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Global Attractivity and Periodic Solutions 
in Delay-Differential Equations Related to 
Models in Physiology and Population Biology

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With appropriate assumptions, the following two general first-order nonlinear differential delay equations may be employed to describe some physiological control systems as well as some population growth processes:

\[ x'(t) = f \left( \int_{-\tau}^{-\sigma} x(t + s)d\mu(s) \right) - g(x(t)) \]

and

\[ x'(t) = x(t) \left[ f \left( \int_{-\tau}^{-\sigma} x(t + s)d\mu(s) \right) - g(x(t)) \right]. \]

It is assumed that \( g(x) \) is strictly increasing, \( g(0) = 0 \), and each of these two equations has a unique positive steady state. Sufficient conditions are obtained for this steady state to be a global attractor (with respect to continuous positive initial functions) when \( f(x) \) is monotone or has only one hump. We also establish the global existence of periodic solutions in these equations.

Key words: global asymptotical stability, oscillation, periodic solutions, nonlinear delay equation, haematopoiesis

1. Introduction

In order to describe some physiological control systems, Mackey and Glass [25] proposed the following three first-order nonlinear differential-delay equations as their appropriate models:

\[ x'(t) = \lambda - \frac{\alpha V_m x(t)x^n(t - \tau)}{\theta^n + x^n(t - \tau)}, \tag{1.1} \]

\[ P'(t) = \frac{\beta_0 \theta^n}{\theta^n + P^n(t - \tau)} - \gamma P(t), \tag{1.2} \]

and

\[ P'(t) = \frac{\beta_0 \theta^n P(t - \tau)}{\theta^n + P^n(t - \tau)} - \gamma P(t). \tag{1.3} \]

Here \( \lambda, \alpha, V_m, n, \tau, \theta, \beta_0 \) and \( \gamma \) are positive constants. Equation (1.1) is used for studying a ‘dynamic disease’ involving respiratory disorders, where \( x(t) \) denotes the arterial CO\(_2\) concentration of a mammal. In (1.1), \( \lambda \) is the CO\(_2\) production
rate, $V_m$ denotes the maximum ‘ventilation’ rate of CO$_2$, and $\tau$ is the time between oxygenation of blood in the lungs and stimulation of chemoreceptors in the brainstem. Equations (1.2) and (1.3) are proposed as models of hematopoiesis (blood cell production). In these two equations, $P(t)$ denotes the density of mature cells in blood circulation, $\tau$ is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream. Details for the derivation of these equations can be found in [25].

Subsequently, these equations are studied by many authors from various angles. Local asymptotic stability analysis, existence of periodic solution through Hopf bifurcation, and numerical simulation are presented in Glass and Mackey [4], Mackey [23, 24]. In [26], Mackey and an der Heiden proposed and studied several related models consisting of differential delay equations, the numerical simulations presented there indicate the existence of apparently aperiodic ("chaotic") solutions (see also the references cited therein). More theoretical explanations of this "chaotic" behavior for equation (1.3) can be found in [13, 15, 31]. Global existence of periodic solutions in (1.2) and (1.3) is first established by Hadeler and Tomiuk [14]. In [27], Mallet-Paret and Nussbaum studied the singularly perturbed differential-delay equation

$$\epsilon x'(t) = -x(t) + f(x(t - 1)), $$

where existence of periodic solutions is shown using global continuation technique based on degree theory. In [12], Hale and Lunel proved that if high frequency perturbations are introduced in a special way, they can eliminate the “chaotic” behavior in equation (1.3).

By letting $x(t) = y^{-1}(t)$, we see that equation (1.1) reduces to

$$y'(t) = y(t) \left[ \frac{\alpha V_m}{\beta y^n(t - \tau) + 1} - \lambda y(t) \right].$$  \hspace{1cm} (1.4)

We notice that equation (1.4) resembles the well-known logistic equation

$$y'(t) = y(t)[\gamma - \lambda y(t)],$$ \hspace{1cm} (1.5)

where in (1.4) $\gamma$ is replaced by a delayed term. Motivated by the mathematical forms and the physiological or biological backgrounds of the above-mentioned equations, we thus propose to study in this paper the following general nonlinear delay equations:

$$x'(t) = f \left( \int_{-\tau}^{t} x(t + s) d\mu(s) \right) - g(x(t))$$ \hspace{1cm} (1.6)

and

$$x'(t) = x(t) \left[ f \left( \int_{-\tau}^{t} x(t + s) d\mu(s) \right) - g(x(t)) \right].$$ \hspace{1cm} (1.7)

Here $0 < \sigma < \tau$ are constants, $f(x)$ and $g(x)$ are continuously differentiable, $\mu(s)$ is nondecreasing and normalized to satisfy $\mu(-\sigma) - \mu(-\tau) = 1$. We always assume
that (1.6) and (1.7) have a unique positive steady state. Clearly, equation (1.6) has
equations (1.2) and (1.3) as special cases, while equation (1.7) is more general than
equation (1.4). When $f\left(\int_{t-\tau}^{t-\sigma} x(t+s) d\mu(s)\right) = px(t-\tau) e^{-ax(t-\tau)}$, $g(x) = \delta x$, where
$p$, $a$ and $\delta$ are positive constants, (1.6) reduces to

$$x'(t) = px(t-\tau) e^{-ax(t-\tau)} - \delta x(t).$$

(1.8)

This is the model proposed by Gurney et al. [8] in order to describe the population
dynamics of Nicholson’s blowflies. For various special forms of equation (1.7) which
have appeared in literature as mathematical population models, the readers are
referred to the recent monograph of Gopalsamy [5]. One apparent advantage of
using equations of form (1.6) or (1.7) as models of real systems is that we can allow
the delay effects to be distributed in a period of time, which is more realistic than
assuming that the delay is of the discrete type.

As pointed out earlier, equations (1.6) and (1.7) are capable of producing
“chaotic” solutions. Therefore, it is very important for us to know when these
equations are less “chaotic” in the sense that no “chaotic” solutions may appear.
An obvious approach for this problem is to derive sufficient conditions for its unique
positive steady state to be globally asymptotically stable. These sufficient condi-
tions are the main objectives of this paper. The most relevant works along this
direction are documented in [6] and [7]. In these works, Gopalsamy et al. obtained
sufficient conditions for the positive steady states in (1.1)–(1.3) to be global
attractors. Their method is to compare these nonlinear differential delay equations
to a nonautonomous linear one, for which the asymptotic stability condition for
its trivial solution is available. Roughly speaking, their main finding is that, if $\tau$
is small enough, then the positive steady states in (1.1)–(1.3) are global attractors.
Sufficient as well as necessary and sufficient conditions for all positive solutions of
(1.1)–(1.3) to oscillate about their positive steady states are also given in [6] and
[7].

The methods to be used in this paper are entirely different from the one used by
Gopalsamy et al. [6, 7]. To some extent, our approaches are related to the method
of Razumikhin-type function. We are able to obtain sufficient conditions for the
positive steady states in (1.6) and (1.7) to be global attractors regardless of the
length of delay.

Finally, we also establish a global existence result of periodic solutions in equa-
tions (1.6) and (1.7) when they have only one discrete delay (which may be state
dependent).

This paper is organized as follows: In the next section, we present some prelimi-
nary results which will be needed in the subsequent sections. Section 3 considers
the global asymptotic stability problem for equations (1.6) and (1.7) when $f(x)$
is strictly decreasing. Section 4 deals with the same problem when $f(x)$ is strictly
increasing or has exactly one hump. Section 5 discusses the global existence of
periodic solutions. The last section is devoted to discussion.
2. Preliminaries

In this paper, we propose to study the global asymptotic stability aspect of equations (1.6) and (1.7). Throughout this paper, we always assume that $f(x)$ and $g(x)$ are positive functions that are continuously differentiable for $x > 0$; $g(0) = 0$, $g(x)$ is strictly increasing, $\lim_{x \to +\infty} g(x) = +\infty$, and $\lim_{x \to +\infty} f(x)/g(x) = 0$; $f(x)$ is either monotone or has only one hump; $0 < \sigma < \tau$ are constants, $\mu(s)$ is nondecreasing, $\mu(-\sigma) - \mu(-\tau) = 1$; there exists a unique $x^* > 0$, such that $g(x^*) = f(x^*)$. Motivated by the physiological or biological backgrounds of these equations, we assume that the initial conditions for (1.6) and (1.7) are of the type:

$$x(s) = \phi(s) > 0, \ s \in [-\tau, 0], \quad (2.1)$$

$\phi(s) \in C([-\tau, 0], R)$, where $C = C([-\tau, 0], R)$ is the set of continuous functions defined on $[-\tau, 0]$.

In the rest of this paper, we say $x(t)$ is a solution of (1.6) or (1.7), if it satisfies an initial condition of the form (2.1). Standard theory [11] implies that the above-mentioned initial value problems have unique (local) solutions which are continuously dependent on related functions and parameters.

We call a function $x(t)$ (defined on $[0, +\infty)$) oscillatory about $x^*$ if there exists a sequence $\{t_n\} \to +\infty$ as $n \to +\infty$, such that $x(t_n) = x^*, \ n = 1, \ldots$. Otherwise, we call it nonoscillatory about $x^*$. When $x^* = 0$, we simply call it oscillatory or nonoscillatory.

