SHARP CONDITIONS FOR OSCILLATIONS IN SOME NONLINEAR NONAUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The oscillatory behavior of the first order delay differential equation has been the subject of many recent investigations. Of particular importance, however, has been the study of oscillations which are caused by retarded arguments and which do not appear in the corresponding ordinary differential equations. In addition to its theoretical interest, this question is also important from the viewpoint of practical applications. This is evidenced by the extensive references in the recent books of Ladde et al. [1] and Győri and Ladas [2].

Our aim in this paper is to study the oscillation of first order nonlinear nonautonomous delay differential equation

\[ x(t) + p(t)f(x(t - \tau(t))) = 0 \]  \hspace{1cm} (1.1)

where

\[ p \in C([0, \infty), [0, \infty)), \hspace{1cm} \tau \in C([0, \infty), (0, \infty)), \hspace{1cm} \lim_{t \to \infty} (t - \tau(t)) = \infty, \]  \hspace{1cm} (1.2)

and

\[ f \in C(\mathbb{R}, \mathbb{R}) \hspace{1cm} \text{and} \hspace{1cm} uf(u) > 0 \text{ for } u \neq 0. \]  \hspace{1cm} (1.3)

For any \( T > 0 \), we define

\[ T_{-1} = \inf_{t \geq T} \{ t - \tau(t) \}. \]  \hspace{1cm} (1.4)

By a solution of equation (1.1) we mean a function \( x \) which is continuously differentiable on \([T, \infty)\), for some \( T \geq 0 \), which is defined and continuous on \([T_{-1}, \infty)\) and which satisfies equation (1.1) for \( t \geq T \).

With equation (1.1) and with a given “initial point” \( T \geq 0 \), one associates an “initial condition” of the form

\[ x(t) = \phi(t) \hspace{1cm} \text{for} \hspace{1cm} T_{-1} \leq t \leq T \]  \hspace{1cm} (1.5)

where \( \phi \in C([T_{-1}, T], \mathbb{R}) \) is a given “initial function”. Then the initial value problem (1.1) and (1.5) has a unique solution \( x \) valid on \([T, \infty)\). That is, \( x \) is defined and continuous on \([T_{-1}, \infty)\), \( x \) is continuously differentiable on \([T, \infty)\), \( x \) satisfies (1.5) and \( x \) satisfies equation (1.1) for \( t \geq T \). See Theorem 1.1.2 in [2].

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As usual, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory. A solution \( x(t) \) of equation (1.1) is said to oscillate about a constant \( x^* \) if \( x(t) - x^* \) has arbitrarily large zeros.

In a recent paper [3], Li obtained an infinite integral condition for oscillation of the linear delay differential equation

\[
 x(t) + p(t)x(t - \tau) = 0
\]

where \( p \in C([0, \infty), [0, \infty)) \) and \( \tau \) is a positive constant. Here we extend this result to nonlinear delay differential equation (1.1). More precisely, we will establish the following result.

**Theorem 1.** Assume that (1.2) and (1.3) hold and that for some \( \varepsilon > 0, M \geq 0, r > 0 \)

\[
 |f(u) - u| \leq M|u|^{1+r} \quad \text{for} \quad u \in (-\varepsilon, \varepsilon).
\]  \hspace{1cm} (1.6)

Suppose that there exists \( t_0 > 0 \) such that

\[
 \int_{\delta(t)}^{t} p(s) \, ds \geq \frac{1}{e} \quad \text{for} \quad t \geq t_0 \quad \text{(1.7)}
\]

and

\[
 \int_{t_0}^{\infty} p(t) \left[ \exp \left( \int_{\delta(t)}^{t} p(s) \, ds - \frac{1}{e} \right) - 1 \right] \, dt = \infty \quad \text{(1.8)}
\]

where \( \delta(t) = \max_{s \in [0, \tau]} \{ s - \tau(s) \} \). Then every solution of equation (1.1) oscillates.

The proof of Theorem 1 is given in Section 2. Section 3 contains some applications and a brief discussion about linearized oscillation.

**2. The Proof of Theorem 1**

First we establish some basic lemmas.

