addition to \( x(s) \), we include most of the state-dependent delays which have been proposed in the literature. In fact, our main result is proved for a more general delay functional \( \tau = \tau(x_t) \) which includes the threshold delay in (0.1) as a special case.

The term \( e^{-\eta \tau} \) in (0.1) (which is absent if \( \eta = 0 \)) requires a word of motivation. This term is not merely a constant since \( \tau = \tau(x_t) \) depends on past history. It is present in many equations arising from mathematical models in the biological sciences (see, e.g., [7], [28], [26]) due to an exponential death or decay rate which discounts a delayed population size. This term has a stabilizing effect on the trivial solution of (0.1) and a large \( \eta \) precludes the existence of periodic solutions of (0.1) as we shall see. From a purely mathematical point of view, the term could be replaced by any positive locally Lipschitz function of \( \tau(x_t) \). Moreover, the term \( e^{-\eta \tau} f(x(t-\tau)) \) could be replaced by \( f(e^{-\eta \tau} x(t-\tau)) \) without affecting our main result.

In stating our main results, it will be convenient to scale variables and eliminate unnecessary parameters. Obviously, the positive constant \( m \) can be incorporated into the function \( k \) so we may as well assume that \( m = 1 \). Scaling time by \( \bar{t} = \delta t \) leaves the form of (0.1) unchanged but with \( \bar{\nu} = \nu \delta, \bar{\eta} = \eta \delta, \bar{f} = \delta f, \bar{k} = \delta k \). Choosing \( \delta = k(0,0)^{-1} \) has the effect of making \( \bar{k}(0,0) = 1 \). Henceforth, we assume that

\[
(0.4) \quad m = 1, \quad k(0,0) = 1.
\]

Observe that \( x \equiv 0 \) is a solution of (0.1) and, by virtue of (0.4), the delay \( \tau \) corresponding to the trivial solution is unity.

It turns out to be possible to relate small solutions of (0.1) to the solutions of the linear equation with constant delay \( \tau = \tau(0) = 1 \)

\[
(0.5) \quad y'(t) = -\nu y(t) - e^{-\eta} f'(0) y(t-1),
\]

associated to which is the characteristic equation

\[
(0.6) \quad \nu + \lambda + \alpha e^{-\lambda} = 0
\]

where \( \alpha = e^{-\eta} f'(0) \). If \( \alpha > \alpha_{\nu} \), where \( \alpha_{\nu} \) is the smallest positive solution of

\[
\nu + \alpha \cos \sqrt{\alpha^2 - \nu^2} = 0,
\]

then (0.6) has a root \( \lambda = \mu + i \gamma \) satisfying \( \mu > 0 \) and \( \frac{\pi}{2} < \gamma < \pi \) (see, e.g., [15], [18]).

The following is a special case of our main result.
Theorem. Suppose that

(i) $e^{-\eta f'(0)} > \alpha_\nu$.

(ii) There exist $M, N > 0$ such that $F_\nu(x) \equiv -\nu^{-1}f(x)$ maps $[-M, N]$ into itself.

(iii) $\frac{\partial k}{\partial x}(x, y)$ exists for $x \neq 0$, $x, y \in [-M, N]$, is continuous and

\begin{equation}
\nu x \frac{\partial k}{\partial x}(x, y) < k(x, y)k(x, x), \quad x, y \in [-M, N], \ x \neq 0.
\end{equation}

Then (0.1) has a nonconstant periodic solution $x(t) = x(t + \omega)$ satisfying:

(a) $x(t)$ has exactly two zeros on $[0, \omega]$, namely $z_1$ and $z_2$, $0 < z_1 < z_2 < \omega$ with $x'(z_1) < 0$ and $x'(z_2) > 0$.

(b) $-M < x(t) < N$, $0 \leq t \leq \omega$.

(c) $e^{\nu t}x(t)$ has exactly two extrema in $(0, \omega]$, namely $t_1$ and $t_2 = \omega$, $0 < z_1 < t_1 < z_2 < t_2$ with $e^{\nu t}x(t)$ increasing on $(t_1, t_2)$ and decreasing on $(0, t_1)$.

Several remarks regarding the Theorem will be helpful. In (ii), it is assumed that $\nu > 0$. If $\nu = 0$, a different assumption is required (see (B) of Section 1).

The assumption (0.7) is crucial to our proof of the theorem for it allows us to establish that $t - \tau \in (z_1, z_2)$ if and only if $t_1 < t < t_2$. Obviously, (0.7) holds if $k(x, y) = k(y)$ is independent of $x$ so (0.7) is not a restriction for the threshold delay (0.3). It may be a restriction, however, for delays of the form (0.2) for which (0.7) requires that $\sigma(x)$ be continuously differentiable for $x \in [-M, N]\{0 \}$ and satisfy

\begin{equation}
\nu x \sigma'(x) > -1, \quad x \neq 0, \ -M \leq x \leq N.
\end{equation}

For example, (0.8) holds for $\sigma(x) = 1 + |x|$ as $x \sigma'(x) > 0$, but (0.8) may fail to hold, depending on the values of $M$ and $N$, for $\sigma(x) = \frac{1}{1 + |x|}$ if $\nu \geq 4$.

We can estimate the period $\omega$, but only clumsily in the full generality of the threshold delay of (0.1). On the other hand, if the delay is given by (0.2), then $z_2 - z_1 > 1$ and $\omega = t_2 > 2$. In this case, the periodic solution of the Theorem could be called a "slowly oscillating" solution of (0.1).

All the assumptions of the Theorem except (i) can be weakened at the expense of a more complicated and lengthy statement (see Section 1).

Threshold delays of the form (0.3) have been used in epidemiological modeling [34], in modeling of the immune response system [34], [11] and in modeling of respiration [16]. The basic idea is that the delay is due to the time required to accumulate an appropriate dosage of infection or antigen concentration. In the context of epidemiology, $x(t)$ may represent the proportion of a population which is infective at time $t$ and (0.3) may reflect that an individual who is first exposed to the disease at time $t - \tau$ becomes infectious at time $t$ if, during the interval from $t - \tau$ to $t$, a threshold level of exposure is accumulated where the per unit time exposure depends on the infective fraction $x(s)$ via $k(x(s))$. See [34] for
a description of such models and [31] for the existence of periodic solutions for such models.

Well-posedness results for systems of differential delay equations with threshold delays given by a slightly more general expression than (0.1) ($k$ is allowed to depend explicitly on $t$) have been proved in [12], [13]. These results are not immediately applicable in our setting, however, since the authors consider initial data for $x(t)$ to be simply the assignment of a value for $x(0)$ and they set $\tau = t$ until such time $t_0 > 0$ when $\int_0^{t_0} k(x(s), x(s)) \, ds = 1$. For $t > t_0$ the delay is determined by (0.1) and hence the equation is viewed as an ordinary differential equation for $0 < t < t_0$ and a delay equation for $t > t_0$. On the other hand, we view (0.1) as providing a delay immediately since we specify $x(t)$ for $t \leq 0$. However, for large $t$, assuming the threshold time $t_0$ exists, the equations are identical and the influence of differing initial data should wane. Hence, the existence of stable periodic solutions will be of interest.

Delay differential equations containing delays of the threshold type (0.3) arise from attempts to simplify structured population models, which usually take the form of hyperbolic partial differential equations. As an example, Nisbet and Gurney [28] consider insect populations which have several life stages (instars). They construct a mathematical model consisting of an equation for the mass density function for the population. Then, under the assumption that the population in any life stage is homogeneous, they are able to reduce the model to a system of delay differential equations for the size of the population in each life stage. These equations have threshold delays (see [28], Appendix 2) due to the assumption that the insect must spend an amount of time in the larval stage sufficient to accumulate a threshold amount of food, where food density is also a dynamical variable. The system of delay equations considered in [28] is considerably more complicated than (0.1).

Metz and Diekmann ([26], p. 236–237) discuss a modification of a structured model for the control of the bone marrow stem cell population which supplies the circulating red blood cell population, due to Kirk, Orr and Forrest [21]. The maturing stem cell population is structured by a maturation variable and the rate of maturation is assumed to depend only on the total mature red blood cell population. Since a threshold level of maturation is required in order for an immature cell to enter the mature population, a threshold delay differential equation arises naturally. See ([26], Exercises 4.30 and 4.31) for the equations of the reduced delay equation model. While there is an error in the equations of Exercise 4.31 ($V(P(t))n(t, 1) = (V(P(t))/V(P(t - \tau)))e^{-\delta t} \gamma P(t - \tau)$), it does not affect our argument that delay equations containing threshold type delays arise naturally from structured population models are therefore worthy of study.

model) is a function of total population size. In these models the delay takes the form (0.2).

Admittedly, (0.1) is extremely simple compared to the equations which arise from the population models described above. We view our efforts here as only a beginning towards understanding the more complicated equations which arise in the applications. In this regard, we hasten to point out that the first steps in this program were taken by Alt [2] who earlier recognized the importance of considering delay equations containing threshold delays. Alt [2] has established the existence of periodic solutions of the equation

$$x'(t) = -f\left(x(t), \int_{-\tau}^{0} g(x(t - \tau + \theta)) d\eta(\theta)\right)$$

where \(\tau\) is determined by (0.3) and \(f\) and \(g\) satisfy suitable conditions, making his equation more general, although his delay is a special case of ours.

