Convergence in Lotka–Volterra-type delay systems without instantaneous feedbacks

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Synopsis

Most of the convergence results appearing so far for delayed Lotka–Volterra-type systems require that undelayed negative feedback dominate both delayed feedback and interspecific interactions. Such a requirement is rarely met in real systems. In this paper we present convergence criteria for systems without instantaneous feedback. Roughly, our results suggest that in a Lotka–Volterra-type system if some of the delays are small, and initial functions are small and smooth, then the convergence of its positive steady state follows that of the undelayed system or the corresponding system whose instantaneous negative feedback dominates. In particular, we establish explicit expressions for allowable delay lengths for such convergence to sustain.

1. Introduction

For many-species Lotka–Volterra systems without delays, it is well known that global stability of a positive equilibrium holds when the intraspecies competition term dominates the interspecific interactions (i.e. the community matrix is diagonally dominant). A result of Goh [7] extends this result to allow a weaker hypothesis on the community matrix and for the equilibrium to not necessarily be positive. See Hofbauer and Sigmund [17] for a nice discussion of such results for Lotka–Volterra ordinary differential equations. The result of Goh continues to hold when continuous diffusion (Neumann boundary conditions) or diffusion among discrete identical habitats is introduced and also when discrete time delays as well as continuous diffusion or diffusion among discrete habitats is allowed (see [22]).

It is also well known that the asymptotic behaviour of solutions of a single species logistic equation, for which the dependence of the per capita growth rate on species density contains both an undelayed contribution as well as a delayed contribution, is particularly sensitive to the relative weight of the delayed versus the nondelayed term (e.g. see [8, 15, 18, 19] and the references cited therein). On the other hand, when the delayed contribution dominates, in order to ensure the global stability of the positive steady state, it is generally required that the length of the delay is short enough (see [19, 29] and the references cited therein). Otherwise, Hopf bifurcation may occur and periodic solutions may appear and persist. This has been shown by Jones [18] for a discrete delay with no undelayed term and by Dunkel [6] and Walther [27] for distributed delays. More elaborate results along this line can be found in Nussbaum’s work (e.g. [25]). For delay

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Lotka–Volterra systems, the interested reader is referred to Cushing [5] for some
detailed Hopf bifurcation and local stability analysis.

In [23], by employing a proper Liapunov–Razumikhin function and choosing a
friendly initial function space \([1, 13, 14, 16]\), we (along with R. H. Martin) were
able to show that: a Lotka–Volterra-like model of \(m\) interacting species which
can disperse among \(n\) discrete habitats and where interaction terms involve
general unbounded delays possess a globally stable steady state if the undelayed
intraspecific competition term dominates interspecific interactions as well as the
delayed intraspecific competition effect and when the \(n\) habitats are nearly
identical. In a subsequent paper [21] this result was extended to allow the system
to be nonautonomous and that the undelayed intraspecific competition to
dominate only the interspecific interactions. In both cases, the existence of strong
undelayed intraspecific competitions (the so-called negative feedback) is crucial.

In view of the fact that in real-life interactions, instantaneous responses are
rare or weak relative to delayed responses, realistic models should consist of
delay differential equations without instantaneous (negative) feedbacks. However,
most of the known convergence results for delayed systems require strong
instantaneous negative feedbacks.

In this paper we establish some useful convergence criteria for the above-
mentioned delay systems. We consider first a class of Lotka–Volterra-type infinite
delay systems, where delays for the negative feedback are expected to be small.
If the system is diagonally dominant (to be defined later), and initial functions are
relatively small, and smooth in the short past, then we can show that the unique
saturated steady state [17] attracts such kind of neighbouring solutions. Similar
results are found to be true for delayed Volterra–Liapunov stable Lotka–
Volterra-type systems. Our approach involves constructing suitable Liapunov–
Razumikhin functions, carefully selecting initial function sets, and estimating the
length of relevant delays.

2. Preliminaries

In this paper we consider the following general autonomous Lotka–Volterra-type
infinite delay system

\[
\dot{u}_i(t) = b_i(u_i(t))G_i(u_i(\cdot)), \quad i = 1, \ldots, n,
\]

where

\[
G_i(u_i(\cdot)) = r_i - a_i \int_{-\tau_i}^0 u_i(t + \theta) \, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\infty}^{0} u_j(t + \theta) \, d\mu_{ij}(\theta)
\]

and \(u_i(\theta) = u(t + \theta)\) for \(\theta \leq 0\). \(u(t) = (u_1(t), u_2(t), \ldots, u_n(t))\). Throughout the
rest of this paper, we assume that, for \(i = 1, \ldots, n\),

(H1) \(b_i(0) = 0\), \(b_i(\cdot)\) is continuously differentiable and \(b'_i(\cdot) > 0\);

(H2) \(r_i, a_i, \tau_i\) are constants, in particular, \(a_i\) and \(\tau_i\) are positive;

(H3) \(\mu_i(\theta)\) are nondecreasing, \(\mu_i(0) - \mu_i(-\tau_i) = 1\);

(H4) \(\mu_{ij}(\theta)\) are bounded real-valued Borel measures on \((-\infty, 0]\) with total
variation \(|\mu_{ij}|\).
System (2.1) may be used to model the population dynamics of a closed ecological system containing \( n \) interacting species (no immigration and emigration). \( u_i(t) \) may represent the population density of the \( i \)th species at time \( t \); \( r = (r_1, r_2, \ldots, r_n) \) may stand for the vector of intrinsic population growth rates; \( a_i \int_{-\tau}^0 u_i(t + \theta) \, d\mu_i(\theta) \) can be interpreted as the delayed negative feedback (due to self-crowding effect); and \( \sum_{j=1}^n \int_{-\infty}^0 u_j(t + \theta) \, d\mu_{ij}(\theta) \) may describe the interactions with the existing species.