**Lemma 1.** Assume that (1.2), (1.3) and (1.6) hold, and

\[
 \int_{0}^{\infty} p(t) \, dt = \infty. \quad \text{(2.1)}
\]

Then every nonoscillatory solution of equation (1.1) converges to zero monotonically for large \( t \) as \( t \to \infty \).

**Proof:** Suppose that \( x(t) \) is a nonoscillatory solution of equation (1.1) which we shall assume to be eventually positive [if \( x(t) \) is eventually negative the proof is similar]. Then there exists \( t_1 \geq 0 \) such that \( x(t) > 0 \) for \( t \geq t_1 \). It follows from equations (1.1), (1.2) and (1.3) that there exists \( t_2 > t_1 \) such that \( t_2 - \tau(t_2) > t_1 \) and \( x(t) \leq 0 \) for \( t > t_2 \). Hence

\[
 \lim_{t \to \infty} x(t) = \alpha \geq 0 \quad \text{exists.} \quad \text{(2.2)}
\]

If \( \alpha > 0 \), there exists \( T \) such that for \( t \geq T \), \( f(x(t) - \tau(t)) \geq f(\alpha)/2 > 0 \). Then for \( t \geq T \)

\[
 x(t) - x(T) \leq -\frac{1}{2} f(\alpha) \int_{T}^{t} p(s) \, ds \quad \text{(2.3)}
\]
From this and (2.1), we find
\[
\lim_{t \to \infty} x(t) = -\infty
\]
contradicting that \(x(t)\) is eventually positive. The proof of Lemma 1 is complete. \(\blacksquare\)

**Lemma 2.** Assume that (1.2), (1.3), (1.6) and (2.1) hold. If \(x(t)\) is a nonoscillatory solution of equation (1.1), there exist \(A > 0\) and \(T \in (0, \infty)\) such that for \(t \geq T\)
\[
|x(t)| \leq A \exp\left(-\frac{1}{2} \int_T^t p(s) \, ds\right).
\] (2.4)

**Proof.** We shall assume \(x(t)\) to be eventually positive [if \(x(t)\) is eventually negative the proof is similar]. By Lemma 1, there exists \(t_1 > 0\) such that
\[
0 < x(t) \leq x(t - \tau(t)) < \varepsilon \quad \text{for} \quad t \geq t_1.
\] (2.5)

From (1.6), we find that for \(t \geq t_1\)
\[
f(x(t - \tau(t))) \geq x(t - \tau(t)) - Mx^{1+r}(t - \tau(t)).
\] (2.6)

On the other hand, by the mean value theorem of differential calculus
\[
x^{1+r}(t - \tau(t)) - x^{1+r}(t) = (1 + r)\xi^r(t)[x(t - \tau(t)) - x(t)]
\] (2.7)
where \(\xi(t)\) is between \(x(t)\) and \(x(t - \tau(t))\). By Lemma 1, \(\lim_{t \to \infty} \xi(t) = 0\). It follows from (2.7) that
\[
[x(t - \tau(t)) - Mx^{1+r}(t - \tau(t))] - [x(t) - Mx^{1+r}](1 + r)\xi^r(t) = 0
\]
for \(t\) sufficiently large. By this, (2.6) and equation (1.1), there exists \(t_2 > t_1\) such that
\[
\dot{x}(t) + p(t)[x(t) - Mx^{1+r}(t)] \leq 0 \quad \text{for} \quad t \geq t_2
\]
or
\[
\dot{x}(t) + p(t)[1 - Mx(t)]x(t) \leq 0 \quad \text{for} \quad t \geq t_2.
\] (2.8)

Since \(\lim_{t \to \infty} x(t) = 0\), there exists \(T \geq t_2\) such that
\[
\dot{x}(t) + \frac{1}{p(t)}x(t) \leq 0 \quad \text{for} \quad t \geq T.
\] (2.9)

This yields
\[
x(t) \leq A \exp\left(-\frac{1}{2} \int_T^t p(s) \, ds\right) \quad \text{for} \quad t \geq T
\] (2.10)
where \(A = x(T)\). The proof of Lemma 2 is complete. \(\blacksquare\)

**Lemma 3.** Assume that (1.2), (1.3), (1.6) and (2.1) hold. If equation (1.1) has a nonoscillatory solution, then
\[
\int_{\delta(t)}^t p(s) \, ds \leq 2
\] (2.11)
eventually.