J. Mallet-Paret and R. Nussbaum have announced very interesting work related to ours at the International Conference on Differential Equations and Applications to Biology and Population Dynamics (Claremont, California, January 1990). They consider a singularly perturbed equation of the form

$$\epsilon x'(t) = -f(x(t), x(t - r))$$

where \(r = r(x(t)) \geq 0\) and \(f\) and \(r\) are given functions. They are able to describe the asymptotic shape of slowly oscillating solutions as \(\epsilon\) tends to zero under appropriate conditions on \(f\) and \(r\). In a private communication, J. Mallet-Paret indicates that they obtain, as part of their work, an existence result for periodic solutions of this equation [27].

The proof of our theorem follows more or less standard lines (see in particular [15] and also [1], [6], [8], [10], [15], [18], [19], [29], [30], [32], [33], [35]). We set up a Poincaré map from a compact convex subset \(K\) of \(C_B\) into itself where \(C_B\) is the space of bounded and continuous functions on \((-\infty, 0]\). The set \(K\) contains the zero function which we show to be an ejective fixed point of the mapping. A theorem of Browder [5] asserts the existence of at least one nonejective fixed point, proving our theorem.

As a simple illustration of the theorem, consider the equation

$$x'(t) = -\nu x(t) + g(x(t - \tau))$$

$$\int_{t-\tau}^{t} k(x(t), x(s)) ds = m$$

(0.9)

where \(g\) is a continuously differentiable nonnegative function on \(x \geq 0\) and \(g'(x) < 0\) for \(x > a\) for some \(a \geq 0\). Assume \(G_\nu(x) = \nu^{-1}g(x)\) has a period two orbit \(G_\nu(x_1) = x_2, G_\nu(x_2) = x_1\), where \(a < x_1 < x_2\). Then \(G_\nu\) has a unique fixed point \(x_0\) satisfying \(x_1 < x_0 < x_2\), which is an equilibrium for (0.9). The change of variables \(y = x - x_0\) transforms (0.9) to (0.1) where

$$f(y) = g(x_0) - g(x_0 + y).$$
For simplicity, assume that \( m = \tilde{k}(x_0, x_0) \) so that on dividing through by \( m \), the transformed equations are as in (0.1) with (0.4) holding for \( \tilde{k}(u, v) \equiv k(x_0 + u, x_0 + v)/m \). It is easily checked that (ii) of the theorem holds \( (M = x_1 - x_0, N = x_2 - x_0) \) for the transformed system and (i) holds provided \( g'(x_0) < -\alpha \nu \). Provided \( \tilde{k} \) has property (iii), the theorem implies the existence of a periodic solution \( x(t) \) of (0.9) which oscillates about \( x = x_0 \) and satisfies \( x_1 < x(t) < x_2 \).

The main result of this paper represents an improvement of earlier results of the authors [23] who considered (0.1) in the case that \( \nu = \eta = 0 \). In an earlier paper [22] the authors considered (0.1) in case \( \nu = \eta = 0 \) and (0.2) holds. This paper contains some numerical simulations and some results on the oscillatory properties of solutions.

Our main result (Theorem 1.9) is described in the next section. In that section, the delay is assumed to be specified by a functional \( \tau : C_B \rightarrow (0, \infty) \) where \( C_B \) is the Banach space of bounded continuous functions on \((-\infty, 0] \) with the uniform norm \( || \cdot || \). Four hypotheses are required to be satisfied by the functional \( \tau \) in order that our result applies. In a brief final section, the threshold delay appearing in (0.1) is shown to have the required properties.

1. Main Results

Consider the delay differential equation

\[
(1.1) \quad x'(t) = -\nu x(t) - e^{-\eta \tau(x_t)} f(x(t - \tau(x_t)))
\]

where \( x_t(s) = x(t + s), \ -\infty \leq s \leq 0 \), and \( \nu, \eta \) are nonnegative constants. Alternatively, the term \( e^{-\eta \tau} \) could multiply \( x(t - \tau) \) inside \( f \).

The following are assumed to hold throughout this section.

(F) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is locally Lipschitz continuous,

\[ x f(x) > 0, \quad x \neq 0, \]

and \( f'(0) \) exists.

The functional \( \tau : C_B \rightarrow (0, \infty) \) satisfies

(D1) \( \tau(0) = 1 \)

(D2) If \( \varphi, \psi \in C_B \) satisfy \( \varphi|_{-\tau(\varphi), 0} = \psi|_{-\tau(\psi), 0} \) then \( \tau(\varphi) = \tau(\psi) \).

(D3) Given \( L > 0 \) there exists \( V > 0 \) such that

\[ |\tau(\phi) - \tau(\psi)| \leq V \max_{-Q \leq s \leq 0} |\phi(s) - \psi(s)| \]

if \( ||\phi||, ||\psi|| \leq L \) where \( Q = \max\{\tau(\varphi), \tau(\psi)\} \).

Note that (D3) implies that \( \tau \) is bounded on bounded subsets of \( C_B \). Indeed, using (D1) and (D3), the estimate \( \tau(\varphi) \leq 1 + V(L_\varphi) ||\varphi|| \) holds for \( ||\varphi|| \leq L \). (D1) is simply a normalization which can be assumed without loss of generality.
Lemma 1.1. For each bounded, Lipschitz function $\phi : (\infty, 0] \to \mathbb{R}$, there exists a unique noncontinuable solution $x(t, \phi)$ of (1.1) defined for $t \in [0, \omega)$, $0 < \omega \leq \infty$, satisfying $x(t) = \phi(t)$ for $t \leq 0$. If $\omega < \infty$, then there exists $t_n \nearrow \omega$ such that $|x(t_n)| \to \infty$ as $n \to \infty$.

Proof. These are well-known results. See [17], [14], [9]. Uniqueness of solutions follows from ([17], Prop. 1.3) and the fact that $f$ and $\tau$ are locally Lipschitz. Since the right-hand side of (1.1) is bounded on bounded subsets of the space of bounded, Lipschitz functions on $(-\infty, 0]$, the behavior of $x(t)$ as $t \to \omega$ follows from ([17], Prop. 1.2). In the case of (1.1), this is easy to see since, if $|x(t)|$ is bounded, then its derivative is bounded and thus $\lim_{t \to \omega} x(t)$ exists. The solution can then be extended to a larger interval.

Define
\[
F^+(x) = \max\{f(s) : 0 \leq s \leq x\}
\]
\[
F^-(x) = \max\{-f(s) : -x \leq s \leq 0\}
\]
for $x \geq 0$. Given $M, N > 0$ let
\[
R = \sup\{\tau(\phi) : \phi \in C_B, -M \leq \phi(s) \leq N\}
\]
\[
R_M = \sup\{\tau(\phi) : \phi \in C_B, \phi(0) = -M \leq \phi(s) \leq N\}
\]
\[
R_N = \sup\{\tau(\phi) : \phi \in C_B, -M \leq \phi(s) \leq N = \phi(0)\}.
\]

$R, R_M$ and $R_N$ depend on both $M$ and $N$ but we suppress this dependence for notational convenience. Obviously, $R_M, R_N \leq R < \infty$, the latter by (D3), and $R \geq 1$ by (D1).

We now make two hypotheses which determine $M$ and $N$ such that solutions corresponding to suitably restricted initial data exist for all $t \geq 0$, take values in the interval $[-M, N]$ and oscillate about $x = 0$.

(B) There exist $M, N > 0$ such that
\[
F^+(N) \left[\frac{1 - e^{-\nu R_M}}{\nu}\right] < M,
\]
and
\[
F^-(M) \left[\frac{1 - e^{-\nu R_N}}{\nu}\right] < N.
\]

If $\nu = 0$, then the term in brackets in (1.2a) ((1.2b)) is understood to be $R_M$ ($R_N$).

Note that (B) is a restriction on $\nu, \tau$ and $f$. We point out some more easily verified sufficient conditions for (B) to hold. If $\nu = 0$, then sufficient conditions for (B) to hold were given in [23]. Here, we will assume $\nu > 0$. If
\[
F_\nu(x) \equiv -\nu^{-1}f(x) \text{ satisfies } F_\nu([-M, N]) \subset [-M, N]
\]
then $\nu^{-1}F^+(N) \leq M$ and $\nu^{-1}F^-(M) \leq N$ implying that (B) holds. Also, (1.3) has the advantage of not involving the delay $\tau$ and being relatively easy to verify.
For the same values of $M$ and $N$ as in (B), we also assume:

(D4) If $\phi \in C_B$ is Lipschitz, takes values in $[-M, N]$ and satisfies $\phi(-\tau(\phi)) = 0$, then for any $\epsilon > 0$ and any extension $y : (-\infty, \epsilon) \to \mathbb{R}$ of $\phi$, $y_0 = \phi$, where $y|[0, \epsilon)$ is continuously differentiable and $y'(0) + \nu \phi(0) = 0$, there exists $\delta \in (0, \epsilon)$ such that

$$\tau(y_t) - \tau(\phi) < t, \quad 0 < t < \delta.$$ 

If, in addition, $\phi'(0)$ exists and $\phi'(0) + \nu \phi(0) = 0$, then

$$\tau(\phi_s) - \tau(\phi) > s, \quad -\delta < s < 0.$$ 

For the $M, N > 0$ such that (B) and (D4) hold, let

$$\tilde{K}(M, N) = \{ \phi \in C_B : \phi(s) = 0, s \leq -R, \phi(s)e^{\nu s} \text{ is nondecreasing on } [-R, 0], \phi(s) \leq N \text{ for } s \leq 0 \text{ and } \phi \text{ is Lipschitz} \}.$$ 

Note that as $\phi(s)e^{\nu s}$ is nondecreasing on $[-R, 0]$, it follows that either $\phi = 0$ or $s_{\phi} \equiv \sup\{s \leq 0 : \phi(s) = 0\}$ satisfies $-R \leq s_{\phi} < 0$ and $0 < \phi(s) \leq N$ holds for $s_{\phi} < s \leq 0$. Hereafter, we write $\bar{K}$ for $\tilde{K}(M, N)$.