It is natural from a biological point of view to seek a solution of (2.1) corresponding to nonnegative initial data belonging to the Banach space \( BC \) of bounded and continuous functions that map \((-\infty, 0]\) into \( \mathbb{R}^n \), with the uniform norm \( \|\phi\|_\infty = \sup_{s \leq 0} |\phi(s)| \), where \( \phi \in BC \) and \(|\cdot|\) is a chosen norm in \( \mathbb{R}^n \). Observe that \( G_i(u_i(\cdot)) \) can be rewritten as

\[
G_i(u_i(\cdot)) = r_i(t) - a_i \int_{-\tau}^0 u_i(t + \theta) \, d\mu_i(\theta) + \sum_{j=1}^n \int_{-\tau}^0 u_j(t + \theta) \, d\mu_{ij}(\theta),
\]

where \( \tau \) is any positive constant, and

\[
r_i(t) = r + \sum_{j=1}^n \int_{-\infty}^{t-\tau} u_j(t + \theta) \, d\mu_{ij}(\theta).
\]

For convenience, we choose \( \tau = \max \{\tau_i: i = 1, \ldots, n\} \).

For fixed \( \tau > 0 \) and \( t \leq \tau \), \( r_i(t) \) is continuous. \( u(t) \) is a solution of (2.1) for \( t \leq \tau \) with initial data in \( BC \) if and only if \( u(t), -\tau \leq t \leq \tau \), is a solution of the following nonautonomous system with bounded delays

\[
u'_i(t) = b_i(u_i(t)) \left[ r_i(t) - a_i \int_{-\tau}^0 u_i(t + \theta) \, d\mu_i(\theta) + \sum_{j=1}^n \int_{-\tau}^0 u_j(t + \theta) \, d\mu_{ij}(\theta) \right],
\]

where \( r_i(t) \) is defined by (2.4). Therefore, existence (local) and uniqueness of \( u(t) \) for (2.1) follows from those for system (2.5) (cf. [15]). However, it is easy to see that bounded solutions of (2.1) may not be precompact in the space \( BC \). To overcome this difficulty, we choose the more friendly space \([1, 13, 14, 16]\)

\[
UCg = \left\{ \phi \in C((-\infty, 0], \mathbb{R}^n): \|\phi\|_g = \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} < \infty, \quad \frac{\phi(s)}{g(s)} \text{ is uniformly continuous on } (-\infty, 0] \right\},
\]

where \( g: (-\infty, 0] \rightarrow [1, \infty) \) satisfies

\( (g1) \ g: (-\infty, 0] \rightarrow [1, \infty) \) is a continuous nonincreasing function on \((-\infty, 0]\) such that \( g(0) = 1; \)

\( (g2) \ g(s + u)/g(s) \rightarrow 1 \) uniformly on \((-\infty, 0]\) as \( u \rightarrow 0^-; \)

\( (g3) \ g(s) \rightarrow \infty \) as \( s \rightarrow -\infty. \)

Clearly, \( UCg \) is a Banach space with norm

\[
\|\phi\|_g = \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)}.
\]
Moreover, $UCg$ is a strong fading memory space (in the sense of [13, 16]) which implies that bounded solutions of an autonomous system corresponding to initial data $\phi \in BC$ have precompact orbits in $UCg$. Thus, positive limit sets are nonempty and have their usual properties (connectedness, compactness and invariance).

A further reason for choosing $UCg$ is that $BC \hookrightarrow UCg$ with continuous inclusion for $g$ satisfying (g1)–(g3).

In the rest of this paper, we always assume our initial conditions satisfy

$$(H5) \quad u_0(s) = \phi(s) \geq 0 \quad \text{for} \quad s \leq 0; \quad \phi \in Bc, \quad \phi(0) > 0.$$ 

Here $\phi(0) > 0$ means $\phi_i(0) > 0$ for $i = 1, \ldots, n$. We call such a $\phi(s)$ an admissible initial function.

The following fundamental lemma is a simple combination of some well-known results.

**Lemma 2.1.** Assume (H1)–(H5) hold. Then

(i) there is a $g(s)$, satisfying (g1)–(g3), such that

$$|\mu_{ij}| = \int_{-\infty}^{0} g(s) \cdot d\mu_{ij}(\theta) | < \infty, \quad i, j = 1, \ldots, n;$$

(ii) (2.1) has a unique solution $u(t)$, such that $u(t) > 0$ for $t \geq 0$ in its maximal interval of existence. If $u(t)$ is a noncontinuable solution, then for any $M > 0$, there is a $t^* > 0$ such that $\Sigma_{i=1}^{n} u_i(t^*) > M$.

**Proof.** Part (i) is rather straightforward and essentially well known. A similar result is proved in [11, Lemma 2.1]. The positivity of $u(t)$ in its maximal interval of existence follows from the fact that (2.1) is of Lotka–Volterra type. The last statement is an immediate result of [16, Theorem 2.3].