**Proof.** Let us suppose that \(x(t)\) is a nonoscillatory solution of equation (1.1) which we shall assume to be eventually positive [if \(x(t)\) is eventually negative the proof is similar]. By Lemma 1, there exists \(t_1 > 0\) such that
\[
x(\delta(\delta(t))) \geq x(\delta(t)) \geq x(t) > 0, \quad x(t - \tau(t)) \geq x(\delta(t)) > 0 \quad \text{for} \quad t \geq t_1.
\] (2.12)
From this, Lemma 1 and (1.6), one can easily show that there exists \( t_2 \geq t_1 \) such that
\[
f(x(t - \tau(t))) \geq \frac{1}{2} x(t - \tau(t)) \geq \frac{1}{2} x(\delta(t)) \quad \text{for } t \geq t_2.
\] (2.13)

It follows from equation (1.1) that
\[
\dot{x}(t) + \frac{1}{2} p(t)x(\delta(t)) \leq 0 \quad \text{for } t \geq t_2
\] (2.14)

Integrating both sides from \( \delta(t) \) to \( t \) yields
\[
x(t) - x(\delta(t)) + \frac{1}{2} \int_{\delta(t)}^{t} p(s)x(\delta(s)) \, ds \leq 0 \quad \text{for } t \geq t_2.
\] (2.15)

By the decreasing nature of \( x(t) \) for large \( t \) and the increasing nature of \( \delta(t) \), there exists \( T \geq t_2 \) such that
\[
x(t) - x(\delta(t)) + \frac{1}{2} \left( \int_{\delta(t)}^{t} p(s) \, ds \right) x(\delta(t)) \leq 0 \quad \text{for } t \geq T
\]

which shows that (2.11) holds for \( t \geq T \). The proof of Lemma 3 is complete. \( \blacksquare \)

**Lemma 4.** Assume that (1.2), (1.3) and (1.6) hold, and
\[
\lim \inf_{t \to \infty} \int_{\delta(t)}^{t} p(s) \, ds > 0. \tag{2.16}
\]

If \( x(t) \) is a nonoscillatory solution of equation (1.1), then \( x(\delta(t))/x(t) \), which is well defined for large \( t \), is bounded.

**Proof.** We shall assume \( x(t) \) to be eventually positive [if \( x(t) \) is eventually negative the proof is similar]. Note that (2.16) implies (2.1). By the same argument as in the proof of Lemma 3, there exists \( t_1 > 0 \) such that
\[
x(\delta(\delta(t))) \geq x(\delta(t)) \geq x(t) > 0 \quad \text{for } t \geq t_1
\] (2.17)

and
\[
\dot{x} + \frac{1}{2} p(t)x(\delta(t)) \leq 0 \quad \text{for } t \geq t_1. \tag{2.18}
\]

By (2.16), there exist \( d > 0 \) and \( t_2 > 0 \) such that
\[
\int_{\delta(t)}^{t} p(s) \, ds \geq d \quad \text{for } t \geq t_2. \tag{2.19}
\]

Then for any \( t > t_2 \) there exists \( \xi(t) > t \) such that
\[
\int_{t}^{\xi(t)} p(s) \, ds = \frac{d}{2} \quad \text{and} \quad \int_{\delta(\xi(t))}^{\xi(t)} p(s) \, ds \geq \frac{d}{2}. \tag{2.20}
\]

Let \( t_3 = \max\{t_1, t_2\} \). For \( t \geq t_3 \), integrating (2.18) over the intervals \([t, \xi(t)]\) and \([\delta(\xi(t)), t]\), we find
\[
x(\xi(t)) - x(t) + \frac{1}{2} \int_{t}^{\xi(t)} p(s)x(\delta(s)) \, ds \leq 0 \quad \text{for } t \geq t_3
\] (2.21)