For the remainder of this section, we assume that (B) and (D4) hold and we only consider initial data $\phi \in \bar{K}$. For such $\phi$, let $\bar{t} = \bar{t}(\phi) = \sup\{t \geq 0 : -M \leq x(s, \phi) \leq N \text{ for } 0 \leq s \leq t\}$. If $\phi \in \bar{K}\setminus\{0\}$, then $\phi(0) \leq N$ and $x'(0) < 0$ so $0 < \bar{t} \leq \omega$. If $\bar{t} < \infty$ then by Lemma 1.1, $x(\bar{t}) \in \{-M, N\}$. Observe that $\tau(x_t) \leq R$ for $0 \leq t < \bar{t}$. By (D3), we have $\tau(x_t) > t_0 - \tau(x_{t_0})$ for $t_0 < t < \bar{t}$, and $\tau(x_t) < t_0 - \tau(x_{t_0})$ for $0 \leq t < t_0$.

**Lemma 1.2.** Let $\phi \in \tilde{K}$ and $x(t) = x(t, \phi)$. If $\frac{dx(t)e^{\nu t}}{dt} = 0$ for some $t_0 \in [0, \bar{t}]$, then

$$t - \tau(x_t) > t_0 - \tau(x_{t_0}) \text{ for } t_0 < t < \bar{t}$$

and

$$t - \tau(x_t) < t_0 - \tau(x_{t_0}) \text{ for } 0 \leq t < t_0.$$ 

**Proof.** By (1.1),

$$\frac{d}{dt}|_{t=t_0} e^{\nu t} x(t) = -e^{\nu t_0 - \eta \tau(x_t)} f(x(t_0 - \tau(x_{t_0}))) = 0$$

implies that $x_{t_0}(-\tau(x_{t_0})) = 0$. By (D4) with $y(t) = x(t_0 + t)$, there exists $\delta > 0$ such that $\tau(x_t) - \tau(x_{t_0}) < t - t_0$ for $0 < t - t_0 < \delta$, if $t_0 < \bar{t}$, and $\tau(x_t) - \tau(x_{t_0}) > t - t_0$ for $-\delta < t - t_0 < \delta$, if $t_0 > 0$. Thus the inequality assertions of the lemma hold locally near $t_0$. If the first inequality above does not hold for all $t \in (t_0, \bar{t}]$, then there exists a point $\tilde{t} \in (t_0, \bar{t}]$ such that

$$\bar{t} - \tau(x_{\tilde{t}}) = t_0 - \tau(x_{t_0}) < t - \tau(x_{t_0})$$

for $t_0 < t < \bar{t}$. As $x_{\tilde{t}}(-\tau(x_{\tilde{t}})) = x_{t_0}(-\tau(x_{t_0})) = 0$ we may argue a contradiction to the last inequality above using (D4) again at $\tilde{t}$. A similar argument applies to show that the second inequality must hold for all $t < t_0$. This completes our proof.
PROPOSITION 1.3. In addition to the assumptions above, assume that $f'(0)e^{-\eta} > 1$. If $\phi \in \tilde{K} \setminus 0$ and $x(t) = x(t, \phi)$, then

(i) There exists $z_1 = z_1(\phi) > 0$ such that $x(t) > 0$ and $x'(t) \leq 0$ on $[0, z_1)$, $x(z_1) = 0$ and $x'(z_1) < 0$. Moreover, there exists $P = P(M, N) > 0$ such that $z_1(\phi) \leq P$ for all $\phi \in \tilde{K} \setminus 0$.

(ii) There exists $t_1 = t_1(\phi) \in (z_1, z_1 + R)$ such that $t_1 - \tau(x_{t_1}) = z_1 > t - \tau(x_t)$ for $z_1 \leq t < t_1$. Consequently, $\frac{d}{dt}(e^{\nu t}x(t))$ is negative on $[z_1, t_1)$ and vanishes at $t_1$. Moreover, $0 > x(t) > -M$ on $[z_1, t_1]$.

(iii) There exists $z_2 = z_2(\phi) > t_1$ such that $-M < x(t) < 0$ on $(z_1, z_2)$, $x(z_2) = 0$, $x'(z_2) > 0$. There exists $Q = Q(M, N) > 0$ such that $z_2(\phi) \leq Q$ for all $\phi \in \tilde{K} \setminus 0$.

(iv) There exists $t_2 = t_2(\phi) \in (z_2, z_2 + R)$ such that $t_2 - \tau(x_{t_2}) = z_2$. For $t_1 < t < t_2$, $z_1 < t - \tau(x_t) < z_2$ and consequently $\frac{d}{dt}(e^{\nu t}x(t)) > 0$ and vanishes at both endpoints. Finally, $0 < x(t) < N$ on $(z_2, t_1]$.

PROOF. Clearly $x'(t) \leq 0$ for all $t \in [0, z_1]$ where $z_1 = \inf\{t > 0 : x(t) = 0\} \leq +\infty$. Define $F(x) = \min\{f(u) : x \leq u \leq N\}$ for $0 \leq x \leq N$ so that $F$ is Lipschitz, $F(0) > 0$ for $0 < x \leq N$, $F$ is nondecreasing and $F(0) = 0$. If $z_1 > R$, then for $R \leq t \leq z_1$, $t - \tau(x_t) \geq 0$ since $\tau(x_t) \leq R$, so

$$x'(t) = -\nu x(t) - e^{-\eta \tau(x_t)}f(x(t - \tau(x_t))) \leq -\nu x(t) - e^{-\eta R}F(x(t))$$

where we have used the fact that $F$ is nondecreasing and $x(t)$ is decreasing. Standard differential inequality arguments imply that $x(t) \leq u(t)$, $R \leq t \leq z_1$, where

$$u'(t) = -\nu u(t) - e^{-\eta R}F(u(t)), \quad u(R) = N.$$

Observe that $u(t)$ is independent of $\phi \in \tilde{K} \setminus 0$. Obviously $u(t) \to 0$ as $t \to +\infty$.

Let $k$ be such that $f'(0)e^{-\eta} > k > 1$ and choose $\gamma \in (0, 1)$ and $\delta > 0$ such that $e^{-\eta(1+\gamma)}f(x)/x > k > 1$ if $0 < x \leq \delta$. If $0 \leq \phi \leq N$ then by (D3), $|\tau(\phi) - 1| \leq V \max_{-R \leq s \leq 0} |\phi(s)|$. Choose $\delta$ smaller if necessary so that $V \delta < \min\{\gamma, 1 - k^{-1}\}$. Now there exists $T > 0$ such that $u(t) \leq \delta$ for $t \geq T$. Suppose $z_1 > 2R + T + k^{-1}$. Then $0 < x(t) < u(t) \leq \delta$ for $T < t \leq 2R + T + k^{-1}$ and $|\tau(x_t) - 1| \leq V \max_{-R \leq s \leq 0} |x(t+s)| \leq V \delta$ for $T < R \leq t < 2R + T + k^{-1}$. Thus $1 + \gamma > \tau(x_t) > k^{-1}$ on the latter interval. For $2R + T < t \leq 2R + T + k^{-1}$, we have $R + T < t - R \leq t - \tau(x_t) < t - k^{-1} \leq 2R + T$ and hence $0 < x(t - \tau(x_t)) < \delta$ and $\tau(x_t) < 1 + \gamma$. Thus

$$x'(t) = -\nu x(t) - e^{-\eta \tau(x_t)}f(x(t - \tau(x_t))) \leq -e^{-\eta(1+\gamma)}f(x(t - \tau(x_t)))$$

$$< -kx(t - \tau(x_t)) \leq -kx(2R + T)$$

for $2R + T \leq t \leq 2R + T + k^{-1}$. This implies that

$$x(2R + T + k^{-1}) < x(2R + T) - kx(2R + T)k^{-1} = 0,$$
a contradiction to our assumption that $z_1 > 2R + T + k^{-1}$. Hence, $z_1 \leq 2R + T + k^{-1} \equiv P(M, N)$.