In the next section we choose the norm $|\cdot|$ in $\mathbb{R}^n$ as

$$|\phi(s)| = \max \{|\phi_i(s)|: i = 1, \ldots, n\}$$

where $\phi(s) = (\phi_1(s), \ldots, \phi_n(s)) \in \mathbb{R}^n$. Thus, for $g(s)$ satisfying (g1)–(g3), we have

$$\|\phi\|_g = \sup_{s \leq 0} \max \left\{|\phi_i(s)|/g(s): i = 1, \ldots, n\right\},$$

where $\phi \in UCg$.

### 3. Convergence in infinite delay systems

We say system (2.1) is diagonally dominant if

$$a_i > \sum_{j=1}^{n} |\mu_{ij}|, \quad i, j = 1, \ldots, n. \quad (3.1)$$

A vector $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ is called a saturated equilibrium for (2.1) if for $i = 1, \ldots, n$,

$$\rho_i \geq 0, \quad G_i(\rho) \leq 0, \quad \text{and} \quad \rho_i G_i(\rho) = 0. \quad (3.2)$$
It is well known that (see [4, 21, 23]) system (2.1) has a unique saturated equilibrium if it is diagonally dominant. It is also easy to see that if (2.1) is diagonally dominant, then there is a \( g(s) \) satisfying (g1)–(g3) such that 
\[ a_i > \sum_{j=1}^n |\mu_{ij}|, \quad i = 1, \ldots, n. \]
For simplicity, we denote 
\[ \tau = \max \{ \tau_i ; i = 1, \ldots, n \}. \] (3.3)

Throughout this section we assume that system (2.1) is diagonally dominant and denote \( g = g(s) \) as a function which satisfies (g1)–(g3), 
\[ a_i > \sum_{j=1}^n |\mu_{ij}|, \quad i = 1, \ldots, n, \] and \( g(s) = 1 \) for \( s \in [-\tau, 0] \). Clearly, such a \( g(s) \) always exists as long as (2.1) is diagonally dominant. Also, we always denote \( p \) as the unique saturated equilibrium of (2.1).

In order to present and prove our main result of this section we need the following lemma.

**Lemma 3.1.** In addition to (H1)–(H5), assume that system (2.1) is diagonally dominant. Assume further that:

(i) there is a constant \( K > 0 \) such that 
\[ a_i (1 - d_i(K)\tau_i) > \sum_{j=1}^n |\mu_{ij}|, \]
where 
\[ d_i(K) = b_i(K + \rho_i) \left[ a_i + \sum_{j=1}^n |\mu_{ij}| \right]; \]
(ii) \( \phi \in BC, \|\phi - \rho\|_g < K \);
(iii) for \( s \geq -\tau_i, \phi_i(s) \) is continuously differentiable, and satisfies 
\[ |\phi'_i(s)| < d_i(K)K. \]

Then for \( t \geq 0, \|u_i(\phi)(\cdot) - \rho\|_g \leq K \) and \( |u'_i(\phi)(t)| \leq d_i(K)K. \)

**Proof.** Clearly, if for \( t \in [0, t^*], \|u_i(\phi)(\cdot) - \rho\|_g \leq K, \) then 
\[ |u'_i(\phi)(t)| = b_i(u_i) \left| -a_i \int_{-\tau_i}^0 (u_i(t + \theta) - \rho_i) d\mu_i(\theta) + \sum_{j=1}^n \int_{-\tau_j}^0 (u_j(t + \theta) - \rho_j) d\mu_j(\theta) \right| \]
\[ \leq b_i(u_i) \left[ a_i \int_{-\tau_i}^0 \left| u_i(t + \theta) - \rho_i \right| g(\theta) d\mu_i(\theta) \right. \]
\[ + \left. \sum_{j=1}^n \int_{-\tau_j}^0 \left| u_j(t + \theta) - \rho_j \right| g(\theta) d\mu_j(\theta) \right] \]
where \( u_i = u_i(\phi)(t), \) \( u_i(t + \theta) = u_i(\phi)(t + \theta), \) \( i = 1, \ldots, n. \) Clearly, \( \|u_i(\phi)(\cdot) - \rho\|_g \leq K \) implies that \( u_i \leq K + \rho_i \) and \( |u_i(t + \theta) - \rho_i|/g(\theta) \leq K \) for \( i = 1, \ldots, n, \theta \leq 0. \) Thus 
\[ |u'_i(\phi)(t)| \leq b_i(K + \rho_i) \left[ a_i + \sum_{j=1}^n |\mu_{ij}| \right] K = d_i(K)K. \]

Note that we have used the assumption that \( g(\theta) = 1 \) for \( \theta \in [-\tau, 0]. \) This shows that \( |u'_i(\phi)(t)| \leq d_i(K)K \) follows from \( \|u_i(\phi)(\cdot) - \rho\|_g \leq K \) for \( t \geq 0. \)
Assume that there is a $i>0$ such that $||u_i(\phi)(\cdot) - \rho||_g = K$ and $||u_i(\phi)(\cdot) - \rho||_g < K$ for $t \in [0, i)$. Then the definition of $||\cdot||_g$ together with the fact that $||\phi - \rho||_g < K$ implies that there is an $i \in \{1, \ldots, n\}$, such that $||u_i(\phi)(\cdot) - \rho||_g = |u_i(i) - \rho_i|$. The previous arguments assure that $|u_i(t)| \leq d_i(K)K$ for $t \in [-\tau, i]$. Denote

$$V_i(t) = (u_i(t) - \rho_i)^2.$$  \hspace{1cm} (3.4)