and
\[
x(t) - x(\delta(\xi(t))) + \frac{1}{2} \int_{\delta(\xi(t))}^{t} p(s)x(\delta(s)) \, ds \leq 0 \quad \text{for } t \geq t_3. \tag{2.22}
\]
By Lemma 1, \(x(t)\) is decreasing for large \(t\). By using the increasing nature of \(\delta(t)\) we find that there exists \(T \geq t_3\) such that
\[
x(\xi(t)) - x(t) + \frac{1}{2} \left( \int_{t_3}^{t} p(s) \, ds \right) x(\delta(\xi(t))) \leq 0 \quad \text{for } t \geq T
\] (2.23)
and
\[
x(t) - x(\delta(\xi(t))) + \frac{1}{2} \left( \int_{t_3}^{t} p(s) \, ds \right) x(\delta(t)) \leq 0 \quad \text{for } t \geq T.
\] (2.24)

By omitting the first terms in (2.23) and (2.24), we find
\[
-x(t) + \frac{d}{4} x(\delta(\xi(t))) \leq 0 \quad \text{and} \quad -x(\delta(\xi(t))) + \frac{d}{4} x(\delta(t)) \leq 0 \quad \text{for } t \geq T
\] (2.25)
or
\[
x(t) \geq \frac{d}{4} x(\delta(\xi(t))) \geq \frac{d^2}{16} x(\delta(t)) \quad \text{for } t \geq T
\] (2.26)
or
\[
\frac{x(\delta(t))}{x(t)} \leq \frac{16}{d^2} \quad \text{for } t \geq T.
\] (2.27)

The proof of Lemma 4 is complete. ■

According to (1.7), there is a \(\rho(t) \in [\delta(t), t]\) such that
\[
\int_{\rho(t)}^{t} p(s) \, ds = \frac{1}{e} \quad \text{for } t \geq t_0.
\] (2.28)

**Lemma 5.** Assume that (1.2), (1.3), (1.6), (1.7) and (1.8) hold. If \(x(t)\) is a nonoscillatory solution of equation (1.1), then for large \(T\)
\[
\lim_{N \to \infty} \int_{T}^{N} p(t) \left( w(t) - e^{\int_{t_0}^{t} p(s) \, ds} \right) dt = \infty
\] (2.29)
where \(\rho(t)\) satisfies (2.28), and \(w(t) = x(\delta(t))/x(t)\).

**Proof.** We shall assume \(x(t)\) to be eventually positive [if \(x(t)\) is eventually negative the proof is similar]. Clearly, (1.7) implies (2.1). It follows from Lemma 1 that for \(\varepsilon\) given by (1.6) there exists \(t_1 \geq 0\) such that
\[
0 < x(t) \leq x(\delta(t)) \leq x(t - \tau(t)) < \varepsilon \quad \text{for } t \geq t_1
\]
leading to
\[
w(t) \geq 1 \quad \text{for } t \geq t_1.
\] (2.30)
It follows from (1.6) that
\[
f(x(t - \tau(t))) \geq x(t - \tau(t)) - Mx^{1+r}(t - \tau(t)) \quad \text{for } t \geq t_1.
\] (2.31)
By using the mean value theorem of differential calculus and that \(\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x(t - \tau(t)) = 0\), one can show that
\[
x(t - \tau(t)) - Mx^{1+r}(t - \tau(t)) \geq x(\delta(t)) - Mx^{1+r}(\delta(t))
\] (2.32)
for $t$ sufficiently large. From this, (2.31) and equation (1.1), we find that there exists $t_2 \geq t_1$ such that
\begin{equation}
\dot{x}(t) + p(t)[x(\delta(t)) - Mx^{1+\gamma}(\delta(t))] \leq 0 \quad \text{for } t \geq t_2. \tag{2.33}
\end{equation}

Dividing both sides by $x(\delta(t))$ and then integrating both sides from $\delta(t)$ to $t$, we obtain
\begin{equation}
w(t) \geq \exp\left(\int_{\delta(t)}^{t} p(s)w(s)[1 - Mx'(\delta(s))] \right) ds \quad \text{for } t \geq t_2. \tag{2.34}
\end{equation}