We say $x(t) \equiv x^*$ is globally asymptotically stable (G.A.S.) in (1.6) (or (1.7)), if, for fixed $\tau$, all solutions of (1.6) and (2.1) (or (1.7) and (2.1)) tend to $x^*$ as $t \to +\infty$. We say $x(t) \equiv x^*$ is absolutely globally asymptotically stable (A.G.A.S.) if it is globally asymptotically stable for all $\tau \geq 0$.

If a function $f(x)$ is strictly monotone, we denote $f^{-1}(x)$ as the inverse function of $f(x)$.

The following facts are used implicitly in the subsequent sections.

**Proposition 2.1.** The solution $x(t)$ of (1.6) (or (1.7)) and (2.1) is positive and bounded for $t \geq 0$.

**Proof.** If $x(t)$ is not positive for $t \geq 0$, then there is a $\bar{t} > 0$, such that $x(t) > 0$ for $t \in [0, \bar{t})$, $x(\bar{t}) = 0$. Thus, we must have $x'(\bar{t}) \leq 0$, which requires that

$$f \left( \int_{-\tau}^{-\sigma} x(\bar{t} + s) d\mu(s) \right) \leq g(x(\bar{t})) = 0.$$

Clearly, this is impossible if $f(x) > 0$ for $x \geq 0$. Our assumption of $f(x)$ implies that this can be true only when $f(0) = 0$, and if

$$\int_{-\tau}^{-\sigma} x(\bar{t} + s) d\mu(s) = 0.$$

However, this is a contradiction to the definition of $\bar{t}$. 
Assume first that \( f(x) \) is decreasing or has exactly one hump. Then there is an \( \overline{M} > 0 \), such that \( f(x) < \overline{M} \) for \( x \geq 0 \). If \( x(t) \) is not bounded, then there is a \( \bar{t} > 0 \), such that \( x'(\bar{t}) > 0 \), \( x(\bar{t}) > g^{-1}(\overline{M}) \), and \( x(t) < x(\bar{t}) \) for \( -\tau \leq t < \bar{t} \). Clearly, \( x'(\bar{t}) > 0 \) implies that

\[
\int_{-\tau}^{\sigma} x(\bar{t} + s)d\mu(s) > g(x(\bar{t})). \tag{2.2}
\]

However, since \( x(\bar{t}) > g^{-1}(\overline{M}) \), we have \( g(x(\bar{t})) > \overline{M} \), thus

\[
f\left(\int_{-\tau}^{\sigma} x(\bar{t} + s)d\mu(s)\right) > \overline{M}.
\]

This is a contradiction.

Assume now that \( f(x) \) is increasing. Since \( \lim_{x \to +\infty} f(x)/g(x) = 0 \), we see that there is an \( N > 0 \), such that for \( x \geq N \), \( f(x) < g(x) \). If \( x(t) \) is not bounded, then there is a \( \bar{t} > 0 \), such that \( x'(\bar{t}) > 0 \), \( x(\bar{t}) > N \), and \( x(t) < x(\bar{t}) \) for \( -\tau \leq t < \bar{t} \). Again, we have (2.2) valid. Since \( \int_{-\tau}^{\sigma} x(\bar{t} + s)d\mu(s) < x(\bar{t}) \), this leads to

\[
f(x(\bar{t})) > g(x(\bar{t})),
\]

a contradiction to the fact that \( x(\bar{t}) > N \). This proves the proposition. \( \square \)

If \( g(x) \leq Mx \) for some constant \( M \), \( 0 \leq x \leq 1 \), then one can show that the solution of (1.6) (or (1.7)) with the following more general initial function is positive for \( t \geq 0 \),

\[
x(s) = \phi(s) \geq 0, \; s \in [-\tau, 0], \; \phi(0) > 0, \; \phi(s) \text{ is}
\]

piecewise continuous on \([-\tau, 0]\).

This can be seen easily from the fact that, for \( x(t) \geq 0 \),

\[
x'(t) \geq -g(x(t)) \geq -Mx(t)
\]

which implies that

\[
x(t) \geq x(0)e^{-\int_0^t Mds} = x(0)e^{-Mt} > 0.
\]

3. **Global Stability When \( f(x) \) is Decreasing**

We consider first the following autonomous equation

\[
x'(t) = f\left(\int_{-\tau}^{\sigma} x(t + s)d\mu(s)\right) - g(x(t)) \tag{3.1}
\]

where \( \tau > \sigma > 0 \), \( \mu(s) \) is nondecreasing and \( \int_{-\tau}^{\sigma} d\mu(s) = 1 \). We assume throughout this section that
(H1a) \( f(x) \) is strictly decreasing, \( f(0) > 0 \), \( \lim_{x \to +\infty} f(x) = 0 \).

(H2) \( g(x) \) is strictly increasing, \( g(0) = 0 \), \( \lim_{x \to +\infty} g(x) = +\infty \).

Clearly, there exists a unique \( x^* > 0 \), such that \( f(x^*) = g(x^*) \).

**Proposition 3.1.** Assume \( x(t) \) is a solution of (3.1) and it is not oscillatory about \( x^* \), then \( \lim_{t \to +\infty} x(t) = x^* \).

**Proof.** Assume first that \( x(t) - x^* \) is eventually positive, then there is a \( T > 0 \), such that \( x(t) > x^* \) for \( t \geq T - \tau \), and hence for \( t \geq T \), \( x'(t) = f(t - \tau) + s d(\mu(s)) - g(x(t)) \leq f(x^*) - g(x(t)) < 0 \) (since \( f(x) \) is strictly decreasing and \( g(x) \) is strictly increasing). Thus, there exists an \( \bar{x} \geq x^* \), such that

\[
\lim_{t \to +\infty} x(t) = \bar{x}.
\]

(3.2)

If \( \bar{x} > x^* \), then we have

\[
\lim_{t \to +\infty} x'(t) = f(\bar{x}) - g(\bar{x}) < 0.
\]

This implies that

\[
\lim_{t \to +\infty} x(t) = -\infty,
\]

a contradiction to (3.2).

The case that \( x(t) - x^* \) is eventually negative can be dealt with similarly.

\[\Box\]

Our first result presents sufficient conditions for the unique positive steady state \( x(t) = x^* \) to be absolutely globally asymptotically stable.

**Theorem 3.1.** In (3.1), assume that

\[
|g^{-1}(f(y)) - x^*| < |y - x^*|, \ y > 0, \ y \neq x^*.
\]

(3.3)

Then, the steady state \( x(t) = x^* \) is absolutely globally asymptotically stable.

**Proof.** Let \( x(t) \) be a solution of (3.1). Since Proposition 3.1 holds, we may assume that \( x(t) \) is oscillatory about \( x^* \). Denote

\[
u = \limsup_{t \to +\infty} |x(t) - x^*|.
\]

Since \( x(t) \) is bounded, we see that \( u < +\infty \). If \( u \neq 0 \), then, by the continuity of \( g^{-1}(\cdot) \) and \( f(\cdot) \), we see that there exists \( \epsilon > 0 \), such that

\[
|g^{-1}(f(x^* \pm u + \delta)) - x^*| < u - \epsilon, \ \text{for} \ \delta \in [-\epsilon, \epsilon] \ \text{and} \ x^* \pm u + \delta > 0.
\]

(3.4)

Clearly, there exists \( T > 0 \), such that for \( t \geq T \),

\[
|x(t) - x^*| < u + \epsilon.
\]

(3.5)
Let \( \bar{t} > T + 2\tau \), such that

\[
|x(\bar{t}) - x^*| > u - \epsilon, \quad \text{and} \quad x'(\bar{t}) = 0. \tag{3.6}
\]

Assume first \( x(\bar{t}) > x^* \). Since \( x'(\bar{t}) = 0 \), we have

\[
g(x(\bar{t})) = f\left( \int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) \right). \tag{3.7}
\]

(3.6), together with (3.7), yields

\[
u - \epsilon < g^{-1}\left( f\left( \int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) \right) \right) - x^*. \tag{3.8}
\]

(3.8), together with (3.3), gives

\[
|\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) - x^*| > u + \epsilon, \tag{3.9}
\]

This, together with (3.4), implies that

\[
\left| \int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) - x^* \right| > u + \epsilon. \tag{3.10}
\]

Otherwise, by (3.4), we have that

\[
\left| g^{-1}\left( f\left( \int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) \right) \right) - x^* \right| < u - \epsilon,
\]

a contradiction to (3.8).

Since

\[
\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) - x^* = \int_{-\tau}^{-\sigma} (x(\bar{t} + s) - x^*)d\mu(s),
\]

we conclude from (3.10) that there is a \( \theta \in [-\tau, -\sigma] \), such that

\[
|x(\bar{t} + \theta) - x^*| > u + \epsilon.
\]

Clearly, this is a contradiction to (3.5).