Then it is easy to see that we must have

$$V_i'(i) \geq 0.$$  \hspace{1cm} (3.5)

We note that

$$V_i'(t) = 2(u_i(i) - \rho_i)b_i(u_i) \left[ -a_i \int_{-\tau_i}^{0} (u_i(\theta) - \rho_i) d\mu_i(\theta) ight. \\ + \sum_{j=1}^{n} \int_{-\tau_i}^{0} (u_j(i) - \rho_i) d\mu_j(\theta) \bigg]$$

$$= 2b_i(u_i) \left[ -a_i(u_i(i) - \rho_i)^2 - a_i(u_i(i) - \rho_i) \int_{-\tau_i}^{0} (u_i(\theta) - u_i(i)) d\mu_i(\theta) ight. \\ + (u_i(i) - \rho_i) \sum_{j=1}^{n} \int_{-\tau_i}^{0} (u_j(\theta) - \rho_j) d\mu_j(\theta) \bigg].$$

If $i \geq \tau_i$, then

$$\int_{-\tau_i}^{0} |u_i(i) + \theta) - u_i(i)| d\mu_i(\theta) = \int_{-\tau_i}^{0} |u_i(i)\xi(\theta)| d\mu_i(\theta) \leq d_i(K)K\tau_i,$$

where $\xi = \xi(\theta) \in [i + \theta, i]$ is determined by mean value theorem. If $i < \tau_i$, then for $\theta \in [-i, i]$,

$$|u_i(i + \theta) - u_i(i)| \leq d_i(K)K\theta < d_i(K)K\tau_i,$$

and for $\theta \in [-\tau_i, -i]$,

$$|u_i(i + \theta) - u_i(i)| \leq |u_i(i + \theta) - u_i(\theta)| + |u_i(0) - u_i(i)| \leq d_i(K)K|i + \theta| + d_i(K)Ki = d_i(K)K\theta \leq d_i(K)K\tau_i.$$

Hence,

$$\left| \int_{-\tau_i}^{0} (u_i(\theta) - u_i(i)) d\mu_i(\theta) \right| \leq d_i(K)K\tau_i.$$

Clearly,

$$\left| \sum_{j=1}^{n} \int_{-\tau_i}^{0} (u_j(\theta) - \rho_j) d\mu_j(\theta) \right| \leq K \sum_{j=1}^{n} |\mu_{ij}|.$$

Therefore,

$$V_i'(i) \leq 2b_i(u_i(i)) \left[ -a_iK^2 + a_iKd_i(K)K\tau_i + K^2 \sum_{j=1}^{n} |\mu_{ij}| \right]$$

$$\leq 2b_i(u_i(i))K^2 \left[ -(a_i - a_iKd_i(K)\tau_i) + \sum_{j=1}^{n} |\mu_{ij}| \right] < 0,$$
which contradicts (3.5). This implies that no such $\bar{t}$ exists and, hence, for $t \geq 0$, $\|u_i(\phi)(\cdot) - \rho\|_g \leq K$. \(\square\)

Now we are ready to state the main result of this section.

**Theorem 3.2.** Assume that all the assumptions of Lemma 3.1 are satisfied. Then

$$\lim_{t \to +\infty} u(\phi)(t) = \rho.$$ 

**Proof.** Let $\alpha \in (0, 1)$ be a constant such that

$$a_i(\alpha - d_i(K)\tau_i) > \sum_{j=1}^{n} |\mu_{ij}|. \quad (3.6)$$

We claim that there is a $T_1 > \tau$ such that for $t \geq T_1$, $\|u_i(\phi)(\cdot) - \rho\|_g \leq \alpha K$.

There are two possibilities: (i) for any large $T$, there is a $i > T, i \in \{1, \ldots, n\}$ such that

$$K^2 \geq V_i(i) > \alpha K^2, \quad V'_i(i) \geq 0. \quad (3.7)$$

However, a similar argument as the proof of Lemma 3.1 yields

$$V'_i(i) \leq 2b_i(u_i(i)) \left[ -a_i\alpha^2 K^2 + a_i Kd_i(K)\alpha K\tau_i + \alpha K^2 \sum_{j=1}^{n} |\mu_{ij}| \right],$$

$$= 2ab_i(u_i(i)) \left[ -a_i(\alpha - d_i(K)\tau_i) + \sum_{j=1}^{n} |\mu_{ij}| \right]K^2 < 0,$$

which contradicts (3.7).

(ii) There is an $i \in \{1, \ldots, n\}$, such that $V'_i(t) < 0$, and

$$\lim_{t \to +\infty} V_i(t) = \alpha^2 K^2.$$ 

In this case, a similar argument as case (i) implies that

$$V'_i(t) \leq 2ab_i(\alpha K) \left[ -a_i(\alpha - a_i(K)\tau_i) + \sum_{j=1}^{n} |\mu_{ij}| \right]K^2 < 0,$$

which leads to $\lim_{t \to +\infty} V_i(t) = -\infty$, a contradiction. This proves the claim.