Then there exists $t_3 \geq \max\{t_0, t_2\}$ where $t_0$ is associated with (1.7) such that, for $t \geq t_3$,
\begin{align*}
w(t) &\geq \exp\left(\int_{\delta(t)}^{t} p(s)w(s)[1 - Mx'(\delta(s))] \right) ds \\
&= \exp\left(\int_{\rho(t)}^{t} p(s)w(s) \right) \exp\left(\int_{\delta(t)}^{\rho(t)} p(s)w(s) ds \right) \exp\left(-M\int_{\delta(t)}^{t} p(s)w(x_\gamma(\delta(s))) ds \right) \\
&\geq \exp\left(\int_{\rho(t)}^{t} p(s)w(s) \right) \exp\left(\int_{\delta(t)}^{t} p(s) ds - \frac{1}{e} \right) \exp\left(-M\int_{\delta(t)}^{t} p(s)w(s)x'(\delta(s)) ds \right)
\end{align*}

where $\rho(t)$ satisfies (2.28). One can easily show that
\begin{equation}
e^c \geq ec \quad \text{for all } c \geq 0
\end{equation}

and so
\begin{align*}
w(t) &\geq e \int_{\rho(t)}^{t} p(s)w(s) ds \exp\left(\int_{\delta(t)}^{t} p(s) ds - \frac{1}{e} \right) \exp\left(-M\int_{\delta(t)}^{t} p(s)w(s)x'(\delta(s)) ds \right) \quad \text{for } t \geq t_3 \tag{2.35}
\end{align*}
or
\begin{align*}
w(t) - e \int_{\rho(t)}^{t} p(s)w(s) ds &\geq e \int_{\rho(t)}^{t} p(s) ds \left[ \exp\left(\int_{\delta(t)}^{t} p(s) ds - \frac{1}{e} \right) \\
&\times \exp\left(-M\int_{\delta(t)}^{t} p(s)w(s)x'(\delta(s)) ds \right) - 1 \right] \tag{2.36}
\end{align*}
or
\begin{align*}
p(t)\left(w(t) - e \int_{\rho(t)}^{t} p(s)w(s) ds \right) &\geq ep(t) \int_{\rho(t)}^{t} p(s)w(s) ds \left[ \exp\left(\int_{\delta(t)}^{t} p(s) ds - \frac{1}{e} \right) \\
&\times \exp\left(-M\int_{\delta(t)}^{t} p(s)w(s)x'(\delta(s)) ds \right) - 1 \right] \quad \text{for } t \geq t_3 \tag{2.37}
\end{align*}
By Lemma 1, the increasing nature of $\delta(t)$ and Lemmas 2–4, there exist $T \geq t_1, A > 0$ and $M_1 > 0$ such that for $t \geq T$

$$x(\delta(s)) \leq x(\delta(\delta(t))) \quad \text{for} \quad \delta(t) \leq s \leq t, \quad (2.38)$$

$$x(\delta(\delta(t))) \leq A \exp\left(-\frac{1}{2} \int_{\delta(t)}^{\delta(\delta(t))} p(s) \, ds \right), \quad (2.39)$$

$$\int_{\delta(t)}^{t} p(s) \, ds \leq 2 \quad \text{and} \quad \int_{\delta(\delta(t))}^{\delta(t)} p(s) \, ds \leq 2 \quad (2.40)$$

and for $t \geq \delta(T)$

$$w(t) \leq M_1. \quad (2.41)$$

Set

$$D(t) = e^p(t) \int_{\rho(t)}^{t} p(s) w(s) \, ds \exp\left(\int_{\delta(t)}^{t} p(s) \, ds - \frac{1}{e}\right) \times \left[1 - \exp\left(-M \int_{\delta(t)}^{t} p(s) x'(\delta(s))w(s) \, ds\right)\right] \quad \text{for} \quad t \geq T. \quad (2.42)$$

One can easily see that

$$0 \leq 1 - e^{-c} \leq c \quad \text{for all} \quad c \geq 0.$$  

From this, (2.28), and (2.41), we obtain

$$D(t) \leq MM_1 p(t) \exp\left(\int_{\delta(t)}^{t} p(s) \, ds - \frac{1}{e}\right) \int_{\delta(t)}^{t} p(s)x'(\delta(s))w(s) \, ds \quad \text{for} \quad t \geq T. \quad (2.43)$$