Note that for $0 \leq t \leq z_1$, $t - \tau(x_t) \geq t - R \geq -R$. If $t - \tau(x_t) \leq -s_\phi$ for $0 \leq t \leq z_1$, then $x'(t) = -\nu x(t)$ on this interval so $x(z_1) = e^{-\nu z_1} \phi(0) > 0$, a contradiction to $x(z_1) = 0$. Hence, there exists $t_0 \in (0, z_1)$ such that $t_0 - \tau(x_{t_0}) \geq -s_\phi$. Lemma 1.2 implies that $t - \tau(x_t) > -s_\phi$ for $z_1 \leq t \leq t_0$. It follows that $x'(z_1) = x'(z_1) + \nu x(z_1) < 0$.

Suppose that there does not exist $t_1 \in [z_1, \bar{t}]$ such that $t_1 - \tau(x_{t_1}) = z_1$. Then $t - \tau(x_t) < z_1$ for all $t \in [z_1, \bar{t}]$ and (1.1) implies $\frac{d}{dt}(e^{\nu t} x(t)) < 0$ for all $t \in [z_1, \bar{t}]$. This implies that $x(t) < 0$ for $z_1 < t \leq \bar{t}$. If $z_1 + R \leq \bar{t}$, then $z_1 + R - \tau(x_{z_1 + R}) \geq z_1$ contradicting our assumption that $t_1$ does not exist. Consequently, $\bar{t} < z_1 + R$ so $x(\bar{t}) = -M$. Then, using (1.1), and considering the case $\nu > 0$ ($\nu = 0$ proceeds similarly, see [23]), we conclude that

$$-x(\bar{t}) = M = \int_{z_1}^{\bar{t}} e^{-\nu(\bar{t} - t)} e^{-\eta \tau(x_t)} f(x(t - \tau(x_t))) \, dt$$

$$\leq F^+(N) \int_{z_1}^{\bar{t}} e^{-\nu(\bar{t} - t)} \, dt = F^+(N) \left( \frac{1 - e^{-\nu(\bar{t} - z_1)}}{\nu} \right)$$

$$\leq F^+(N) \left( \frac{1 - e^{-\nu \tau(x_{t_1})}}{\nu} \right)$$

$$\leq F^+(N) \left( \frac{1 - e^{-\nu R_M}}{\nu} \right)$$

where we have used $\bar{t} - \tau(x_{\bar{t}}) < z_1$ and $\tau(x_{\bar{t}}) \leq R_M$. As this contradicts (1.2a) of (B), our assumption that $t_1$ does not exist must be discarded. We denote by $t_1$ the smallest $t$ for which $t - \tau(x_t) = z_1$. Then $x(t) < 0$ on $(z_1, t_1]$ since $\frac{d}{dt}(x(t)e^{\nu t}) < 0$ and the argument above implies that $x(t) > -M$ on $[z_1, t_1]$. This in turn implies that $t_1 \leq z_1 + R$ as noted above.

By Lemma 1.2, $t - \tau(x_t) > z_1 = t_1 - \tau(x_{t_1})$ for $t_1 < t \leq \bar{t}$. Thus for $t_1 < t < z_2 \equiv \inf \{ t > t_1 : x(t) = 0 \}$, $x(t) < 0$, $x(t - \tau(x_t)) < 0$ so $x' > 0$. The argument that $z_2 < \infty$ is entirely symmetrical to the argument that $z_1 < \infty$. Indeed, this analogous argument establishes the existence of $Q > 0$, independent of $\phi \in \tilde{K} \backslash 0$, such that $z_2(\phi) \leq Q$.

If there does not exist $t_2 \in (z_2, \bar{t}]$ such that $t_2 - \tau(x_{t_2}) = z_2$ then $z_1 < t - \tau(x_t) < z_2$ for $z_2 \leq t \leq \bar{t}$. Consequently, $\frac{d}{dt}(e^{\nu t} x(t)) > 0$ and $x(t) > 0$ on $z_2 < t \leq \bar{t}$. If $z_2 + R \leq \bar{t}$, then $z_2 + R - \tau(x_{z_2 + R}) \geq z_2$ in contradiction to our assumption that $t_2$ doesn't exist. Hence $\bar{t} < z_2 + R$ which implies that $x(\bar{t}) = N$. 

Then, using (1.1) and considering only the case $\nu > 0$, we conclude that

$$
N = \int_{z_2}^{\tilde{t}} e^{-\nu(\tilde{t}-t)} e^{-\eta \tau(x_t)} f(x(t - \tau(x_t))) \, dt
\leq F^{-}(M) \int_{z_2}^{\tilde{t}} e^{-\nu(\tilde{t}-t)} \, dt = F^{-}(M) \left[ \frac{1 - e^{-\nu(\tilde{t}-z_2)}}{\nu} \right]
\leq F^{-}(M) \left[ \frac{1 - e^{-\nu \tau(x_t)}}{\nu} \right] \leq F^{-}(M) \left[ \frac{1 - e^{-\nu R_N}}{\nu} \right],
$$

since $\tilde{t} - \tau(x_t) < z_2$ and $\tau(x_t) \leq R_N$. As this violates (1.2b) we have a contradiction to our assumption that there exists no $t_2 \in (z_2, \tilde{t})$ such that $t_2 - \tau(x_{t_2}) = z_2$. Consequently, such a $t_2$ exists, $x(t) < N$ on $(z_2, t_2]$ and $t_2 \leq z_2 + R$ as the previous arguments show.

**Corollary 1.4.** Let the hypotheses of Proposition 1.3 hold. For each $\phi \in \tilde{K} \setminus 0$, $x(t) = x(t, \phi)$ satisfies the following properties.

(i) $x(t)$ is defined for all $t > 0$ and $-M < x(t) < N$ for $t > 0$.

(ii) There exist sequences $\{z_n\}$ and $\{t_n\}$ such that

$$
z_1 < t_1 < z_2 < t_2 < \cdots < z_n < t_n < z_{n+1} < \cdots
$$

satisfying $z_n \to +\infty$ as $n \to \infty$ with the properties:

(a) $\{z_n\} = \{t \geq 0 : x(t) = 0\}$, $x'(z_{2k-1}) < 0$, $x'(z_{2k}) > 0$,

(b) $t_n$ is the unique solution $t$ of $t - \tau(x_t) = z_n$ in $(z_n, z_{n+1})$,

(c) $t - \tau(x_t) \in (z_n, z_{n+1})$ for $n \geq 1$ if and only if $t \in (t_n, t_{n+1})$.

(iii) $e^{
u t} x(t)$ is increasing on $(t_{2k-1}, t_{2k})$ and decreasing on $(t_{2k}, t_{2k+1})$.

**Proof.** The arguments in the proof of Proposition 1.3 can be continued beyond $t_2$, by an induction argument, in order to obtain the sequences $\{z_n\}$ and $\{t_n\}$ where $x(z_n) = 0$, $x'(z_{2k-1}) < 0$, $x'(z_{2k}) > 0$, $t_n - \tau(x_{t_n}) = z_n$, $t - \tau(x_t) \leq z_n$ as $t \leq t_n$, and $-M < x(t) < N$. The keys to extending the results of Proposition 1.3 are Lemma 1.2 and (D2). Lemma 1.2 asserts that for $t > t_2$, $t - \tau(x_t) > z_2$, thus only $x_{t_2}[t_2 - t_2, 0] = x_{t_2}[t - \tau(x_{t_2}), 0]$ is relevant for extending $x(t, \phi)$ to the right of $t_2$. By Lemma 1.2 and (D2), we may as well replace $x_{t_2}$ by $\psi \in C_B$ defined by $\psi[\tau(x_{t_2}), 0] = x_{t_2}[\tau(x_{t_2}), 0]$ and $\psi \equiv 0$ on $(-\infty, -\tau(x_{t_2}))$, in order to extend $x(t, \phi)$ to the right of $t_2$. As $\psi \in \tilde{K}$, we see that Proposition 1.3 itself applies to extend $x(t, \phi)$ to $[t_2, t_4]$. In order to see that $z_n, t_n \to +\infty$ we show that $z_{n+1} - t_n$ is bounded below, independent of $n$. Using the differential equation and the fact that $z_n < t - \tau(x_t) < z_{n+1}$ for $t_n < t < z_{n+1}$ we have

$$
x(t_n) = \int_{t_n}^{t_{n+1}} e^{\nu(t-t_n)} e^{-\eta \tau(x_t)} f(x(t - \tau(x_t))) \, dt.
$$
For definiteness, assume $n$ is even so $x(t_n) > 0$ and $x(t - \tau(x_t)) > 0$ on $t_n < t < z_{n+1}$; the other case is similar. If $p$ is a Lipschitz constant for $f|_{[-M, N]}$, then

$$x(t_n) \leq p \int_{t_n}^{z_{n+1}} e^{\nu(t-t_n)} x(t - \tau(x_t)) \, dt$$

$$\leq p \int_{t_n}^{z_{n+1}} e^{\nu(\tau(x_t)-t_n)} e^{\nu(t-\tau(x_t))} x(t - \tau(x_t)) \, dt$$

$$\leq p \int_{t_n}^{z_{n+1}} e^{\nu(\tau(x_t)-t_n)} e^{\nu t_n} x(t_n) \, dt$$

$$\leq p e^{\nu R} x(t_n)(z_{n+1} - t_n)$$

where we have used the fact that $e^{\nu t} x(t)$ achieves its maximum on $[z_n, z_{n+1}]$ at $t_n$ and $z_n < t - \tau(x_t) < z_{n+1}$. Hence,

$$(1.4) \quad z_{n+1} - t_n \geq p^{-1} e^{-\nu R}$$

for all even $n$. It is now clear that $z_n, t_n \to +\infty$ as $n \to \infty$.