The case that \( x(\bar{t}) < x^* \) can be dealt with similarly. This proves the theorem.

\[
\square
\]

It is easy to see that condition (3.3) is equivalent to

\[
|g^{-1}(z) - x^*| < |f^{-1}(z) - x^*|, \quad 0 < z \leq f(0), \quad z \neq g(x^*). \tag{3.11}
\]

Intuitively, this implies that, if we denote \( A, B \) and \( C \) as the intersection points of \( z = z_0, \; 0 < z_0 \leq f(0) \) with \( z = f(x), \; x = x^* \) and \( z = g(x) \) in the \( xz \)-plane, respectively, then the distance from \( A \) to \( B \) is greater than the one from \( B \) to \( C \). It
is also easy to see intuitively that, if \( \max \{ |f'(x)| : x \in [0, 2x^*] \} < \min \{ g'(x) : x \in [0, 2x^*] \} \), then (3.11) holds.

Let \( f(s) = \beta \theta^n/(\theta^n + s^n) \), \( \int_{-\tau}^{\sigma} x(t+s)d\mu(s) = x(t-\tau) \), and \( g(x) = \gamma x \), where \( \beta, \theta, n \) and \( \gamma \) are positive constants. Then equation (3.1) reduces to

\[
x'(t) = \frac{\beta \theta^n}{\theta^n + [x(t-\tau)]^n} - \gamma x(t).
\]  

(3.12)

By applying Theorem 3.1 to (3.12), we obtain

**Corollary 3.1.** Let \( x(t) = x^* \) be the unique positive steady state in (3.12). Assume that one of the following two conditions holds, then \( x(t) = x^* \) is absolutely globally asymptotically stable:

(i) \( n \leq 1 \) and \( \beta < 2\gamma x^* \),
(ii) \( n > 1, \frac{1}{4} \beta \theta^{-1} n^{-1} (n+1)^{n-1} (n-1)^{n-1} < \gamma \).

**Proof.** Clearly, equation (3.12) satisfies (H1a) and (H2). Therefore, Theorem 3.1 can be applied.

We assume first that (i) holds. We have

\[
f'(s) = -n \beta \theta^n s^{n-1} (\theta^n + s^n)^{-2} < 0, \quad \text{for } s > 0,
\]

and

\[
f''(s) = n \beta \theta^n s^{n-2} [(n+1)s^n + (1-n)\theta^n](\theta^n + s^n)^{-3} > 0, \quad \text{for } s > 0.
\]

Denote \( A = (0, \beta) \), \( B = (x^*, \beta) \), \( C = (\beta \gamma^{-1}, \beta) \). Since \( f''(s) > 0 \) for \( s > 0 \), it is easy to see that if \( AB > BC \), then condition (3.3) holds (see Fig. 1). This is equivalent to

\[
x^* > \beta \gamma^{-1} - x^*,
\]

i.e., \( \beta < 2\gamma x^* \).

Assume now that \( n > 1 \). For \( s > 0 \), \( f(s) \) has a unique inflection point at

\[
\bar{s} = \left( \frac{n-1}{n+1} \right)^{\frac{1}{n}} \theta.
\]

Denote

\[
F = \max \{ |f'(s)| : s \geq 0 \}.
\]

Then we see that \( F = |f'(\bar{s})| \). A straightforward calculation shows that

\[
F = \frac{1}{4} \beta \theta^{-1} n^{-1} (n+1)^{n+1} (n-1)^{n-1}.
\]
Fig. 1. The graph for the proof of Corollary 3.3.

Now, it is easy to see that, if $F < \gamma$, then condition (3.3) holds. This proves the corollary. □

**Lemma 3.1.** Let $x(t)$ be a solution of (3.1), then there is a $T > 0$, such that for $t \geq T$, $x(t) < g^{-1}(f(0))$.

**Proof.** We note

$$g^{-1}(f(0)) > g^{-1}(f(x^*)) = g^{-1}(g(x^*)) = x^*.$$  

If $x(t) \geq x^*$ for all large $t$, say $t \geq T$, then we see that $x'(t) \leq 0$ for $t \geq T$, and hence $\lim_{t \to +\infty} x(t) = x^*$. Thus, we may assume $x(t)$ is oscillatory about $x^*$. Let $\bar{t} > 2\tau$, such that $x(\bar{t}) > x^*$ and $x'(\bar{t}) = 0$. Then (3.1) implies that

$$g(x(\bar{t})) = f\left(\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s)\right).$$

Therefore,

$$x(\bar{t}) = g^{-1}\left(f\left(\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s)\right)\right) < g^{-1}(f(0)).$$
This completes the proof. □

In order to state our next result, we need the following definitions:

\[
\overline{x} = g^{-1}(f(0)),
\]
\[
\overline{f} = \max\{-f'(x) : x \in [0, \overline{x}]\},
\]
\[
\overline{g} = \max\{g'(x) : x \in [0, \overline{x}]\}.
\]

**Theorem 3.2.** In (3.1), if \((\overline{f} + \overline{g})\tau < 1\), then \(x(t) = x^*\) is globally asymptotically stable.

**Proof.** Since Proposition 3.1 holds, we may assume \(x(t)\) is a solution of (3.1) which is oscillatory about \(x^*\). By Lemma 3.1, we see that there exists a \(T > 0\), such that for \(t \geq T\), \(x(t) < \overline{x}\). Let \(t > T\), such that \(x(t) > x^*\) and \(x(t)\) is a local maximum. Then, \(x'(t) = 0\), which leads to

\[
g(x(t)) = f \left( \int_{-\tau}^{t} x(t + s)d\mu(s) \right).
\]

Since \(g(x(t)) > g(x^*) = f(x^*)\), and \(f(\cdot)\) is strictly decreasing, we have

\[
\int_{-\tau}^{t} x(t + s)d\mu(s) < x^*.
\]

This implies there is a \(\theta \in [-\tau, 0]\), such that

\[
x(t + \theta) = x^*.
\]

Similarly, we can show that, if \(x(t) < x^*\) and \(x(t)\) is a local minimum, then there is a \(\theta \in [-\tau, 0]\), such that \(x(t + \theta) = x^*\).

We denote

\[
u = \limsup_{t \to +\infty} |x(t) - x^*|.
\]

If \(u > 0\), then there exists \(\epsilon > 0\), such that

\[
u - \epsilon > (\overline{f} + \overline{g})\tau(u + \epsilon).
\]

Let \(T_0 > T\), such that, for \(t \geq T_0 - 2\tau\),

\[
|x(t) - x^*| < u + \epsilon.
\]

We assume first that there is a \(t > T_0\), \(x(t) > x^*\), and \(x(t)\) is a local maximum such that

\[
x(t) - x^* > u - \epsilon.
\]
Let \( \theta \in [-\tau, 0] \), such that \( x(\bar{t} + \theta) = x^* \). By integrating (3.1) from \( \bar{t} + \theta \) to \( \bar{t} \), we arrive at

\[
x(\bar{t}) - x(\bar{t} + \theta) = \int_{\bar{t} + \theta}^{\bar{t}} \left[ f\left( \int_{-\sigma}^{-\tau} x(t + s)d\mu(s) \right) - g(x(t)) \right] dt
\]

\[
= \int_{\bar{t} + \theta}^{\bar{t}} \left[ f\left( \int_{-\sigma}^{-\tau} x(t + s)d\mu(s) \right) - f(x^*) \right] dt + \int_{\bar{t} + \theta}^{\bar{t}} \left[ g(x^*) - g(x(t)) \right] dt
\]

\[
\leq \bar{f}\theta(u + \epsilon) + \bar{g}\theta(u + \epsilon) \leq (\bar{f} + \bar{g})\tau(u + \epsilon).
\]

That is,

\[
u - \epsilon < (\bar{f} + \bar{g})\tau(u + \epsilon),
\]

a contradiction to (3.16). Hence, \( u = 0 \).

Similarly, we can show that \( u = 0 \) if there is a \( \bar{t} > T_0 \), such that \( x(\bar{t}) < x^* \), \( x(\bar{t}) \) is a local minimum and that \( x^* - x(\bar{t}) > u - \epsilon \). The proof is complete. \( \square \)

**Corollary 3.2.** In (3.12), assume \( n > 1 \) and

\[
\left[ \gamma + \frac{1}{4} \beta \theta^{-1} n^{-1}(n + 1) \frac{n + 1}{n} (n - 1) \frac{n - 1}{n} \right] \tau < 1.
\]

Then its unique positive steady state is globally asymptotically stable.

**Proof.** From the proof of Corollary 3.1, we see that

\[
\bar{f} \leq \frac{1}{4} \beta \theta^{-1} n^{-1}(n + 1) \frac{n + 1}{n} (n - 1) \frac{n - 1}{n}.
\]

Clearly, \( \bar{g} = \gamma \). Hence, (3.18) implies that \( (\bar{f} + \bar{g})\tau < 1 \). The conclusion follows from Theorem 3.2. \( \square \)

In the rest of this section, we consider the following Lotka-Volterra type delay equation

\[
x'(t) = x(t) \left[ f\left( \int_{-\sigma}^{-\tau} x(t + s)d\mu(s) \right) - g(x(t)) \right],
\]

where \( f, g \) satisfies (H1a), (H2), respectively, and \( \mu, \sigma \) and \( \tau \) are the same constants as those that appear in (3.1). By a similar argument as the proof of Theorem 3.1, we obtain

**Theorem 3.3.** In (3.19), assume that the assumptions of Theorem 3.1 are satisfied, then its unique positive steady state is absolutely globally asymptotically stable.