By Lemma 3.1, we know that for $t \geq 0, i = 1, \ldots, n$, $|u_i(t) - \rho_i| \leq K$. Since $\phi \in BC$ and $\lim_{s \to -\infty} g(s) = +\infty$, we see that there is a $S_1 > 0$ such that

$$\frac{(||\phi - \rho||_\infty + K)}{g(-S_1)} < \alpha K.$$ 

Thus for $t \geq T_1 + S_1 = \sigma_1$,

$$||u_i - \rho||_g \leq \alpha K.$$

Observe that $d_i(K)$ is strictly increasing with respect to $K$. If we replace $K$ by $\alpha K$ in Lemma 3.1, we see that all its assumptions are satisfied. Clearly,

$$a_i(\alpha - d_i(\alpha K)\tau_i) > a_i(\alpha - d_i(K)\tau_i) > \sum_{j=1}^{n} |\mu_{ij}|.$$
Hence, we can repeat the above argument and conclude that there is a \( \sigma_2 > \sigma_1 \) such that for \( t \geq \sigma_2 \),
\[
\|u_t - \rho\|_g \leq \alpha^2 K.
\]
By repeating such an argument again and again, we obtain a sequence \( \sigma_1 < \sigma_2 < \cdots < \sigma_i < \sigma_{i+1} \), \( \lim_{i \to +\infty} \sigma_i = +\infty \), such that for \( t \geq \sigma_i \),
\[
\|u_t - \rho\|_g \leq \alpha^i K.
\]
This clearly implies that
\[
\lim_{t \to +\infty} u(\phi)(t) = \rho. \quad \Box
\]
Equivalently, we can state the above theorem as:

**Theorem 3.3.** In addition of (H1)-(H5), assume that system (2.1) is diagonally dominant. Denote for \( i = 1, \ldots, n, K > 0 \),
\[
\tau_i(K) = \frac{a_i - \sum_{j=1}^n |\mu_{ij}|}{a_i b_i(K + \rho_i)(a_i + \sum_{j=1}^n |\mu_{ij}|)}.
\]
Assume further that for some \( K > 0 \), \( \tau_i \leq \tau_i(K) \), \( i = 1, \ldots, n \), and \( \phi \) satisfies the assumptions (ii) and (iii) of Lemma 3.1. Then
\[
\lim_{t \to +\infty} u(\phi)(t) = \rho.
\]

**Proof.** Note that \( \tau_i \leq \tau_i(K) \) is equivalent to (i) in Lemma 3.1. The rest is trivial. \( \Box \)

An immediate consequence of the above theorem is the following corollary:

**Corollary 3.4.** In addition to the assumptions of Theorem 3.3, assume further that
\[
\limsup_{t \to +\infty} |u(\phi)(t)| < K.
\]
Then \( \tau_i \leq \tau_i(K) \) implies that
\[
\tau_i(k) = \tau_i(K) \leq \tau_i(K) \implies \lim_{t \to +\infty} u(\phi)(t) = \rho.
\]

The more general version of Theorem 3.3 takes the form:

**Theorem 3.5.** In addition to (H1)-(H5), assume that in (2.1) there exist \( \delta_i > 0 \), \( i = 1, 2, \ldots, n \), such that
\[
a_i > \delta_i \sum_{j=1}^n \delta_j^{-1} |\mu_{ij}|.
\]
Assume further that there is a \( K > 0 \) such that \( \tau_i \leq \tau_i(K) \), where
\[
\tau_i(K) = \frac{a_i - \delta_i \sum_{j=1}^n \delta_j^{-1} |\mu_{ij}|}{a_i b_i(K + \rho_i)(a_i + \delta_i \sum_{j=1}^n \delta_j^{-1} |\mu_{ij}|)}
\]
i = 1, 2, \ldots, n, and \( \phi \) satisfies the assumptions (ii) and (iii) of Lemma 3.1. Then
\[
\lim_{t \to +\infty} u(\phi)(t) = \rho.
\]

**Proof.** If we denote \( U_i(t) = \delta_i u_i(t) \), then \( U(t) = (U_1(t), \ldots, U_n(t)) \) satisfies
\[
U_i'(t) = \delta_i b_i(\delta_i^{-1} U_i(t)) \tilde{G}_i(U_i(\cdot)), \quad i = 1, \ldots, n,
\]
(3.10)
where
\[
\bar{G}_i(U_i(\cdot)) = r_i - a_i \delta^{-1}_i \int_{-\tau_i}^{0} U_i(t + \theta) \, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} U_j(t + \theta) \, d(\delta^{-1}_i \mu_{ij}(\theta)).
\] (3.11)

If we denote \( \bar{a}_i = a_i \delta^{-1}_i, \bar{\mu}_{ij}(\theta) = \delta^{-1}_i \mu_{ij}(\theta) \) and \( \bar{b}_i(U_i(t)) = \delta_i b_i(\delta^{-1}_i U_i(t)) \), then (3.10) and (3.11) is identical to (2.1) and (2.2) (except the bars). The condition (3.9) thus implies that (3.10) is diagonally dominant. The rest follows Theorem 3.3.