It follows from (2.38) to (2.41) and (2.43) that for $N \geq T$

$$\int_{T}^{T} D(t) \, dt \leq 2MM_1^2 \exp\left(2 - \frac{1}{e}\right) \int_{T}^{T} p(t) x'(\delta(t))) \, dt$$

$$\leq L \int_{T}^{T} p(t) \exp\left(-\frac{r}{2} \int_{\delta(t)}^{t} p(s) \, ds\right) \quad \text{for} \quad t \geq T. \quad (2.44)$$

where $L = 2MM_1^2 \exp(2 - (1/e))A'$. On the other hand,

$$\exp\left(-\frac{r}{2} \int_{T}^{t} p(s) \, ds\right) = \exp\left(-\frac{r}{2} \int_{T}^{t} p(s) \, ds + \frac{r}{2} \int_{\delta(t)}^{t} p(s) \, ds + \frac{r}{2} \int_{\delta(\delta(t))}^{\delta(\delta(t))} p(s) \, ds\right)$$

$$\leq e^{2r} \exp\left(-\frac{r}{2} \int_{T}^{t} p(s) \, ds\right) \quad \text{for} \quad t \geq T.$$  

It follows that for $N \geq T$

$$\int_{T}^{T} D(t) \, dt \leq Le^{2r} \int_{T}^{T} p(t) \exp\left(-\frac{r}{2} \int_{T}^{t} p(s) \, ds\right) \, dt. \quad (2.45)$$

Since (1.7) implies (2.1), we have

$$\int_{T}^{\infty} p(t) \exp\left(-\frac{r}{2} \int_{T}^{t} p(s) \, ds\right) \, dt < \infty. \quad (2.46)$$
From this and (2.45), we find
\[ \int_r^\infty D(t) \, dt < \infty. \] (2.47)

Since
\[
eq p(t) \left[ \exp \left( \int_{\rho(t)}^{t} p(s) \, ds \right) - 1 \right] - D(t)
\]
from (2.37), (2.47) and (1.8), we have
\[
\lim_{N \to \infty} \int_T^N p(t) \left( w(t) - e \int_{\rho(t)}^{t} p(s) w(s) \, ds \right) \, dt = \infty;
\]
the proof of Lemma 5 is complete. \qed

Now we are ready to prove our main theorem.

**Proof of Theorem 1.** Assume, for the sake of contradiction, that equation (1.1) has a nonoscillatory solution \( x(t) \). According to Lemma 5, we get that for large \( T \)
\[ \lim_{N \to \infty} \int_T^N p(t) \left( w(t) - e \int_{\rho(t)}^{t} p(s) w(s) \, ds \right) \, dt = \infty. \] (2.48)
By Lemmas 3 and 4, we choose \( T \) such that, for \( t \geq T \), (2.11) holds and for \( t > \delta(T) \), \( w(t) \) is bounded by a positive number \( M_1 \).