A simple modification of the proof of Theorem 3.2 yields:

**Theorem 3.4.** In (3.19), if \( \bar{x}(\bar{f} + \bar{g})\tau < 1 \), then its unique positive steady state is globally asymptotically stable.
Proof. We note that, in order to use the proof of Theorem 3.2, we need only replace \((\bar{f} + \bar{g})\tau\) by \(\bar{x}(\bar{f} + \bar{g})\tau\) and change (3.17) to

\[
\begin{align*}
x(t) - x(t + \theta) = & \int_{t+\theta}^{t} x(t) \left[ f \left( \int_{-\tau}^{-\sigma} x(t + s) \mu(s) \right) - f(x^*) \right] dt \\
& + \int_{t+\theta}^{t} x(t)[g(x^*) - g(x(t))] dt \\
& \leq \bar{x}(\bar{f} + \bar{g})\tau(u + \epsilon).
\end{align*}
\]

Observe, similar statements to Corollaries 3.1 and 3.2 can be established for

\[
x'(t) = x(t) \left[ \frac{\beta \theta^n}{\theta^n + [x(t - \tau)]^n} - \gamma x(t) \right]. 
\]  

(3.20)

To conclude this section, we consider the following delay differential equation

\[
x'(t) = \lambda - \frac{\alpha V_m x(t)x^n(t - \tau)}{\sigma^n + x^n(t - \tau)}, 
\]  

(3.21)

where \(\lambda, \alpha, V_m, \sigma\) and \(\tau\) are positive constants. This is the model proposed by Mackey and Glass [25] for studying a ‘dynamic disease’ involving respiratory disorders. The interested reader is referred to [25] for more details on the physiological background of this model. A nice discussion on the oscillations and global attractivity of this equation is given in Gopalsamy et al. [7].

It is easy to see that the initial value problem of equation (3.21) has a unique solution. If \(x(0) > 0\), then \(x(t) > 0\) for \(t > 0\). In the following, we assume that \(x(0) > 0\). Denote \(y(t) = 1/x(t)\), then we have

\[
y'(t) = y(t) \left[ \frac{\alpha V_m}{\sigma^n y^n(t - \tau) + 1} - \lambda y(t) \right]. 
\]

(3.22)

Let \(z(t) = \sigma y(t)\), then

\[
z'(t) = z(t) \left[ \frac{\alpha V_m}{z^n(t - \tau) + 1} - \lambda \sigma^{-1} z(t) \right].
\]

(3.23)

Clearly, (3.23) is equivalent to equation (3.20). Therefore, Theorems 3.3 and 3.4 can be applied to obtain:

**Corollary 3.3.** Let \(x(t) = x^*\) be the unique positive steady state in (3.21). Assume that one of the following two conditions holds, then it is absolutely globally asymptotically stable:

(i) \(n \leq 1\) and \(\alpha V_m < 2x^*\lambda\sigma^{-1}\),

(ii) \(n > 1\), \(\frac{1}{4} \alpha V_m \sigma n^{-1}(n + 1)^{n+1} (n - 1)^{n-1} < \lambda \sigma^{-1}\).

If \(n > 1\) and

\[
\left[ \lambda \sigma^{-1} + \frac{1}{4} \alpha V_m \sigma n^{-1}(n + 1)^{n+1} (n - 1)^{n-1} \right] \sigma \lambda^{-1} \alpha V_m \tau < 1,
\]
then \( x(t) = x^* \) is globally asymptotically stable.

**Proof.** The first half of the corollary follows from Theorem 3.3 and Corollary 3.1, where one notes that \( \theta = 1, \beta = \alpha V_m, \gamma = \lambda \sigma^{-1} \). The second half follows from Theorem 3.4 and Corollary 3.2, where one notes that \( \lambda = \sigma^{-1} \).

\[ \square \]

4. Global Stability When \( f(x) \) Is Increasing or Has a Hump

In this section, we assume first that the function \( f(\cdot) \) in equation (3.1) satisfies

\[ (H1b) \quad f(\cdot) \text{ is strictly increasing, } f(0) = 0; \text{ there is a unique } x^* > 0, \text{ such that } f(x) > g(x) \text{ for } x \in (0, x^*), \text{ and } f(x) < g(x) \text{ for } x > x^*. \]

All other assumptions remain the same as in the previous section. It is easy to verify that the following equation meets all these requirements:

\[ x'(t) = \frac{\beta \theta^n x(t-\tau)}{\theta^n + [x(t-\tau)]^n} - \gamma x(t), \quad (4.1) \]

where \( \beta, \theta, \tau, n \) and \( \gamma \) are positive constants, \( 0 < n \leq 1 \), and \( \beta > \gamma \).

**Lemma 4.1.** Let \( x(t) \) be a solution of (3.1) such that \( 0 < x(0) < x^* \). Then, for \( t \geq 0 \), \( x(t) > \min \{ x(t) : t \in [0, \tau] \} \).

**Proof.** Denote

\[ x_m = \min \{ x(t) : t \in [0, \tau] \}. \quad (4.2) \]

By Proposition 2.1, we see that \( x_m > 0 \). Assume that the conclusion of the above lemma is false, then there is a \( \bar{t} \geq \tau \), \( x(\bar{t}) = x_m, x(t) \geq x_m \), for \( 0 \leq t \leq \bar{t} \), and \( x'(\bar{t}) \leq 0 \). This leads to

\[ f\left( \int_{-\tau}^{-\sigma} x(\bar{t}+s) d\mu(s) \right) \leq g(x(\bar{t})). \]

By \( (H1b) \), this implies

\[ \int_{-\tau}^{-\sigma} x(\bar{t}+s) d\mu(s) < x(\bar{t}) = x_m, \]

a contradiction to the definition of \( x_m \). The proof is thus completed. \( \square \)

We are ready to state our main result in this section.

**Theorem 4.1.** Assume \( (H1b) \) and \( (H2) \) in (3.1), then \( x = x^* \) is absolutely globally asymptotically stable.

**Proof.** Denote

\[ u = \limsup_{t \to +\infty} |x(t) - x^*|. \quad (4.3) \]

Since \( x(t) \) is bounded, we see that \( u < +\infty \). If \( u > 0 \), then at least one of the following two statements is true:
(i) there is a sequence \(\{t_i\}, \ t_i > t_{i-1}, \ \lim_{i \to +\infty} t_i = +\infty, \) and
\[ \lim_{x \to +\infty} x(t_i) = x^* + u; \]
(ii) there is a sequence \(\{t_i\}, \ t_i > t_{i-1}, \ \lim_{i \to +\infty} t_i = +\infty, \) and
\[ \lim_{x \to +\infty} x(t_i) = x^* - u, \) provided that \(u \leq x^*.\)

Assume first that (i) is true. Then there is an \(\epsilon > 0, \) such that
\[ f(u + \epsilon + x^*) < g(u - \epsilon + x^*). \tag{4.4} \]

For this \(\epsilon, \) there exists a \(T = T(\epsilon) > \tau, \) such that for \(t \geq T - \tau, \) we have
\[ x(t) < u + \epsilon + x^*. \tag{4.5} \]

We have two subcases to consider:

(i) \(x(t)\) is not monotone,

(ii) \(x(t)\) is monotone.

Suppose first that \(x(t)\) is not monotone. Then there is a \(\bar{t} > T, \) such that \(x'((\bar{t})) = 0, \ x(\bar{t}) - x^* > u - \epsilon. \) This implies that
\[ f\left(\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s)\right) = g(x(\bar{t})). \]

Thus,
\[ f\left(\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s)\right) < g(u - \epsilon + x^*). \]

By (4.3), we have
\[ \int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) > u + \epsilon + x^*. \]

Hence, there is a \(\theta \in [-\tau, -\sigma], \) such that
\[ x(\bar{t} + \theta) > u + \epsilon + x^*, \]

a contradiction to (4.5).

Suppose now that \(x(t)\) is monotone. Thus, we must have \(x'(t) \leq 0\) for large \(t\) and
\[ \lim_{t \to +\infty} x(t) = x^* + u. \tag{4.6} \]

However, this leads to
\[ \lim_{t \to +\infty} x'(t) = f(x^* + u) - g(x^* + u) < 0, \]

which implies that
\[ \lim_{t \to +\infty} x(t) = -\infty, \]

a contradiction to (4.6).

Assume now that (ii) is true. Without loss of generality, we may assume that \(0 < x(0) < x^*. \) We can thus choose \(0 < \epsilon < x_m, \) where \(x_m\) is defined as in (4.2), such that
\[ f(x^* - u - \varepsilon) > g(x^* - u + \varepsilon). \]

The rest of the proof is similar to the one for case (i). We omit it here to avoid repetition. \(\square\)

An immediate consequence of the above theorem is

**Corollary 4.1.** In (4.1), if \(0 < n \leq 1\), and \(\beta > \gamma\), then its unique positive steady state is absolutely asymptotically stable.