Remark 3.6. If we replace \( a_i \int_{-\tau_i}^{0} u_i(t + \theta) \, d\mu_i(\theta) \) by \( a_i u_i(t) \) in system (2.1), then the assumption of diagonal dominance implies that all solutions of (2.1) tend to the unique saturated equilibrium \( \rho \) [23]. Equivalently, this means that \( \rho \) is globally asymptotically stable with respect to admissible initial functions. Our results indicate that this global asymptotical stability of \( \rho \) is maintained provided that (i) initial functions are bounded (in BC norm) and the derivatives of their \( i \)-th component are also bounded properly in the interval \( [-\tau_i, 0] \), (ii) \( \tau_i \) are small enough. In view of the fact that many real systems are studied with bounded initial functions with bounded initial derivatives, and the time delays \( \tau_i \) are usually regarded as small, our results suggest that when the system is diagonally dominant, instantaneous negative feedbacks \( a_i u_i(t) \) are suitable approximations of delayed negative feedbacks \( (a_i \int_{-\tau_i}^{0} u_i(t + \theta) \, d\mu_i(\theta)) \). It should be pointed out that the infinite delays appearing in (2.2) do not create any real difficulties in our analysis; therefore restricting them to a finite delay case will not provide any new results from our method.

If a uniform bound can be found for solutions of (2.1), then Corollary 3.4 asserts that \( \rho \) is globally asymptotically stable as long as the \( \tau_i \)s are small enough. If the initial functions are not differentiable, then one can replace \( \phi \) by \( u_T(\phi) \) for some \( T > \max \{ \tau_i, i = 1, \ldots, n \} \). This way the initial function becomes differentiable for \( s \geq -\tau_i \). The resulting estimate may change, though. Also, our estimate of the size of delay is not optimum even in scalar cases.

4. Convergence in finite delay systems

We consider now a finite delay version of system (2.1).
\[
u_i(t) = b_i(u_i(t)) \left[ r_i - a_i \int_{-\tau_i}^{0} u_i(t + \theta) \, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} u_j(t + \theta) \, d\mu_{ij}(\theta) \right],
\] (4.1)
where \( \tau_{ij} > 0, i, j = 1, \ldots, n \) (H4) thus reduces to
\[
(H4)' \; \; \mu_{ij}(\theta) \; \; \text{are bounded real-valued Borel measures on } [-\tau_{ij}, 0] \; \; \text{with total variation } |\mu_{ij}|.
\]

Denote
\[
\tau = \max \{ \tau_i, \tau_{ij}: i, j = 1, \ldots, n \}.
\]

Clearly, (H5) should be replaced by
\[
(H5)' \; \; u_0(s) = \phi(s), s \in [-\tau, 0], \phi \in \text{BC}([-\tau, 0], \mathbb{R}^n), \phi(0) > 0.
\]

Our objective of this section is to establish convergence criterion for system
(4.1) when it is not diagonally dominant. For notational convenience, we define
\[
\mu_{ij} = \int_{\tau_{ij}}^{0} d\mu_{ij}(\theta), \quad i, j = 1, \ldots, n,
\]
\[
D = \text{diag}(d_1, d_2, \ldots, d_n), \quad d_i > 0, \quad i = 1, \ldots, n.
\]
For such a positive diagonal matrix \(D\) and a nonnegative steady state \(\rho = (\rho_1, \ldots, \rho_n)\) of (4.1), we define
\[
V_{D,\rho}(u) = \sum_{i=1}^{n} d_i \int_{\rho_i}^{\rho_i + u_i} \frac{s - \rho_i}{b_i(s)} ds, \quad u \in \text{Int } R^n_+,
\]
where \(R^n_+ = \{(u_1, \ldots, u_n): u_i \geq 0, i = 1, \ldots, n\}\) and \(\text{Int } R^n_+\) denotes the interior of \(R^n_+\). It is easy to show that \(V_{D,\rho}(u) > V_{D,\rho}(\rho) = 0\) if \(u \neq \rho\) and \(u \in \text{Int } R^n_+\). In the rest of this section, we assume that \(\rho\) is the unique positive steady state of system (4.1).

For any number \(V_0 > 0\), we denote
\[
A(V_0) = \{u: u \in R^n_+, V_{D,\rho}(u) = V_0\}.
\]
In this section, we adopt the standard \(R^n\) norm; if \(u = (u_1, \ldots, u_n) \in R^n\), then \(|u| = (\sum_{i=1}^{n} u_i^2)^{\frac{1}{2}}\). Since \(\lim_{|u| \to \infty} V_{D,\rho}(u) = +\infty\), we see that \(A(V_0)\) compact. We can thus define
\[
p(V_0) = \frac{\max \{|u - \rho|: u \in A(V_0)\}}{\min \{|u - \rho|: u \in A(V_0)\}}.
\]
It is easy to see that \(p(V_0)\) is strictly increasing with respect to \(V_0\) and \(p(V_0) \to 1\) as \(V_0 \to 0\). It should be pointed out here that \(p(V_0)\) also depends on \(D\) and \(\rho\). Denote
\[
A = (a_{ij})_{n \times n},
\]
where
\[
a_{ii} = -a_i + \mu_{ii}; \quad \text{and} \quad a_{ij} = \mu_{ij} \text{ when } i \neq j, \quad i, j = 1, \ldots, n.
\]
We call system (4.1) \(VL\)-stable (Volterra–Liapunov stable [17]), if there is a positive diagonal matrix \(D = \text{diag}(d_1, \ldots, d_n)\) such that \(DA + A^TD\) is negative definite.