Let $L = \nu \max\{M, N\} + \max\{|f(x)| : -M \leq x \leq N\}$ and define

$$K = \{ \phi \in \bar{K} : \text{lip} \phi \leq L \}.$$

$K$ is a compact, convex subset of $C_B$ containing the zero function. Define $T : K \to C_B$ by

$$(T \phi)(\theta) = \begin{cases} x(t_2(\phi) + \theta), & z_2 - t_2 \leq \theta \leq 0 \\ 0, & \theta < z_2 - t_2 \end{cases}$$

if $\phi \in K \setminus 0$ and $T 0 = 0$. Our choice of $L$ insures that lip $x(\cdot, \phi) \leq L$ for all $\phi \in K$.

**Proposition 1.5.** $T$ maps $K$ into $K$. If $T \phi = \phi$ for some $\phi \in K \setminus 0$, then $x(t, \phi)$ is a nonconstant periodic solution of (1.1) with minimal period $t_2(\phi)$.

**Proof.** If $\phi \in K \setminus 0$, then $0 < x(t) < N$ on $(z_2, t_2)$ so $0 < (T \phi)(\theta) < N$ on $z_2 - t_2 < \theta < 0$ and $(T \phi)(\theta) = 0$ for $\theta \leq z_2 - t_2$. As $t_2 - \tau(x_{t_2}) = z_2$, $z_2 - t_2 \geq -R$ and hence $(T \phi)(\theta) = 0$ for $\theta \leq -R$. Since $e^{\nu t} x(t)$ is increasing on $[z_2, t_2]$, $e^{\nu \theta} x(t_2 + \theta)$ is increasing on $z_2 - t_2 \leq \theta \leq 0$ and so $e^{\nu \theta} (T \phi)(\theta)$ is nondecreasing on $-R \leq \theta \leq 0$. Obviously, $T \phi$ is Lipschitz and lip $(T \phi) \leq L$. It follows that $T \phi \in K$.

If $T \phi = \phi \in K \setminus 0$, then $s_\phi = z_2 - t_2$ and $\phi(\theta) = x(t_2 + \theta)$ for $z_2 - t_2 \leq \theta \leq 0$. As $t_2 - \tau(x_{t_2}) = z_2$ and $\phi|_{[-\tau(x_{t_2}), 0]} = x_{t_2}|_{[-\tau(x_{t_2}), 0]}$ it follows from (D2) that $\tau(x_{t_2}) = \tau(\phi)$ and hence $-\tau(\phi) = s_\phi$. By Lemma 1.2 and the fact that $\phi'(0) + \nu \phi(0) = x'(t_2) + \nu x(t_2) = 0, t - \tau(x_t) > s_\phi$ for $0 < t \leq t_2$. We can now verify that the function $y : \mathbb{R} \to \mathbb{R}$ defined by $y_0 = \phi, y(t) = x(s)$ for $t \geq 0$ where $t = nt_2 + s, 0 \leq s < t_2, n \geq 0$ an integer, is a $t_2$-periodic solution of (1.1). Note that $y_{t_2} = x_{t_2}$. Thus, for $0 \leq s \leq t_2, x_s|_{[-\tau(x_s), 0]} = y_s|_{[-\tau(x_s), 0]}$ so by (D2) $\tau(x_s) = \tau(y_s)$. Then $y_s|_{[-\tau(x_s), 0]} = y_{nt_2+s}|_{[-\tau(x_s), 0]}$ for $n \geq 1$ so
\[ \tau(x_s) = \tau(y_s) = \tau(y_t) \text{ by (D2)}, \text{ where } t = nt_2 + s. \text{ But this implies that} \]

\[ y'(t) = x'(s) = -\nu x(s) - e^{-\eta \tau(x_s)} f(x(s - \tau(x_s))) \]
\[ = -\nu y(t) - e^{-\eta \tau(y_s)} f(y_s(-\tau(x_s))) \]
\[ = -\nu y(t) - e^{-\eta \tau(y_s)} f(y_t(-\tau(y_t))). \]

Thus \( y(t) \) satisfies (1.1) for all \( t \geq 0 \). Now \( y(t) = x(t, \phi) \) for \( t \geq 0 \) by the uniqueness of solutions of initial value problems asserted in Lemma 1.1.

**Proposition 1.6.** \( T : K \rightarrow K \) is continuous.

**Proof.** We begin by establishing an estimate on \( |x(t, \phi) - y(t, \phi)| \) for \( \phi, \psi \in K \). Let \( m = \max \{|f(x)| : -M \leq x \leq N\} \) and \( |f(x) - f(y)| \leq p|x - y| \) for all \( x, y \in [-M, N] \). If \( \phi, \psi \in K \), put \( Z(t) = \sup \{e^{\nu s}|x(s) - y(s)| : -\infty \leq s \leq t\} \), where \( x(t) = x(t, \phi) \) and \( y(t) = x(t, \psi) \). From (1.1) we have

\[ |e^{\nu t}(x(t) - y(t))| \leq |x(0) - y(0)| + \int_0^t e^{\nu s}|e^{-\eta \tau(x_s)} f(x(s - \tau(x_s))) \]
\[ - e^{-\eta \tau(y_s)} f(y(s - \tau(y_s)))| \, ds \]
\[ \leq Z(0) + \int_0^t e^{\nu s}|f(x(s - \tau(x_s)))| e^{-\eta \tau(x_s)} - e^{-\eta \tau(y_s)}| \, ds \]
\[ + \int_0^t e^{\nu s} e^{-\eta \tau(y_s)} |f(x(s - \tau(x_s))) - f(y(s - \tau(y_s)))| \, ds \]
\[ \leq Z(0) + m \eta \int_0^t e^{\nu s}|\tau(x_s) - \tau(y_s)| \, ds + \int_0^t e^{\nu s} p|x(s - \tau(x_s)) - y(s - \tau(y_s))| \, ds \]
\[ \leq Z(0) + V m \eta \int_0^t e^{\nu s} \max_{-R \leq r \leq 0} |x(s + r) - y(s + r)| \, ds \]
\[ + p \int_0^t e^{\nu s}|x(s - \tau(x_s)) - x(s - \tau(y_s))| \, ds \]
\[ + p \int_0^t e^{\nu s}|x(s - \tau(y_s)) - y(s - \tau(y_s))| \, ds \]
\[ \leq Z(0) + V m \eta e^{\nu R} \int_0^t Z(s) \, ds + p \int_0^t L e^{\nu s}|\tau(x_s) - \tau(y_s)| \, ds \]
\[ + p e^{\nu R} \int_0^t Z(s) \, ds \]
\[ \leq Z(0) + [V m \eta e^{\nu R} + p e^{\nu R} + p L V e^{\nu R}] \int_0^t Z(s) \, ds. \]

Hence,

\[ Z(t) \leq Z(0) + A \int_0^t Z(s) \, ds \]

and Gronwall's inequality yields \( Z(t) \leq Z(0)e^{A t} \) where \( A = e^{\nu R}[V m \eta + p + p L V] \).

This implies

\[ |x(t) - y(t)| \leq \max_{-R \leq s \leq 0} e^{\nu s} |\phi(s) - \psi(s)| e^{(A - \nu) t} \text{ for } t \geq 0. \]
Hence, for each $\bar{t} > 0$, $x(t, \phi)$ is uniformly close to $x(t, \psi)$ on $[0, \bar{t}]$ if $\phi$ is uniformly close to $\psi$ on $[-R, 0]$ and $\phi, \psi \in K$.

It is now easy to establish the continuity of the maps $\phi \to z_i(\phi)$, $i = 1, 2$, at each $\phi \in K \setminus \{0\}$, using the fact that these are simple zeros of $x(t, \phi)$ and that the $z_i$ are uniformly bounded from above. One can then use the fact that $t_2(\phi)$ is the unique solution of $t - \tau(x_t(\phi)) = z_2(\phi)$ together with the continuity of $\tau$ to show that $\phi \to t_2(\phi)$ is also continuous at each $\phi \in K \setminus \{0\}$. The continuity of $T$ on $K$ follows easily from this.

**Proposition 1.7.** If $T \phi = \phi \in K \setminus \{0\}$, then the $t_2(\phi)$-periodic solution $x(t, \phi)$ has the following properties:

(i) $x(t)$ has precisely two zeros in $[0, t_2]$, namely $z_1$ and $z_2$, $0 < z_1 < z_2 < t_2$ with $x'(z_1) < 0$ and $x'(z_2) > 0$,

(ii) $e^{\nu t} x(t)$ has two extrema in $(0, t_2]$, namely $t_1$ and $t_2$, $0 < z_1 < t_1 < z_2 < t_2$ with $e^{\nu t} x(t)$ increasing on $(t_1, t_2)$ and decreasing on $(0, t_1)$,

(iii) $-M < x(t) < N$ for all $t$,

(iv) If $\tau(x_t) = \tau(x(t))$, then $z_2 - z_1 > 1$ and $t_2 > 2$.