It is easy to see that the same argument as the proof of Theorem 4.1 can be applied to

\[ x'(t) = x(t) \left[ f \left( \int_{-\tau}^{\sigma} x(t + s) d\mu(s) \right) - g(x(t)) \right], \tag{4.7} \]

where \(f, g\) satisfies (H1a) and (H2), respectively. We have

**Theorem 4.2.** Assume (H1a) and (H2) hold in (4.7), then its unique positive steady state \(x(t) = x^*\) is absolutely globally asymptotically stable.

In the rest of this section, we always assume that the function \(f(\cdot)\) in equation (3.1) satisfies

\((H1c)\) \(f(0) = 0;\) there is an \(x_M > 0,\) such that \(f(\cdot)\) is strictly increasing in \([0, x_M]\) and strictly decreasing in \([x_M, +\infty)\); \(\lim_{x \to +\infty} f(x) \geq 0.\) There is a unique \(x^* > 0,\) such that \(f(x) > g(x)\) for \(x \in (0, x^*)\) and \(f(x) < g(x)\) for \(x > x^*\).

We keep the rest of the assumptions made in the previous section. It is easy to verify that, if \(n > 1,\) then equation (4.1) meets all these requirements. Another interesting example is the following model which has been used in describing the dynamics of Nicholson's blowflies [8],

\[ x'(t) = px(t - \tau)e^{-ax(t - \tau)} - \delta x(t), \tag{4.8} \]

where \(p, \tau, a, \) and \(\delta\) are positive constants.

In order to state our next theorem, we need the following notations.

\[ f_1 = \lim_{x \to +\infty} f(x(t)), \quad f_2 = f(x_M); \tag{4.9} \]

\[ F_1(y) = f^{-1}(y)_{|[0, x_M]}, \quad y \in [0, f_2] \tag{4.10} \]

\[ F_2(y) = f^{-1}(y)_{|[x_M, +\infty)}, \quad y \in (f_1, f_2], \tag{4.11} \]

\[ G(y) = g^{-1}(y), \quad y \in [0, \infty). \tag{4.12} \]

We also need the following lemma.

**Lemma 4.2.** Assume (H1c) and (H2) hold in (3.1), and \(0 < x(0) < x^*\). Then there is a \(\delta > 0, \) \(T > 0,\) such that for \(t \geq T,\) \(x(t) \geq \delta.\)

**Proof.** Since \(x(t)\) is bounded, there is an \(\bar{x} > x^*,\) such that \(x(t) < \bar{x}\) for \(t \geq 0.\) Let \(0 < \bar{x} < x^*\) be so small, such that \(f(\bar{x})\) is strictly increasing on \([0, \bar{x}],[\bar{x}, x^*]\), and

\[ \max\{f(x) : x \in [0, \bar{x}]\} = f(\bar{x}) = \min\{f(x) : x \in [\bar{x}, x^*]\}. \]
If the conclusion of the lemma is false, then there is a \( \bar{t} > \tau \), \( x(\bar{t}) = \min\{x(t) : t \in [0, \bar{t}]\} \), \( x(\bar{t}) < g^{-1}(f(x)) \), and \( x'(\bar{t}) \leq 0 \). However, this leads to
\[
f\left(\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s)\right) < g(x(\bar{t})) < f(x).
\]

By the definition of \( x \) and the choice of \( \bar{t} \), we see that
\[
\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) < x.
\]

By (H1c), we have
\[
\int_{-\tau}^{-\sigma} x(\bar{t} + s)d\mu(s) < x(\bar{t}),
\]
which implies that there is a \( \theta \in [-\tau, -\sigma] \), such that
\[
x(\bar{t} + \theta) < x(\bar{t}),
\]
a contradiction to the definition of \( x(\bar{t}) \). This proves the lemma.

**Theorem 4.3.** Assume (H1c) and (H2) hold in (3.1). If
\[
|G(y) - x^*| < |F_2(y) - x^*| \quad \text{for} \quad y \in (f_1, f_2), \quad y \neq g(x^*), \quad (4.13)
\]
then \( x(t) = x^* \) is absolutely globally asymptotically stable.

**Proof.** Denote
\[
u = \limsup_{t \to +\infty} |x(t) - x^*|;
\]
again, we have \( u < +\infty \). In the following, we assume that \( u > 0 \). It is easy to see that (4.13) is equivalent to
\[
|g^{-1}(f(z)) - x^*| < |z - x^*|, \quad z \in [x_M, \infty), \quad z \neq x^*. \quad (4.14)
\]
If \( x_M < x^* \), then we have
\[
|g^{-1}(f(z)) - x^*| < |z - x^*|, \quad z \in (0, F_1(f(x^*))) \quad (4.15)
\]
since \( f(x) > g(x) \) for \( x \in [0, x^*) \). Also, we have for \( y \in [f(x^*), f(x_M)] \),
\[
0 \leq x^* - F_2(y) \leq x^* - F_1(y). \quad (4.16)
\]
Thus, by (4.13), we have for \( y \in [f(x^*), f(x_M)] \),
\[
0 \leq G(y) - x^* < x^* - F_1(y), \quad (4.17)
\]
which is equivalent to
\[
0 \leq g^{-1}(f(z)) - x^* < x^* - z, \quad z \in [F_1(f(x^*)), x_M]. \quad (4.18)
\]
Combining (4.14), (4.15) and (4.18), we arrive at
\[ |g^{-1}(f(z)) - x^*| < |z - x^*|, \quad z > 0, \quad z \neq x^*. \] (4.19)

If \( x_M \geq x^* \), we can also arrive at (4.19) by a similar argument.

If \( x(t) \) is monotone, then by an argument similar to the one included in the proof of Theorem 4.1 (dealing with the monotone case), we conclude that \( u = 0 \). In case that \( x(t) \) is oscillatory about \( x^* \), the rest of the proof follows from Lemma 4.2 and the proof of Theorem 3.1. We omit the details here to avoid repetition. \( \square \)

Clearly, the above theorem is equally true for equation (4.7).

Let \( x(t) = \theta y(t) \), then (4.1) reduces to
\[ y'(t) = \frac{\beta y(t - \tau)}{1 + y^n(t - \tau)} - \gamma y(t). \] (4.20)

**Corollary 4.2.** In (4.20), assume \( \beta > \gamma \), \( n > 1 \), and
\[ \frac{1}{4} \beta n^{-1}(n - 1)^2 < \gamma, \] (4.21)
then its unique positive steady state is absolutely globally asymptotically stable.

**Proof.** Denote
\[ f(y) = \beta y/(1 + y^n), \quad y \geq 0. \]

We have
\[ f'(y) = \beta \frac{1 - (n - 1)y^n}{(1 + y^n)^2} \] (4.22)
and
\[ f''(y) = \beta n y^{n-1} \frac{(n - 1)y^n - (n + 1)}{(1 + y^n)^3}. \] (4.23)

It is easy to see that \( f(y) \) achieves its maximum value at \( y_M = (n - 1)^{-n-1} \).
Equation (4.23) indicates that the minimum of \( f'(y) \), for \( y \geq y_M \), is achieved at \( y_m \), where \( (n - 1)y_m^n = n + 1 \). A simple algebraic computation yields
\[ |f'(y_m)| = \frac{1}{4} \beta n^{-1}(n - 1)^2. \]

Therefore, if (4.21) is satisfied, then (4.13) holds for equation (4.20). The conclusion now follows from Theorem 4.3. \( \square \)

We now consider equation (4.8). By letting \( y(t) = ax(t) \), it reduces to
\[ y'(t) = py(t - \tau)e^{-\gamma(t - \tau)} - \delta y(t). \] (4.24)

**Corollary 4.3.** In (4.24), assume \( pe^{-2} < \delta < p \), then its unique positive steady state is absolutely globally asymptotically stable.
Proof. Let \( f(y) = pye^{-y} \). We have \( f'(y) = pe^{-y}(1-y) \) and \( f''(y) = pe^{-y}(y-2) \). Hence, its maximum is achieved at \( y = 1 \), and its minimum derivative for \( y \geq 1 \) is achieved at \( y = 2 \). \( |f'(2)| = pe^{-2} \). Now, the conclusion follows from Theorem 4.3.

The next simple theorem deals with the situation when (4.13) fails.

**Theorem 4.4.** In Eq. (3.1), assume that (H1c) and (H2) hold and \( x_M \geq x^* \). Then \( x(t) = x^* \) is absolutely globally asymptotically stable.

**Proof.** If \( x(t) \) is monotone, then by a similar argument as in the proof of Theorem 4.1, we can show that \( \lim_{t \to +\infty} x(t) = x^* \). Thus, we assume in the following that \( x(t) \) is not monotone.

Assume first that \( x_M = x^* \). One can show easily that, if, for some \( t_0 > 0 \), \( x(t_0) \leq x^* \), then \( x(t) \leq x^* \) for \( t \geq t_0 \). If \( x(t) > x^* \) for \( t \geq 0 \), then we see that \( x'(t) < 0 \), which implies that \( x(t) \) is monotone and, therefore, \( \lim_{t \to +\infty} x(t) = x^* \). Hence, without loss of any generality, we may assume that \( x(t) \leq x^* \), \( t \geq 0 \). Using the notations as defined in (4.10) and (4.12), we thus have

\[
|G(y) - x^*| < |F_1(y) - x^*|, \; y \in (0, f(x^*)).
\] (4.25)

Now, a similar argument as the proof of Theorem 4.3 can be made to show that \( \lim_{t \to +\infty} x(t) = x^* \).