Finally, we define for any positive numbers \(M, N,\)
\[
F(M, N) = \{\phi: \phi \in C([-\tau, 0], R^n_+), \|\phi(\cdot) - \rho\| < M, \|\phi'(\cdot)\| < N\},
\]
where \(|\cdot|\) is the uniform norm (with respect to \(|\cdot|\) in \(R^n\)) in \(C([-\tau, 0], R^n_+)\).

We are now ready to state and prove our main result of this section.

**Theorem 4.1.** In addition to (H1)–(H3) and (H4)', assume that (4.1) is \(VL\)-stable. Then for any positive number \(M\), there is an \(N = N(M)\) and \(\tau(M)\) such that \(\tau < \tau(M)\) and \(\phi \in F(M, N)\) implies that \(\lim_{t \to +\infty} u(\phi)(t) = \rho\), where \(\rho\) is the unique positive steady state of (4.1).

**Proof.** Since (4.1) is \(VL\)-stable, there is a \(D = \text{diag}(d_1, \ldots, d_n)\), such that \(E = DA + A^TD\) is negative definite. Thus there is a \(\lambda > 0\) such that for any
Convergence in Lotka–Volterra-type delay systems

\[ u = (u, \ldots, u_n), \]

\[ uE u^T < -\lambda u \cdot u^T = -\lambda \sum_{i=1}^{n} u_i^2. \]  
(4.4)

Denote for \( u(t) \in \text{Int} \ R^n_+ \)

\[ V(u) = \sum_{i=1}^{n} d_i \int_{\rho_i}^{\infty} \frac{u_i(s) - \rho_i}{b_i(s)} ds, \]

\[ W(u(t)) = \max_{\theta \in [-\tau, 0]} V(u(t + \theta)). \]

The derivative of \( V(u)(t) \) along a solution \( u(t) \) of (4.1) takes the form

\[ \dot{V}(u)(t) = \sum_{i=1}^{n} d_i (u_i(t) - \rho_i) Q_i(u, \cdot), \]  
(4.5)

where

\[ Q_i(u, \cdot) = r_i - a_i \int_{-\tau_i}^{0} u_i(t + \theta) d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} u_j(t + \theta) d\mu_{ij}(\theta) \]

\[ = r_i - a_i u_i(t) + \sum_{j=1}^{n} \mu_{ij} u_j(t) - a_i \int_{-\tau_i}^{0} [u_i(t + \theta) - u_i(t)] d\mu_i(\theta) \]

\[ + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} [u_j(t + \theta) - u_j(t)] d\mu_{ij}(\theta). \]  
(4.6)

Note that

\[ r_i - a_i u_i(t) + \sum_{j=1}^{n} \mu_{ij} u_j(t) = -a_i (u_i(t) - \rho_i) + \sum_{j=1}^{n} \mu_{ij} (u_j(t) - \rho_j). \]

Denote

\[ V_0 = V_0(M) = \max \{ V(u): |u - \rho| \leq M \}, \]  
(4.7)

\[ \bar{u} = \bar{u}(M) = \max \{|u - \rho|: V(u) = V_0\}, \]  
(4.8)

\[ q = q(M) = \max \left\{ b_i (\rho_i + \bar{u}) (a_i + \sum_{j=1}^{n} |\mu_{ij}|) : i = 1, 2, \ldots, n \right\}, \]  
(4.9)

\[ N = N(M) = q(M) \bar{u}(M). \]  
(4.10)

Clearly \( \bar{u} \geq M \). If \( |u(t) - \rho| \leq \bar{u} \) for \( t \in [-\tau, t^*] \), then \( |u_i(t)| \leq b_i (\rho_i + \bar{u}) (a_i + \sum_{j=1}^{n} |\mu_{ij}|) \bar{u} \leq N \), for \( t \in [0, t^*] \).

We claim that if \( \phi \in F(M, N) \) and

\[ \tau < \frac{\lambda p(V_0)^{-1}}{2q \sum_{i=1}^{n} d_i (a_i + \sum_{j=1}^{n} \mu_{ij})} = \tau(M), \]  
(4.11)

then \( |u(\phi)(t) - \rho| < \bar{u}, t \geq 0 \).

Otherwise, there is a \( t^* > 0 \) such that \( V_0 = V(u(t^*)) = W(u(t^*)), \) and \( \dot{W}(u(t^*)) \leq 0 \). We note from the definition of \( p(V_0) \) that

\[ p(V_0)^{-1} \bar{u} \leq |u(t^*) - \rho| \leq \bar{u}. \]  
(4.12)
Hence, we have
\[
\dot{W}(u(t^*)) = \dot{V}(u(t^*)) + \sum_{i=1}^{n} d_i(u_i(t^*) - \rho_i)Q_i(u_i(t))
\]
\[
= \sum_{i=1}^{n} d_i(u_i(t^*) - \rho_i) \left[ -a_i(u_i(t^*) - \rho_i) + \sum_{j=1}^{n} \mu_{ij}(u_j(t^*) - \rho_j) \right]
\]
\[
+ \sum_{i=1}^{n} d_i(u_i(t^*) - \rho_i) \left[ -a_i \int_{-\tau_i}^{0} (u_i(t^* + \theta) - u_i(t^*)) d\mu_i(\theta) \right]
\]
\[
+ \sum_{j=1}^{n} \int_{-\tau_j}^{0} (u_j(t^* + \theta) - u_j(t^*)) d\mu_{ij}(\theta)
\]
\[
\leq \frac{1}{2}(u(t^*) - \rho)(DA + A^T D)(u(t^*) - \rho)^T + \sum_{i=1}^{n} d_i \left[ a_i N\tau_i + \sum_{j=1}^{n} N\mu_{ij}\tau_{ij} \right]
\]
\[
\leq -\frac{1}{2}\lambda p(V_0)^{-1}\dot{u}^2 + \ddot{u} \sum_{i=1}^{n} d_i \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right) \tau q
\]
\[
= \ddot{u} \left[ -\frac{1}{2}\lambda p(V_0)^{-1} + \tau q \sum_{i=1}^{n} d_i \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right) \right] < 0,
\]
i.e. \(\dot{W}(u(t^*)) < 0\), a desired contradiction. This proves the claim.