Let \( R(t) \in C^1([0, \infty)) \) such that
\[ \dot{R}(t) = p(t) + 1. \] (2.49)
Set
\[ Q(t) = p(t) + \exp(-R(t)). \]
Then, for \( N > T \),
\[
\int_T^N Q(t) \left( w(t) - e \int_{\rho(t)}^{t} Q(s) w(s) \, ds \right) \, dt = \int_T^N p(t) \left( w(t) - e \int_{\rho(t)}^{t} p(s) w(s) \, ds \right) \, dt
\]
\[ - e \int_T^N p(t) \left( \int_{\rho(t)}^{t} \exp(-R(s)) w(s) \, ds \right) \, dt
\]
\[ + \int_T^N \exp(-R(t)) \left( \int_{\rho(t)}^{t} p(s) w(s) \, ds \right) \, dt
\]
\[ - e \int_T^N \exp(-R(t)) \left( \int_{\rho(t)}^{t} \exp(-R(s)) w(s) \, ds \right) \, dt. \] (2.50)
We have
\[
\int_T^\infty p(t) \left( \int_{\rho(t)}^{t'} \exp(-R(s))w(s) \, ds \right) \, dt \leq M_1 \int_T^\infty p(t) \exp \left( -\int_0^{\rho(t)} p(s) \, ds \right) \left( \int_{\rho(t)}^{t'} e^{-s} \, ds \right) \, dt \\
\leq M_1 e^{\int_T^\infty p(t) \exp \left( -\int_0^{t'} p(s) \, ds \right) \, dt} < \infty,
\]
and
\[
\int_T^\infty \exp(-R(t)) \left( \int_{\rho(t)}^{t'} \exp(-R(s))w(s) \, ds \right) \, dt \leq M_1 \int_T^\infty \exp(-R(t)) \, dt < \infty,
\]
\[
\left| \int_T^\infty \exp(-R(t)) \left( w(t) - e \int_{\rho(t)}^{t'} p(s)w(s) \, ds \right) \, dt \right| \leq \int_T^\infty \exp(-R(t)) \left( w(t) + e \int_{\rho(t)}^{t'} p(s)w(s) \, ds \right) \, dt < \infty
\]
because \( w(t) + e \int_{\rho(t)}^{t'} p(s)w(s) \, ds \) is bounded for \( t \geq T \). From this, (2.48) and (2.5), we find
\[
\lim_{N \to \infty} \int_T^N Q(t) \left( w(t) - e \int_{\rho(t)}^{t'} Q(s)w(s) \, ds \right) \, dt = \infty. \tag{2.51}
\]
Set
\[
u(u) = w(u^\sigma^{-1}(u)), \tag{2.53}
\]
Then \( \sigma(t) \to \infty \) as \( t \to \infty \); \( \sigma(t) \) is strictly increasing, and thus \( \sigma^{-1} \) exists. Set
\[
\sigma(u) = w(u^\sigma^{-1}(u)). \tag{2.54}
\]
Then
\[
\int_T^N Q(t) \left( w(t) - e \int_{\rho(t)}^{t'} Q(s)w(s) \, ds \right) \, dt \leq \int_{\sigma(T)}^{\sigma(N)} \left( v(u) - e \int_{u-1/e}^u v(s) \, ds \right) \, du,
\]
because
\[
\sigma(\rho(t)) = \int_{\rho(t)}^\infty Q(s) \, ds \leq \int_{\rho(t)}^{t'} Q(s) \, ds - \int_{\rho(t)}^{t'} p(s) \, ds = u - \frac{1}{e}.
\]
From (2.51) and (2.54), we have
\[
\lim_{H \to \infty} \int_{\sigma(T)}^H \left( v(u) - e \int_{u-1/e}^u v(s) \, ds \right) \, du = \infty. \tag{2.55}
\]
By interchanging the order of integration, we find that for $H > \sigma(T)$
\[
\int_{\sigma(T)}^{H} e^{\int_{u-1/e}^{u} v(s) \, ds} \, du = \int_{\sigma(T)-1/e}^{\sigma(T)} e^{\left(\int_{u-1/e}^{u} v(s) \, ds\right)} \, ds + \int_{\sigma(T)}^{H-1/e} e^{\left(\int_{u-1/e}^{u} v(s) \, ds\right)} \, ds + \int_{H-1/e}^{H} e^{\int_{s}^{H} v(s) \, ds} \, ds
\]
or
\[
\int_{\sigma(T)}^{H} e^{\left(\int_{u-1/e}^{u} v(s) \, ds\right)} \, du = \int_{\sigma(T)-1/e}^{\sigma(T)} [es + 1 - e\sigma(T)]v(s) \, ds + \int_{\sigma(T)}^{H-1/e} v(s) \, ds + \int_{H-1/e}^{H} e(H - s)v(s) \, ds. \tag{2.56}
\]
On the right-hand side of (2.56), the first term is a constant independent of $H$ and the last term is positive. From (2.55) and (2.56) we obtain
\[
\lim_{H \to \infty} \int_{H-1/e}^{H} v(s) \, ds = \infty. \tag{2.57}
\]
This shows
\[
\lim \sup_{u \to \infty} v(u) = \infty \tag{2.58}
\]
and thus
\[
\lim \sup_{t \to \infty} w(t) = \infty \tag{2.59}
\]
which contradicts that $w(t)$ is bounded for $t \geq T$.