Suppose now that \( x_M > x^* \). Denote

\[
u = \limsup_{t \to +\infty} x(t).
\] (4.26)

Assume first that \( u \geq x_M \). Since \( x_M > x^* \), we see there is an \( \epsilon > 0 \), such that

\[
g(u - \epsilon) > f(x_M).
\] (4.27)

Clearly, by the definition of (4.26), we see that there is a \( \tau > \tau \), such that \( x(\tau) > u - \epsilon \) and \( x'(\tau) \geq 0 \). This implies that

\[
g(u - \epsilon) < g(x(\tau)) \leq f \left( \int_{-\tau}^{-\sigma} x(\tau + s) d\mu(s) \right) \leq f(x_M),
\]

a contradiction to (4.27). This proves that \( u < x_M \). This implies that there is a \( T > \tau \), such that, for \( t \geq T \), \( x(t) < x_M \). Again, by using the notations as defined in (4.10) and (4.12), we have

\[
|G(y) - x^*| < |F_1(y) - x^*|, \; y \in (0, f(x_M)), \; y \neq f(x^*).
\] (4.28)

Thus, by a similar argument as the proof of Theorem 4.3, we can show that \( \lim_{t \to +\infty} x(t) = x^* \). This completes the proof.

**Remark 4.1.** For equations (4.20) and (4.24), the results obtained from Theorem 4.4 are not as sharp as the ones derived from Theorem 4.3. However, Theorem 4.4 is very useful when \( f(x) \) decreases rapidly for \( x \geq x_M \).
Our next result is similar to Theorem 3.2, which relates the delay length $\tau$ to the global asymptotic stability of $x(t) = x^*$. Because of Theorem 4.4, we assume in the following that $x_M < x^*$. We need the following definitions:

\begin{align}
\bar{x} &= g^{-1}(f(x_M)), \quad \underline{x} = F_1(g(x^*)); \\
\bar{f} &= \max\{|f'(x)| : x \in [\underline{x}, \bar{x}]\}; \\
\bar{g} &= \max\{g'(x) : x \in [0, \bar{x}]\}.
\end{align}

(4.29) (4.30) (4.31)

We also need the following simple lemma:

**Lemma 4.3.** In (3.1), assume that (H1c) and (H2) hold, then $\limsup_{t \to +\infty} x(t) \leq \bar{x}$.

**Proof.** Denote

\[ u = \limsup_{t \to +\infty} x(t). \]

Assume $u > \bar{x}$, then there is a $\bar{t} > \tau$, such that $x(\bar{t}) > \bar{x}, \ x'(\bar{t}) \geq 0$. This implies that

\[ g(\bar{x}) < g(x(\bar{t})) \leq f \left( \int_{-\tau}^{\bar{t}} x(\bar{t} + s)d\mu(s) \right) \leq f(x_M), \]

a contradiction to (4.29). The proof is complete. \qed

**Theorem 4.5.** In (3.1), assume that (H1c) and (H2) hold, $x_M < x^*$ and $(\bar{f} + \bar{g})\tau < 1$. Then $x(t) = x^*$ is globally asymptotically stable.

**Proof.** By the continuity of $f'(\cdot)$ and $g'(\cdot)$, we see there exists $\epsilon > 0$, such that

\[ (\bar{f}_\epsilon + \bar{g}_\epsilon)\tau < 1, \]

(4.32)

where

\begin{align}
\bar{f}_\epsilon &= \max\{|f'(x)| : x \in [\underline{x}, \bar{x} + \epsilon]\}, \\
\bar{g}_\epsilon &= \max\{g'(x) : x \in [0, \bar{x} + \epsilon]\}.
\end{align}

(4.33) (4.34)

By Lemma 4.3, we see that there is a $T > \tau$, such that

\[ x(t) \leq \bar{x} + \epsilon, \quad t \geq T. \]

(4.35)

Denote

\[ u = \limsup_{t \to +\infty} |x(t) - x^*|. \]

(4.36)

We assume in the following that $u > 0$. We assume first that there is a sequence of $t_i, \ t_{i+1} > t_i > \tau, \ \lim_{i \to \infty} t_i = +\infty, \ x'(t_i) \leq 0,$ and $\lim_{i \to +\infty} x(t_i) = x^* - u$. By Lemma 4.2, we must have $u < x^*$ in this case. Clearly, there exists $0 < \delta < \epsilon$ such that

\[ f_\delta \triangleq \min\{f(x^* - u + \theta\delta) : \theta \in [-1, 1]\} > g(x^* - u + \delta), \]

(4.37)
and
\[ u - \delta > (\bar{f}_e + \bar{g}_e) \tau (u + \delta). \]

Let \( T_1 > T + 2\tau \), such that
\[ x(t) > x^* - u - \delta, \quad t \geq T_1 - 2\tau. \] (4.39)

Let \( \bar{t} = t_{i0} \), such that \( t_{i0} > T_1 \), \( x(\bar{t}) < x^* - u + \delta \). Since \( x'(\bar{t}) \leq 0 \),
\[ f \left( \int_{-\tau}^{\tau} x(\bar{t} + s) d\mu(s) \right) \leq g(x(\bar{t})) < g(x^* - u + \delta). \] (4.40)

Hence, we have from (4.37)
\[ i \quad \int_{-\tau}^{\tau} x(\bar{t} + s) d\mu(s) < x^* - u - \delta \]
or, because of the nonmonotonicity of \( f(\cdot) \),
\[ (ii) \quad \int_{-\tau}^{\tau} x(\bar{t} + s) d\mu(s) > x^*. \]

If (i) holds, then there is a \( \theta \in [-\tau, -\sigma] \), such that \( x(\bar{t} + \theta) < x^* - u - \delta \), a contradiction to (4.39). If (ii) holds, then we see that there is a \( \theta \in [-\tau, 0] \), such that \( x(\bar{t} + \theta) = x^* \). By integrating (3.1) from \( \bar{t} + \theta \) to \( \bar{t} \), we obtain
\[ |x(\bar{t}) - x^*| = \left| \int_{\bar{t} + \theta}^{\bar{t}} \left[ f \left( \int_{-\tau}^{\tau} x(t + s) d\mu(s) \right) - g(x(t)) \right] dt \right| \]
\[ \leq (\bar{f}_e + \bar{g}_e) \theta (u + \delta). \]

That is,
\[ u - \delta \leq (\bar{f}_e + \bar{g}_e) \tau (u + \delta), \]
a contradiction to (4.38).

Assume now that there is a sequence of \( t_i \), \( t_{i+1} > t_i > \tau \), \( \lim_{i \to +\infty} t_i = +\infty \), \( x'(t_i) \geq 0 \), \( x(t_i) > x^* \), and \( \lim_{i \to +\infty} x(t_i) = x^* + u \). In this case, we can show easily that there are \( \theta_i \in [-\tau, \theta] \), such that \( x(t_i + \theta_i) = x^* \). By a similar argument as the one made in the previous case, we again arrive at a contradiction. Therefore, \( u \) must be zero, proving the theorem. \( \square \)

By the virtue of the proof of Theorem 4.5, we have

**Theorem 4.6.** In (4.7), assume that (H1c) and (H2) hold, \( x_M < x^* \) and \( \bar{x}(\bar{f} + \bar{g}) \tau < 1 \). Then \( x(t) = x^* \) is globally asymptotically stable.

Clearly, both Theorems 4.3 and 4.4 are valid for equation (4.7), provided that (H1c) and (H2) hold. Direct applications of Theorem 4.5 can easily lead to explicit conditions for the unique positive steady states to be globally asymptotically stable in both equations (4.20) and (4.24).
5. **Global Existence of Periodic Solutions**

In this section we establish the global existence of periodic solutions in

\[ x'(t) = f(x(t - \tau)) - g(x(t)), \]  

(5.1)

which is a special case of (1.6). Our approach can be easily modified to cover the equation

\[ x'(t) = x(t)[f(x(t - \tau)) - g(x(t))]. \]  

(5.2)

Without loss of generality, we assume the unique positive steady state is 1, i.e.,

\[ f(1) = g(1). \]  

(5.3)

As we have shown in previous sections, if \( f(x) \) is increasing or has a hump located at the right-hand side of \( x = 1 \), then \( x(t) \equiv 1 \) is A.G.A.S. Thus, in order to have periodic solutions, it is necessary for us to assume that either \( f(x) \) is decreasing or has a hump to the left of \( x = 1 \). More precisely, we will assume, throughout the rest of this section, that

(A1) \( f(x) \) satisfies (H1a) or (H1c) with \( x_M < x^* = 1 \).

For convenience, we denote \( x_M = 0 \) when \( f \) satisfies (H1a). In the following, we also assume

(A2) There is an \( \bar{x} \in \langle x_M, 1 \rangle \) such that \( f(\bar{x}) > g(\bar{x}) \), where \( \bar{x} = g^{-1}(f(\bar{x})) \).

The accompanying Figure 2 will be helpful in understanding the assumption (A2). The following simple lemma is necessary in our subsequent discussion.