**Proof.** (i), (ii) and (iii) follow from Proposition 1.3 and the periodicity of $x(t)$. If $\tau(x_t) = \tau(x(t))$, then since $\tau(0) = 1$ and $z_2 - \tau(x(z_2)) = z_2 - 1 \in (z_1, z_2)$, we have $z_2 - z_1 > 1$. Hence $t_2 = z_2 - s_\phi = z_2 - z_1 + z_1 - s_\phi > 1 + 1 = 2$ as $z_1 - s_\phi > 1$ by the same argument as above.

Finally, we assume

(H) $e^{-\eta} f'(0) > \alpha_\nu$ where $\alpha_\nu$ is the smallest positive solution of

$$\nu + \alpha \cos \sqrt{\alpha^2 - \nu^2} = 0.$$

As noted in [15], $\alpha_\nu \geq \pi/2 = \alpha_0$ and $\alpha_\nu$ increases with $\nu$. With this assumption we can prove the following Lemma, using classical ideas of ([35], [29], [30], [15]), with slight modifications.

**Lemma 1.8.** Assume that (H) holds. Then there exists a positive number $a$, independent of $\phi \in K \setminus \{0\}$ such that

$$\sup_{t \geq z_n} |x(t, \phi)| \geq a$$

for all positive integers $n$ where $z_n$ is the $n$'th positive zero of $x(t, \phi)$.

**Proof.** Let $\alpha = e^{-\eta} f'(0)$. Hypothesis (H) implies, see [15], that the characteristic equation $\nu + \lambda + \alpha e^{-\lambda} = 0$ has a root $\lambda = \mu + i\gamma$ satisfying $\mu > 0$ and $\frac{\pi}{2} < \gamma < \pi$. Express (1.1) as

$$x'(t) = -\nu x(t) - \alpha x(t - 1) + g(x_t) + h(x_t) + k(x_t)$$

where $g(x_t) = \alpha x(t - 1) - e^{-\eta} f(x(t - 1))$, $h(x_t) = [e^{-\eta} - e^{-\eta \tau(x_t)}] f(x(t - 1))$ and $k(x_t) = e^{-\eta \tau(x_t)} [f(x(t - 1)) - f(x(t - \tau(x_t)))]$. Choose $\epsilon > 0$ such that $\epsilon < \frac{1}{2} \mu \cos \frac{\pi}{2} e^{-3\nu/2}$ and choose $\alpha$ such that $|\alpha x - e^{-\eta} f(x)| \leq \frac{\epsilon}{3} |x|$ if $|x| \leq \alpha$. We will choose $\alpha$ still smaller below. Let $p$ be a Lipschitz constant for $f([-M, N])$. 
If $\phi \in K \setminus 0$, then $-M \leq x(t, \phi) \leq N$ so we can estimate the functions $h(x_t), k(x_t)$ along the solution by

$$|h(x_t)| \leq p|x(t-1)||x(0) - x(t)| \leq p|x(t-1)|\eta|x(t) - \tau(0)|$$
$$\leq p\eta V|x(t-1)|\max_{-Q \leq s \leq 0}|x(t+s)|,$$

and

$$|k(x_t)| \leq p|x(t-1)| - x(t) - \tau(x_t)| \leq p|x''(x)|\max_{-Q \leq s \leq 0}|x(t+s)|$$
$$\leq p|x''(x)|V\max_{-Q \leq s \leq 0}|x(t+s)|$$
$$\leq pV\max_{-Q \leq s \leq 0}|x(t+s)||\nu|x(x)| + p|x(x - \tau(x))||$$
$$\leq pV(\nu + p)\max_{-Q \leq s \leq 0}|x(t+s)|\max\{|x(x)|, |x(x - \tau(x))||\},$$

where $\xi$ is a point between $t - 1$ and $t - \tau(x_t)$ and $Q = \max\{1, \tau(x_t)\} \leq R$. Choose a smaller if necessary so that the following hold:

\begin{equation}
(1.5) \quad p\eta V a \leq \frac{\varepsilon}{3},
\end{equation}

\begin{equation}
(1.6) \quad pV(\nu + p)a \leq \frac{\varepsilon}{3},
\end{equation}

\begin{equation}
(1.7) \quad pVe^{3\nu/2} \leq \frac{1}{3}\cos\frac{\gamma}{2}e^{-\mu/2},
\end{equation}

\begin{equation}
(1.8) \quad aV \leq \frac{1}{2},
\end{equation}

\begin{equation}
(1.9) \quad apV < e^{-\nu R},
\end{equation}

\begin{equation}
(1.10) \quad (\nu + p)aVe^{3\nu/2} \leq \frac{1}{3},
\end{equation}

and

\begin{equation}
(1.11) \quad \gamma(\frac{1}{2} + aV) < \frac{\pi}{2}.
\end{equation}

Using (1.5) and (1.6) and the estimates above, we have

\begin{equation}
(1.12) \quad |g(x_t) + h(x_t) + k(x_t)| \leq \frac{\varepsilon}{3}|x(t-1)| + \frac{\varepsilon}{3}|x(t-1)| + \frac{\varepsilon}{3}\max_{-Q \leq s \leq 0}|x(t+s)|
\end{equation}

provided $\max_{2R \leq s \leq 0}|x(t+s)| \leq a$ since $\xi$ and $\xi - \tau(x) \geq t - 2R$.

If $\sup_{t \geq z_n}|x(t, \phi)| \geq a$ fails to hold for some integer, then clearly it fails to hold for all larger integers so we may choose an $n$ such that $\sup_{t \geq z_n}|x(t, \phi)| = \delta < a$ and such that $|x(t)| \leq a$ for $t \geq z_n - 2R$ and such that there exists $t_0 \in (z_n, z_{n+1})$ for which $|x(t_0)| = \max_{z_n < t < z_{n+1}}|x(t)| \geq 3\delta/4$ and $x'(t_0) = 0$. We assume $x(t_0) > 0$ for definiteness; the other case is handled similarly. As $x'(t) < 0$ on $[t_n, z_{n+1}]$, it follows that $t_0 < t_n$. 
We first argue that \( z_{n+1} - z_n > 1 \). From (1.4), \( z_{n+1} - t_n \geq p^{-1}e^{-\nu R} \). As \( t_n - z_n = \tau(x_{t_n}) \geq 1 - aV \) by (D3), we have
\[
z_{n+1} - z_n \geq p^{-1}e^{-\nu R} + 1 - aV > 1
\]
by (1.9).

Now put \( T = 1 + z_n \in (z_n, z_{n+1}) \). The estimate
\[
(1.13) \quad |T - t_n| = |\tau(0) - \tau(x_{t_n})| \leq V a
\]
will be used routinely.

We estimate \( x(t_n) \) by noting that \( e^{\nu t}x(t) \) is increasing on \([z_n, t_n]\) and \( t_0 \) is a point in this interval. Hence,
\[
x(t_n) \geq x(t_0)e^{-\nu(t_n-t_0)} \geq x(t_0)e^{-\nu(t_n-z_n)} = x(t_0)e^{-\nu\tau(x_{t_n})}
\]
\[
\geq x(t_0)e^{-\nu(1+Va)} \geq x(t_0)e^{-3\nu/2} \geq \frac{3\delta}{4}e^{-3\nu/2},
\]
where (1.8) was used.