The proof of Theorem 1 is complete. □

3. APPLICATIONS

In this section we apply Theorem 1 to some delay differential equations. For the linear delay differential equations
\[
\dot{x}(t) + p(t)x(t - \tau(t)) = 0 \tag{3.1}
\]
where $p$ and $\tau$ satisfy (1.2), a well-known oscillation criterion is
\[
\lim \inf_{t \to \infty} \int_{t-\tau(t)}^{t} p(s) \, ds > \frac{1}{e} \tag{3.2}
\]
see [4] and [5]. It is easy to prove that (3.2) is equivalent to
\[
\lim \inf_{t \to \infty} \int_{t-\delta(t)}^{t} p(s) \, ds > \frac{1}{e} \tag{3.3}
\]
where $\delta(t) = \max_{s \in [0, t]} \{s - \tau(s)\}$.

Zhang and Gopalsamy [6] discussed the nonautonomous delay–logistic equations of the form
\[
\dot{x} = p(t)x(t) \left[1 - \frac{x(t - \tau(t))}{K}\right] \tag{3.4}
\]
where \( p(t) \) and \( \tau(t) \) satisfy (1.2), representing intrinsic growth rate and gestation delay, respectively, and \( K \) is a constant which is the carrying capacity of the environment. They showed that every solution of equation (3.4) oscillates about \( K \) if (3.2) or (3.3) holds. By setting

\[
x(t) = K \exp(y(t))
\]  

(3.5)

Equation (3.4) is reduced to the nonlinear differential equation

\[
\dot{y} = p(t)[1 - \exp(y(t - \tau(t))].
\]  

(3.6)

Clearly, \( x(t) \) oscillates about \( K \) if and only if \( y(t) \) oscillates. Therefore, every solution of equation (3.6) oscillates if (3.3) holds. Set

\[
f(u) = 1 - e^{-u}
\]  

(3.7)

and note that equation (3.6) can be written in the form of equation (1.1). Obviously \( f(u) \) given by (3.7) satisfies (1.3) and (1.6).

One can easily see that condition (3.3) implies (1.7) and (1.8). Therefore, Theorem 1 extends sufficient conditions for oscillations of equations (3.1) and (3.4) in [4–6].

**Example.** Consider the delay differential equation

\[
\dot{x}(t) + \left( \frac{1}{e \ln 2(t + 2)} + \frac{1}{(t + 2) \ln(t + 2)} \right) x\left( \frac{t}{2} - 1 \right) = 0
\]  

(3.8)

where \( p(t) = (1/(e \ln 2(t + 2))) + (1/((t + 2) \ln(t + 2))) \) and \( \tau(t) = t/2 + 1 \). For \( t \geq 4 \),

\[
\int_{t/2-1}^{t} p(t) \, dt = \frac{1}{e} + \ln \frac{\ln(t/2 + 1)}{\ln(t/2 + 1)} > \frac{1}{e}
\]

and

\[
\lim_{t \to \infty} \int_{t/2-1}^{t} p(t) \, dt = \frac{1}{e}.
\]

Hence, condition (3.2) is not satisfied. On the other hand,

\[
\int_{4}^{\infty} p(t) \left( \int_{t/2-1}^{t} p(s) \, ds - \frac{1}{e} \right) - 1 \, dt \geq \int_{4}^{\infty} p(t) \left( \int_{t/2-1}^{t} p(s) \, ds - \frac{1}{e} \right) dt
\]

\[
\geq \frac{1}{e \ln 2} \int_{4}^{\infty} \frac{1}{t + 2} \ln \frac{\ln(t/2 + 1)}{\ln(t/2 + 1)} \, dt
\]

\[
= \frac{1}{2e \ln 2} \int_{4}^{\infty} \frac{1}{t/2 + 1} \ln \left( 1 + \frac{\ln 2}{\ln(t/2 + 1)} \right) \, dt
\]

\[
= \infty.
\]

Then, by Theorem 1, every solution of (3.8) oscillates.

In [7] Kocić et al. obtained linearized oscillation results for quite general nonlinear non-autonomous delay differential equations. However, it appears that the proof of Theorem 1 in [7] is incomplete. We will provide more details to address this in a forthcoming paper.

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