**Lemma 5.1.** Assume that \( f(x) \) in (5.1) satisfies (A1) and (A2), and \( x(s) \in [\bar{x}, \bar{x}] \) for \( s \in [t_0 - \tau, t_0] \), then \( x(t) \in [\bar{x}, \bar{x}] \) for \( t \geq t_0 \).

**Proof.** Let \( 0 \leq \bar{x} < x \) such that for \( x \in [\bar{x}, \bar{x}] \), \( f(g^{-1}(f(x))) > g(x) \). If the lemma is false, then there are two cases to consider

(i) there is a \( \bar{t} > t_0 \), \( \bar{x} < x(\bar{t}) < \bar{x} \), \( x(\bar{t}) < x(t) \leq \bar{x} \) for \( t_0 - \tau < t < \bar{t} \) and \( x'(\bar{t}) < 0 \);

(ii) there is a \( \bar{t} > t_0 \), \( x(\bar{t}) > \bar{x} \), \( \bar{x} \leq x(t) < x(\bar{t}) \) for \( t_0 - \tau < t < \bar{t} \), and \( x'(\bar{t}) > 0 \).

We consider first the case (i). Since \( x(\bar{t}) < \bar{x} \), \( x'(\bar{t}) < 0 \), we have

\[ f(x(\bar{t} - \tau)) < g(x(\bar{t})). \]  

(5.4)

Since \( f(x) \geq g(x) \) for \( x \in [0, 1] \), and \( g(x) \) is strictly increasing, (5.4) thus implies that \( x(\bar{t} - \tau) > 1 \). If \( 1 < x(\bar{t} - \tau) \leq \bar{x} \), then the monotonicity of \( f(x) \) for \( x \geq x_M \) together with (A2) imply

\[ f(x(\bar{t} - \tau)) \geq f(\bar{x}) > g(\bar{x}). \]  

(5.5)
Fig. 2. \( f(g^{-1}(f(x))) \geq g(x) \) implies that \([x, \bar{x}]\) is invariant.

And the monotonicity of \( g(x) \) leads to
\[
f(x(\bar{t} - \tau)) > g(x) > g(x(\bar{t})), \tag{5.6}
\]
a contradiction to (5.4).

Assume now that (ii) holds. Clearly, \( x'(\bar{t}) > 0 \) implies that
\[
f(x(\bar{t} - \tau)) > g(x(\bar{t})). \tag{5.7}
\]
The monotonicity of \( g(x) \) thus implies that
\[
f(x(\bar{t} - \tau)) > g(\bar{x}) = g(g^{-1}(f(x))) = f(x). \tag{5.8}
\]
Now the monotonicity of \( f(x) \) for \( x > x_M \) clearly indicates that
\[
x(\bar{t} - \tau) < \bar{x}
\]
which contradicts our assumption that \( \bar{x} \leq x(t) < x(\bar{t}) \) for \( t_0 - \tau < t < \bar{t} \). This completes the proof of the lemma. \qed
In the rest of this section we assume that the initial condition for (5.1) satisfies

\[ x(s) = \phi(s), \quad s \in [-\tau, 0], \quad \underline{x} \leq \phi(s) \leq \overline{x}, \quad (5.9) \]

and \( \phi(s) \in C([-\tau, 0], R) \). Lemma 5.1 thus implies that \( \underline{x} \leq x(t) \leq \overline{x} \) for \( t \geq 0 \). Clearly in \([\underline{x}, \overline{x}]\), both \(-f(x)\) and \(g(x)\) are strictly increasing. The readers thus, without loss of generality, may assume in the following that \( f(x) \) satisfies (H1a). Also, for convenience, we assume from now that \( \tau = 1 \). (5.1) thus reduces to

\[ x'(t) = f(x(t-1)) - g(x(t)). \quad (5.10) \]

We denote that \( \alpha = -f'(1), \quad \beta = g'(1) \). Then the linearized equation of (5.10) at \( x = 1 \) takes the form

\[ x'(t) = -\alpha x(t - 1) - \beta x(t). \quad (5.11) \]

It has the characteristic equation

\[ \lambda + \beta + \alpha e^{-\lambda} = 0. \quad (5.12) \]

The following lemma can be found in Haderler and Tomiuk [14]. We note that \( \alpha > 0, \beta > 0 \).

**Lemma 5.2.** Let \( \alpha > \alpha_\beta \), where \( \alpha_\beta \) is the smallest positive solution of

\[ \beta + \alpha \cos \sqrt{\alpha^2 - \beta^2} = 0. \quad (5.13) \]

Then (5.12) has a solution \( \lambda \) with \( \Re \lambda > 0, \quad \pi/2 < \Im \lambda < \pi \).

In order to present our main results, we need the following notations:

\[ \nu = \max\{g'(x) : x \in [\underline{x}, \overline{x}]\}; \quad (5.14) \]

\[ K = \{\phi(s) : \phi \in C, \phi(-1) = 1, \underline{x} \leq \phi(s) \leq \overline{x}, (\phi(s) - 1) \exp(\nu s) \}
\]

a nondecreasing function of \( s \in [-1, 0] \} \);

\[ K_1 = K \setminus \{1\}, \quad (5.15) \]

where 1 is the function \( \phi(s) \equiv 1, \ s \in [-1, 0] \). It is easy to see that \( K \) is a closed, bounded and convex subset of the Banach space \( C([-1, 0], R) \) with the standard supremum norm \( \| \cdot \| \). This set is of crucial importance in the proof of the following theorem, which is the main result of this section:

**Theorem 5.1.** Assume (A1) and (A2) hold, and \( \alpha > \max\{1, \alpha_\beta\} \). Then the equation (5.10) has a nonconstant periodic solution \( x(t) \) with period greater than 2, and satisfies \( \underline{x} \leq x(t) \leq \overline{x} \).

Roughly, our approach involves showing that initial conditions near 1 in \( K \) are taken away from it, and those away from it tend in some sense to approach it (with
the help of Lemma 5.1). This leads to the existence of nonconstant fixed points of an operator \( A \) of the form \( A\phi = x_\sigma(\cdot, \phi) \), where \( \sigma = \sigma(\phi) \) is a nonnegative number (to be defined), \( \phi \in K \) and \( x_\sigma(s, \phi) = x(\sigma + s, \phi) \), \( s \in [-1, 0] \). We need the following well-known definition due to Browder (see Hale [11, p. 249]).

**Definition 5.1.** Let \( X \) be a Banach space, \( U \) a subset of \( X \) and \( x \in U \). The point \( x \) is said to be an ejective point of a map \( A : U \setminus \{x\} \to X \), if there is an open neighborhood \( G \subset X \) of \( x \) such that if \( y \in G \cap U \), \( y \neq x \), there is an integer \( m = m(y) > 0 \), such that \( A^m y \in G \cap U \).

The following key lemma is again due to Browder (see Hale [11, p. 249]).

**Lemma 5.3.** Let \( K \) be a closed, bounded and convex set of infinite dimension in a Banach space \( X \), and let \( A : K \to K \) be completely continuous. Then \( A \) has a fixed point in \( K \) which is not ejective.

The following lemma is needed to show that there is an operator \( A \) that maps \( K \) into itself.

**Lemma 5.4.** Assume that all conditions of Theorem 5.1 are satisfied and \( x(t) = x(t, \phi) \), \( \phi \in K_1 \) is a solution of (5.10). Then

1. There is a sequence \( \{z_i\}_{i=1}^\infty \), \( 0 < z_1 < z_2 < \cdots \), such that \( x(z_i) = 1 \), \( z_{i+1} > z_i \), \( i = 1, 2, \ldots \).
2. \( x'(z_{2k-1}) < 0 \), \( x'(z_{2k}) > 0 \) for \( k = 1, 2, \ldots \).
3. The function \( e^{\nu t}(x(t) - 1) \) is nonincreasing on each of the intervals \( (z_{2k-1}, z_{2k-1} + 1) \) and nondecreasing on each of the intervals \( (z_{2k}, z_{2k} + 1) \), \( k = 1, 2, 3, \ldots \).
4. There is a constant \( q > 0 \), such that for \( \phi \in K \), \( z_2 \leq q \).