Now we estimate \( x(T) \) as follows
\[
S = \frac{|x(T) - x(t_n)|}{x(t_n)} = \frac{|x'(s)||T - t_n|}{x(t_n)} = \frac{|x'(s)| |1 + z_n - t_n|}{x(t_n)} \leq (\nu + p) \max\{|x'(s)|, |x(s - \tau(x_s))|\} \frac{|1 - \tau(x_{t_n})|}{x(t_n)}
\]
where \( s \) lies between \( t_n \) and \( 1 + z_n \). If \( t_n < 1 + z_n \), then \( s > t_n \) so \( s - \tau(x_s) > z_n \) by Corollary 1.4 and \( 0 < x(s - \tau(x_s)) < x(t_0) \). Hence
\[
S \leq (\nu + p)x(t_0) \frac{Va}{x(t_n)} \leq (\nu + p)Vae^{3\nu/2} \leq \frac{1}{3}
\]
where we have used (1.14). If \( t_n \geq 1 + z_n \), then \( \tau(x_{t_n}) = t_n - z_n \geq 1 \) so \(|\tau(x_{t_n}) - 1| \leq V \max_{-\tau(x_{t_n}) \leq r \leq 0} |x(t_n + r)| \leq V x(t_0) \). Hence, again
\[
S \leq (\nu + p)a \frac{Vx(t_0)}{x(t_n)} \leq (\nu + p)Vae^{3\nu/2} \leq \frac{1}{3}.
\]
The last inequality follows from (1.10). Finally, we have
\[
|x(T)| \geq |x(t_n)||1 - S| \geq \frac{2}{3}|x(t_n)| \geq \frac{\delta}{2}e^{-3\nu/2}.
\]
Now we proceed as in [15]. If \( \lambda \) is the root of the characteristic equation as described above, then an integration by parts yields
\[
\int_T^\infty x'(t)e^{-\lambda t} \, dt = -x(T)e^{-\lambda T} + \lambda \int_T^\infty x(t)e^{-\lambda t} \, dt.
\]
Using the differential equation in the left side, we get
\[
\int_T^\infty x'(t)e^{-\lambda t} \, dt = \int_T^\infty \left[-\nu x(t) - \alpha x(t - 1) + g(x_t) + h(x_t) + k(x_t)\right]e^{-\lambda t} \, dt
\]
\[
= -\nu \int_T^\infty x(t)e^{-\lambda t} \, dt - \alpha e^{-\lambda} \int_{T-1}^\infty x(t)e^{-\lambda t} \, dt + \int_T^\infty \left[g + h + k\right]e^{-\lambda t} \, dt
\]
\[
= \nu \int_{T-1}^T x(t)e^{-\lambda t} \, dt + \lambda \int_{T-1}^\infty x(t)e^{-\lambda t} \, dt + \int_T^\infty \left[g + h + k\right]e^{-\lambda t} \, dt
\]
where we have used the characteristic equation \(\nu + \alpha e^{-\lambda} = -\lambda\). Putting this in the equality above, we have
\[
-x(T)e^{-\lambda T} - (\lambda + \nu) \int_{T-1}^T x(t)e^{-\lambda t} \, dt = \int_T^\infty \left[g + h + k\right]e^{-\lambda t} \, dt.
\]
Writing the integrand in the integral on the left as \([x(t)e^{\nu t}]e^{-(\lambda + \nu)t}\) and integrating by parts, using \(x(T - 1) = 0\), gives
\[
-\int_{T-1}^T y'(t)e^{-(\lambda + \nu)t} \, dt = \int_T^\infty \left[g + h + k\right]e^{-\lambda t} \, dt.
\]
where \(y(t) = x(t)e^{\nu t}\). Finally, multiply through by \(e^{\lambda(T-1)/2}\) to get
\[
\int_{T-1}^T y'(t)e^{-\lambda(t-T+1/2)}e^{-\nu t} \, dt = -\int_T^\infty \left[g + h + k\right]e^{-\lambda(t-T+1/2)} \, dt. 
\]
(1.15)

There are two cases to consider. If \(T = 1 + z_n \leq t_n\), then \(y' \geq 0\) on \([T - 1, T] = [z_n, 1 + z_n]\) so
\[
\text{Re} \int_{T-1}^T y'(t)e^{-\lambda(t-T+1/2)}e^{-\nu t} \, dt
\]
\[
= \int_{T-1}^T y'(t)e^{-\mu(t-T+1/2)} \cos \gamma(t - T + 1/2)e^{-\nu t} \, dt
\]
\[
\geq e^{-\mu/2} e^\gamma T \left[ y(T) - y(T - 1) \right]
\]
\[
\geq e^{-\mu/2} e^\gamma T\frac{\gamma}{2} x(T)
\]
\[
\geq e^{-\mu/2} e^\gamma T\frac{\gamma}{2} e^{-3\nu/2}.
\]

On the other hand, estimating the real part of the right side of (1.15),
\[
\text{Re} \left[ -\int_T^\infty \left[g + h + k\right]e^{-\lambda(t-T+1/2)} \, dt \right]
\]
\[
\leq \int_T^\infty \left[|g| + |h|\right]e^{-\mu(t-T+1/2)} \, dt - \int_T^{t_n} ke^{-\mu(t-T+1/2)} \cos \gamma(t - T + 1/2) \, dt
\]
\[
+ \int_{t_n}^\infty |k|e^{-\mu(t-T+1/2)} \, dt.
\]
The integrand in the second integral is nonnegative on \([T, t_n]\) since on that interval \(t - \tau(x_t) < z_n < t - 1\) so \(f(x(t-1)) > 0\) and \(f(x(t-\tau(x_t))) < 0\) (see the definition of \(k\)). Also note that \(\cos \gamma (t - T + 1/2) > 0\) on \([T, t_n]\) by virtue of (1.11), \(\pi/2 < \gamma < \pi\) and (1.13). Using (1.12) in the above estimate yields

\[
\text{Re} \left[ - \int_T^\infty [g + h + k] e^{-\lambda(t-T+1/2)} dt \right] \leq \frac{\epsilon}{3} \frac{\delta e^{-\mu/2}}{\mu} + \frac{\epsilon}{3} \frac{\delta e^{-\mu/2}}{\mu} + \frac{\epsilon}{3} \frac{\delta e^{-\mu/2}}{\mu} = \frac{\epsilon \delta e^{-\mu/2}}{\mu},
\]

where we have used the fact that \(t - \tau(x_t) > z_n\) for \(t > t_n\) and thus \(|k| \leq pV(\nu + p)a \max_{\tau(x_t) \leq s \leq 0} |x(t + s)| \leq \frac{\epsilon}{5} \delta\) by (1.6). But these two estimates imply that

\[
\frac{\delta}{2} e^{-\mu/2} \cos \gamma e^{-3\nu/2} \leq \epsilon \delta e^{-\mu/2} \mu^{-1}
\]

contradicting our choice of \(\epsilon\).

If \(T > t_n\), then \(y' \geq 0\) on \([T - 1, t_n]\) and \(y' \leq 0\) on \([t_n, T]\) so we must estimate the two sides of (1.15) as follows.

\[
I = \text{Re} \int_T^{T-1} y'(t)e^{-\lambda(t-T+1/2)} e^{-\nu t} dt
\]

\[
= \int_{T-1}^{t_n} y'(t)e^{-\mu(t-T+1/2)} \cos \gamma(t - T + 1/2)e^{-\nu t} dt
\]

\[
+ \int_{t_n}^T y'(t)e^{-\mu(t-T+1/2)} \cos \gamma(t - T + 1/2)e^{-\nu t} dt
\]

\[= I_1 + I_2.
\]

We may estimate \(I_1\) as above,

\[
I_1 \geq \cos \gamma \frac{\nu}{2} e^{-\mu/2} e^{-\nu t}n y(t_n) = \cos \gamma \frac{\nu}{2} e^{-\mu/2} x(t_n).
\]

To estimate \(I_2\), note that by (1.13),

\[
e^{-\mu(t-T+1/2)} \cos \gamma(t - T + 1/2) \leq e^{-\mu(t_n-T+1/2)} \leq 1.
\]

Therefore

\[
I_2 \geq \int_{t_n}^T y'(t)e^{-\nu t} dt = \int_{t_n}^T e^{-\eta\tau(x_t)} f(x(t - \tau(x_t))) dt
\]

\[
\geq \int_{t_n}^T -px(t - \tau(x_t)) dt \geq -px(t_0)[T - t_n]
\]

\[
\geq -pe^{3\nu/2} x(t_n)[T - t_n] \geq -paV e^{3\nu/2} x(t_n),
\]
where we used (1.14) and \( t - \tau(x_t) \geq z_n \) for \( t_n \leq t \leq T \). Putting the estimates of \( I_1 \) and \( I_2 \) together and using (1.7) and (1.14), we have

\[
I \geq (\cos \frac{\gamma}{2} e^{-\mu/2} - p_a V e^{3\nu/2}) x(t_n) \\
\geq \frac{2}{3} \cos \frac{\gamma}{2} e^{-\mu/2} x(t_n) \\
\geq \frac{\delta}{2} \cos \frac{\gamma}{2} e^{-\mu/2} e^{-3\nu/2}.
\]

Note that this estimate is the same estimate obtained in the previous case.

We can estimate the real part of the right-hand side of (1.15), in this case, using (1.2) since \( T - Q \geq z_n \)

\[
\text{Re} \left[ -\int_T^{\infty} (g + h + k) e^{-\lambda(t-T+1/2)} \, dt \right] \\
\leq \int_T^{\infty} [||g|| + ||h|| + ||k||] e^{-\mu(t-T+1/2)} \, dt \\
\leq \int_T^{\infty} \left[ \frac{2\epsilon}{3} |x(t-1)| + \frac{\epsilon}{3} \max_{-Q \leq s \leq 0} |x(t+s)| \right] e^{-\mu(t-T+1/2)} \, dt \\
\leq \epsilon \frac{e^{-\mu/2}}{\mu}.
\]

Thus we reach the same contradiction as before.

**THEOREM 1.9.** If (H) holds, then \( T : K \rightarrow K \) has a nonzero fixed point. In particular, (1.1) has a nonconstant periodic solution with the properties described in Proposition 1.7.