Denote
\[
\dot{u} = \dot{u}(\phi) = \limsup_{t \to +\infty} |u(\phi)(t) - \rho|, \quad \phi \in F(M, N).
\] (4.13)

If the Theorem is false, then for some \(\phi \in F(M, N)\), \(\dot{u} > 0\). Assume in the following that this is the case. Denote
\[
V^* = \limsup_{t \to +\infty} V(u(\phi)(t)),
\]
\[
u^* = \max \{|u - \rho|: V(u) = V^*, \quad \nu_* = \min \{|u - \rho|: V(u) = V^*\}.
\]

The proof of the above claim clearly implies that \(V^* \leq V_0\). It is trivial to see that \(u^* \geq \dot{u} \geq \nu_* > 0\). Since \(-\frac{1}{2}\lambda p(V_0)^{-1} + \tau q \sum_{i=1}^{n} d_i (a_i + \sum_{j=1}^{n} \mu_{ij}) < 0\), there is an \(\varepsilon_0 = \varepsilon_0(\ddot{u}) > 0\), such that
\[
-\frac{1}{2}\lambda p(V_0 + \varepsilon_0)^{-1}(\dot{u} - \varepsilon_0) + \left[ \tau q \sum_{i=1}^{n} d_i \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right) \right](\dot{u} + \varepsilon_0) < 0. \] (4.14)

By the definition of \(\dot{u}\), we see that there is a \(t_1 > 0\) such that for \(t \geq t_1\), \(|u(\phi)(t) - \rho| \leq \min \{\ddot{u}, \dot{u} + \varepsilon_0\}\). Since \(u_* = p(V^*)^{-1}u^*\), we can choose a small positive \(\varepsilon_1\), such that
\[
u^* - \varepsilon_1 > p(V^*)^{-1}(u^* - \varepsilon_0) \geq p(V_0 + \varepsilon_0)^{-1}(\dot{u} - \varepsilon_0).
\] (4.15)

Clearly, there is a \(t_2 = t_2(\varepsilon_1) > t_1 + \tau\), such that
\[
|u(\phi)(t_2) - \rho| > \nu_* - \varepsilon_1, \quad \dot{V}(u(\phi)(t_2)) \geq 0. \] (4.16)

Note that for \(t \geq t_1 + \tau\),
\[
u'_i(t) \leq q(\dot{u} + \varepsilon_0).
\]
By a similar argument as the proof of the above claim, we obtain that
\[
\dot{V}(u(\phi)(t_2)) \\
\leq \frac{1}{2}(u(\phi)(t_2) - \rho)(DA + A^TD)(u(\phi)(t_2) - \rho)^T \\
+ \sum_{i=1}^{n} d_i |u(\phi)(t_2) - \rho| \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right) \tau q(\hat{u} + \varepsilon_0)
\]
\[
\leq - \frac{1}{2} \lambda |u(\phi)(t_2) - \rho|^2 + |u(\phi)(t_2) - \rho| \tau q(\hat{u} + \varepsilon_0) \sum_{i=1}^{n} d_i \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right)
\]
\[
= |u(\phi)(t_2) - \rho| \left\{ - \frac{1}{2} \lambda |u(\phi)(t_2) - \rho| + \left[ \tau q \sum_{i=1}^{n} d_i \left( a_i + \sum_{j=1}^{n} \mu_{ij} \right) \right](\hat{u} + \varepsilon_0) \right\}.
\]
Now the fact that \(|u(\phi)(t_2) - \rho| > u_* - \varepsilon_1 > p(V_0 + \varepsilon_0)^{-1}(\hat{u} + \varepsilon_0)|
\]
[together with (4.14) imply that
\[
\dot{V}(u(\phi)(t_2)) < 0,
\]
which contradicts (4.16). This indicates that \(\hat{u}\) must be zero, proving the theorem. \(\square\)

Similar comments to those included in Remark 3.6 can be made for the above theorem. It is well known that [17] if a nondelayed Lotka–Volterra-type system is VL-stable, then its unique positive steady state (if any) is globally asymptotically stable with respect to positive initial data. Roughly, Theorem 4.1 asserts that if (i) all involved delays are small, and (ii) the initial (positive) functions are small and smooth, then this positive steady state continues to attract neighbouring solutions. This partially justifies that in some real-life systems if delays are expected to be small, one can approximate these systems by models consisting of only ordinary differential equations.

In theory, one may also make use of the friendly space $UC_g$ to allow infinite delays in the analysis of this section. The resulting findings may generalise and improve Theorem 4.1. However, the details may become tedious. Relevant arguments can be found in [21].

References


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