**PROOF.** We use the theorem of Browder [5] (see also [18], Thm. 11.2.4) on the existence of a nonejective fixed point, showing that 0 is an ejective point of \( T \) and therefore there must exist a nonzero fixed point of \( T \). The point 0 of \( K \) is an ejective point if there is an open neighborhood \( G \) of 0 in \( C_B \) such that for every \( \phi \in G \cap K \), \( \phi \neq 0 \), there is an integer \( m = m(\phi) \) such that \( T^m \phi \notin G \cap K \). Recall that \( K \) is a compact and convex subset of \( C_B \). If \( \phi \in K \setminus 0 \) we denote by \( r_n \), a point of \((z_n, z_{n+1})\) at which \( |x(t, \phi)| \) achieves its maximum value, \( n \geq 1 \). Then \( ||T^n \phi|| = x(r_{2n}, \phi), n \geq 1 \). Lemma 1.8 implies that \( |x(r_n, \phi)| \geq a/2 \) for infinitely many values of \( n \), if \( \phi \in K \setminus 0 \). As \( r_n < t_n \), an estimate as in (1.14) gives

\[
|x(t_n, \phi)| \geq e^{-\nu R} |x(r_n, \phi)|.
\]

Arguing as in the estimates yielding (1.4),

\[
x(t_{2n+1}) = \int_{x_{2n+1}}^{t_{2n+1}} e^{\nu(t-t_{2n+1})} e^{-\eta \tau(x_t)} f(x(t - \tau(x_t))) \, dt
\]
so that, as \( z_{2n} < t - \tau(x_t) < z_{2n+1} \) for \( z_{2n+1} < t < t_{2n+1} \),

\[
|z(t_{2n+1})| \leq p \int_{t_{2n+1}}^{z_{2n+1}} e^{\nu(t-x_t)} e^{\nu(t-\tau(x_t))}|z(t - \tau(x_t))| dt \\
\leq px(t_{2n})e^{\nu R} e^{t_{2n} - t_{2n+1}}(z_{2n+1} - t_{2n+1}) \\
\leq pRe^{\nu R}x(t_{2n}).
\]

Hence,

\[
x(r_{2n}) \geq x(t_{2n}) \geq (pRe^{\nu R})^{-1}|x(t_{2n+1})| \\
\geq (pRe^{\nu R})^{-1}e^{-\nu R}|x(r_{2n+1})|.
\]

It follows that \( x(r_{2n}) \geq (pRe^{\nu R})^{-1}e^{-\nu R}/2 \) for infinitely many values of \( n \), proving that 0 is an ejective fixed point of \( T \). This proves the Theorem.

### 2. Generalized Threshold Delays

In this section we show that our main result applies to (1.1) in the case that \( \tau(x_t) \) is determined implicitly by

\[
(2.1) \quad \int_{t-\tau}^{t} k(x(t), x(s)) ds = m
\]

where \( k : \mathbb{R}^2 \to (0, \infty) \) is locally Lipschitz and \( m > 0 \). Dividing through by \( m \), incorporating \( m \) into \( k \) and rescaling time, it may be assumed that

\[
m = 1 \quad \text{and} \quad k(0, 0) = 1.
\]

We assume the above normalization throughout this section. The delay (2.1) was treated extensively in [23] and we will quote results from there as necessary.

The delay appearing in (2.1) can be viewed as a delay functional \( \tau : C_B \to (0, \infty) \) defined implicitly by the formula

\[
(2.2) \quad \int_{-\tau(\phi)}^{0} k(\phi(0), \phi(s)) ds - 1 = 0.
\]

**Lemma 2.1.** Given \( L > 0 \), let \( p, q, m, M > 0 \), depending on \( L \), be such that

\[
|k(x, y) - k(u, v)| \leq p|x - u| + q|y - v|
\]

and

\[
m \leq k(x, y) \leq M
\]

for all \((x, y), (u, v)\) such that \(|x|, |y|, |u|, |v| \leq L\). Then (2.2) defines a functional \( \tau : C_B \to (0, \infty) \) satisfying:

(i) \( \tau(0) = 1 \),

(ii) \( ||\phi|| \leq L \) implies \( M^{-1} \leq \tau(\phi) \leq m^{-1} \),

(iii) There exists \( P = P(L) > 0 \) such that if \( ||\phi||, ||\psi|| \leq L \), then

\[
|\tau(\phi) - \tau(\psi)| \leq P \max_{-Q \leq s \leq 0} |\phi(s) - \psi(s)|
\]

where \( Q = \max\{\tau(\phi), \tau(\psi)\} \) and \( \phi, \psi \in C_B \).
PROOF. See ([23], Lemma 2.1 and its proof).

As a consequence of Lemma 2.1 and the definition of $\tau(\phi)$, the latter satisfies hypotheses (D1), (D2) and (D3).

Hypothesis (B) concerning the choice of $M, N > 0$ such that solutions $x(t, \phi)$ corresponding to $\phi \in \hat{K}(M, N)$ satisfy $-M < x(t, \phi) < N$, involves $\tau$ only through the numbers $R_M, R_N$. It is easy to verify that

$$R_N = \left[ \min_{-M \leq y \leq N} k(N, y) \right]^{-1},$$

$$R_M = \left[ \min_{-M \leq y \leq N} k(-M, y) \right]^{-1}.$$

The next Lemma provides sufficient conditions for the delay defined by (2.1) to satisfy (D4). The sufficient condition, (2.3), may be easily derived from formally differentiating (2.1) with respect to $t$.

**Lemma 2.2.** In addition to the assumptions above, assume that $\frac{\partial k}{\partial x}(x, y)$ exists when $x \neq 0, x, y \in [-M, N]$ and is continuous. If

$$\nu x \frac{\partial k}{\partial x}(x, y) < k(x, x)k(x, y), \quad -M \leq x, y \leq N, \quad x \neq 0,$$

then $\tau$ satisfies (D4).

**Proof.** The proof follows a similar one in [23]. Let $\phi \in C_B$ be Lipschitz, take values in $[-M, N]$ and satisfy $\phi(-\tau(\phi)) = 0$. Actually, we do not use the last assumption on $\phi$ nor that it is Lipschitz. Suppose that $y$ is an extension of $\phi$ as in (D4) satisfying $y(0) + \nu y(0) = 0$. Let $\tau_0 = \tau(\phi)$ and $\tau(y_t) = \tau_0 + r(t)$ for simplicity in notation. Then from

$$1 = \int_{-\tau_0}^{0} k(\phi(0), \phi(s)) \, ds = \int_{-\tau_0 - r(t)}^{0} k(y(t), y(t + s)) \, ds$$

we have

$$\int_{-\tau_0}^{0} k(\phi(0), \phi(s)) \, ds = \int_{-\tau_0 - r}^{-t} k(y(t), y(t + s)) \, ds + \int_{-t}^{0} k(y(t), y(t + s)) \, ds$$

$$= \int_{t - \tau_0 - r}^{0} k(y(t), \phi(s)) \, ds + \int_{-t}^{0} k(y(t), y(t + s)) \, ds.$$

This leads to

$$0 = \int_{-\tau_0}^{0} t^{-1} [k(y(t), \phi(s)) - k(\phi(0), \phi(s))] \, ds + t^{-1} \int_{t - \tau_0 - r}^{-\tau_0} k(y(t), \phi(s)) \, ds$$

$$+ t^{-1} \int_{-t}^{0} k(y(t), y(t + s)) \, ds.$$

Letting $t \to 0+$, we find, assuming $\phi(0) \neq 0$, that

$$\lim_{t \to 0^+} t^{-1} \int_{t - \tau_0 - r}^{-\tau_0} k(y(t), \phi(s)) \, ds = \int_{-\tau_0}^{0} \nu \phi(0) \frac{\partial k}{\partial x}(\phi(0), \phi(s)) \, ds - k(\phi(0), \phi(0)).$$

(2.4)
If $\phi(0) = 0$, then as $y'(0) = 0$ and $k$ is Lipschitz, the right side of (2.4) is $-k(\phi(0), \phi(0))$. If we can show that the right-hand side is negative then it follows that $t - \tau(t) > 0$ for small positive $t$ and hence $\tau(y_t) - \tau(\phi) = \tau(t) - t$ for small positive $t$. Now, using (2.3)

$$\int_{-\tau_0}^0 \nu(0) \frac{\partial k}{\partial x}(\phi(0), \phi(s)) \, ds$$

$$= \int_{-\tau_0}^0 \nu(0) \left( \frac{\partial k}{\partial x}(\phi(0), \phi(s))/k(\phi(0), \phi(s)) \right) k(\phi(0), \phi(s)) \, ds$$

$$< \int_{-\tau_0}^0 k(\phi(0), \phi(0)) k(\phi(0), \phi(s)) \, ds$$

$$= k(\phi(0), \phi(0)).$$

Hence the integral on the right side of (2.4) is negative. Thus the first requirement of (D4) holds. The second is established in a similar manner.

In the special case that

$$k(x, y) = \frac{1}{\sigma(x)} \quad \text{i.e.} \quad \tau(x_t) = \sigma(x(t))$$

then (2.3) becomes

$$\nu x \sigma'(x) > -1 \quad \text{for} \quad x \neq 0.$$ 

In case

$$k(x, y) = k(y) \quad \text{so} \quad \int_{-\tau(x_t)}^0 k(x(t + s)) \, ds = 1$$

then (2.3) is automatically satisfied since $\frac{\partial k}{\partial x} = 0$.

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