**Proof.** Since \( \phi \in K_1 \), we must have \( \phi(0) = x(0) > 1 \). We shall show that there is a finite time \( z_1 > 0 \), such that

\[
z_1 = \inf\{t : t \geq 0, \ x(t) = 1\}. \tag{5.17}
\]

Since \( -f'(1) = \alpha > 1 \), we can choose a constant \( \delta > 0 \) such that \( \delta < \min\{\bar{x} - 1, 1 - \underline{x}\} \), and

\[
|f(x) - f(1)| \geq |x - 1|, \quad \text{for} \quad |x - 1| \leq \delta. \tag{5.18}
\]

Clearly, if \( x(t) \geq 1 \) for \( t \in [0, t_0] \), then \( x'(t) \leq 0 \) for \( t \in [0, t_0] \). Let

\[
t_1 = \inf\{t : t \geq 0, \ x(t) \leq 1 + \delta\}. \tag{5.19}
\]

If \( x(0) \leq 1 + \delta \), then \( t_1 = 0 \). Now assume that \( x(0) > 1 + \delta \). For \( 0 \leq t \leq t_1 \), we have

\[
x'(t) = -[g(x(t)) - g(1)] + [f(x(t - 1)) - f(1)]. \tag{5.20}
\]

Hence

\[
x'(t) \leq -[g(x(t)) - g(1)] \leq -[g(1 + \delta) - g(1)]. \tag{5.21}
\]
Thus, we must have

\[ t_1 \leq (\bar{x} - 1 - \delta)/(g(1 + \delta) - g(1)). \]  

(5.22)

Suppose \( z_1 > t_1 + 1 \), then \( 0 \leq x(t) - 1 \leq \delta \) if \( t \in [t_1, t_1 + 1] \). From (5.20), we have

\[ x'(t) \leq f(x(t - 1)) - f(1) \leq 1 - x(t - 1), \quad x \in [t_1, t_1 + 1]. \]  

(5.23)

Since \( x(t - 1) \geq 1 + \delta \) for \( t \in [t_1, t_1 + 1] \), we thus have

\[ x'(t) \leq -\delta \quad \text{for} \quad t \in [t_1, t_1 + 1], \]  

(5.24)

which leads to

\[ x(t_1 + 1) \leq x(t_1) - \delta = 0, \]  

(5.25)

a contradiction. Thus, we have shown that \( z_1 \leq t_1 + 1 \).

If \( z_1 \geq 1 \), then \( x'(z_1) = f(x(t - 1)) - f(1) < 0 \). On the other hand, suppose that \( z_1 < 1 \) and \( x'(z_1) = 0 \). Then we must have \( \phi(t) \equiv 1 \) if \( -1 \leq t \leq -1 + z_1 \) (since \( \phi \in K_1 \)). For \( t \in [0, z_1] \), we have

\[ x'(t) = -[g(x(t)) - g(t)] \geq -\nu(x(t) - 1), \]  

(5.26)

which clearly implies that

\[ x(z_1) - 1 \geq (x(0) - 1)e^{-\nu} > 0, \]  

(5.27)

a contradiction.

We denote

\[ z_2 = \inf\{t : t > z_1, \ z(t) = 1\}, \]  

(5.28)

\[ z^* = \min\{z_2, z_1 + 1\}. \]  

(5.29)

For \( t \in (z_1, z^*) \), we have \( x(t) < 1 \), and

\[ x'(t) \leq -\nu(x(t) - 1) + [f(x(t - 1)) - f(1)]. \]  

(5.30)

Hence,

\[ [(x(t) - 1)e^{\nu t}]' = e^{\nu t}[f(x(t - 1)) - f(1)] < 0, \]  

(5.31)

which implies that \( (x(t) - 1)e^{\nu t} \) is decreasing on \((z_1, z^*)\), and therefore \( z_2 > z_1 + 1 \) and \( z^* = z_1 + 1 \). Thus, we have proved that \( \bar{x} \leq x(t) < 1 \) for \( t \in (z_1, z_1 + 1) \).

For \( t \in (z_1 + 1, z_2) \), we have \( x'(t) \geq 0 \) from (5.20). We shall show that \( z_2 \) is finite. Let

\[ t_2 = \inf\{t : t \geq z_1 + 1, x(t) \geq -\delta + 1\}. \]  

(5.32)
If \( x(z_1 + 1) \geq 1 - \delta \), then \( t_2 = z_1 + 1 \). On the other hand, suppose that \( x(z_1 + 1) < 1 - \delta \), then for \( z_1 + 1 \leq t \leq t_2 \), we have from (5.20),

\[
x'(t) \geq -[g(1 - \delta) - g(1)] = g(1) - g(1 - \delta).
\]

(5.33)

Hence, we must have

\[
t_2 - z_1 - 1 \leq (1 - \bar{x} - \delta)/[g(1) - g(1 - \delta)].
\]

(5.34)

Now suppose that \( z_2 \geq t_2 + 1 \). Then for \( t \in [t_2, t_2 + 1] \),

\[
x'(t) \geq f(x(t - 1)) - f(1) \geq 1 - x(t - 1) \geq \delta,
\]

which leads to

\[
x(t_2 + 1) \geq x(t_2) + \delta = 1,
\]

(5.36)

a contradiction. Hence, we must have \( z_2 \leq t_2 + 1 \). By summarizing the above argument, we have

\[
1 < z_2 \leq 3 + \frac{\bar{x} - 1 - \delta}{g(1 + \delta) - g(1)} + \frac{1 - \bar{x} - \delta}{g(1) - g(1 - \delta)} \equiv q.
\]

(5.37)

Similarly, we can show that

\[
[(x(t - 1)e^{\mu t})'] > 0, \quad \text{for} \quad t \in (z_2, z_2 + 1).
\]

(5.38)

Now the function \( \phi_1 \), where \( \phi_1(s) = x(z_2 + s + 1) \), \(-1 \leq s \leq 0\) is again in \( K_1 \).

Therefore, the argument can be repeated and the existence of \( z_i \), \( i = 1, 2, \ldots \) is thus established. This completes the proof. \( \square \)

We are now ready to define the operator \( A : K \to K \) as:

\[
A(\phi(s)) = x(z_2(\phi) + 1 + s, \phi), \quad s \in [-1, 0], \quad \phi \in K_1,
\]

(5.39)

\[
A(1) = 1.
\]

The following lemma is quite straightforward.

**Lemma 5.5.** The mapping \( \phi \to z_2(\phi) \) of \( K_1 \) into \((1, +\infty)\) and \( A : K \to K \) are completely continuous.

**Proof.** The continuous dependence on the initial data together with the fact that \( x'(z_1(\phi), \phi) < 0 \), \( x'(z_2(\phi), \phi) > 0 \) clearly indicates that if \( \|\varphi - \phi\| \) is very small, then the function \( x(t, \varphi) \) has two zeros \( \bar{z}_1, \bar{z}_2 \) very close to \( z_1(\phi), z_2(\phi) \), and \( x'(\bar{z}_1, \varphi) < 0 \), \( x'(\bar{z}_2, \varphi) > 0 \), and cannot have any other zeros for \( t \leq \bar{z}_2 \). The complete continuity follows from the fact that \( z_2(\phi) \leq q \) for \( \phi \in K_1 \). The continuity of \( A \) follows from the continuity of \( z_2(\phi) \) and again the continuous dependence on the initial data.
Since $z_2 : K_1 \to (1, +\infty)$ is completely continuous, we see that for any bounded set $B \subset K_1$, $A(B)$ is bounded and equicontinuous (since $z_2 > 1$) and, thus, compact. Therefore, $A$ is completely continuous.

Finally, we shall show that $\phi(s) \equiv 1$ is an ejective fixed point of $A$. Suppose $L : C \to R^n$ is linear and continuous, $f : C \to R^n$ is completely continuous together with a continuous derivative $f'$ and $f(0) = 0, f'(0) = 0$. Consider

\begin{align*}
x'(t) &= Lx_t + f(x_t), \quad (5.40) \\
y'(t) &= Ly_t. \quad (5.41)
\end{align*}

For any characteristic root $\lambda$ of (5.41), there is a decomposition of $C$ as $C = P_\lambda \oplus Q_\lambda$, where $P_\lambda$ and $Q_\lambda$ are invariant under the solution operator $T_L(t)$ of (5.41), $T_L(t)\phi = y_t(\phi), \phi \in C$. Let the projection operators defined by the above decomposition of $C$ be $\pi_\lambda, I - \pi_\lambda$ with the range of $\pi_\lambda$ equal to $P_\lambda$. The following result can be found in Hale [11, p. 250].

**Lemma 5.6.** Suppose the following conditions are satisfied:

(i) There is a characteristic root $\lambda$ of (5.41) satisfying $\Re \lambda > 0$.

(ii) There is a closed convex set $K \subseteq C$, $0 \in K$, and $\delta > 0$, such that

$$\mu = \mu(\delta) \equiv \inf\{\|\pi_\lambda \phi\| : \phi \in K, \|\phi\| = \delta\} > 0.$$ 

(iii) There is a completely continuous function $\tau : K\{0\} \to [\alpha, \infty)$, $\alpha \geq 0$, such that the map defined by

$$A\phi = x_{\tau(\phi)}(\phi), \quad \phi \in K\{0\}$$

takes $K\{0\}$ into $K$ and is completely continuous. Then $0$ is an ejective point of $A$.

Replacing $\{0\}$ by $\{1\}$, we see that to show $\phi(s) \equiv 1$ is ejective, it suffices to show that

**Lemma 5.7.** Let $\bar{\alpha} = \max\{1, \alpha_\beta\}$, $\delta = \min\{\bar{\alpha} - 1, 1 - \bar{\alpha}\}$, and $J$ is a compact set of $(\bar{\alpha}, \infty)$, then

$$\mu = \inf\{\pi_{\lambda(\alpha)} \phi : \phi \in K, \|\phi - 1\| = \delta, \alpha \in J\} > 0. \quad (5.42)$$

**Proof.** Let $\lambda = \lambda(\alpha)$ be the solution of (5.12) given by Lemma 5.2 (where $\beta$ is fixed), $\phi(\theta) = e^{\lambda\theta}/(1 + \lambda + \beta)$, $\theta \in [-1, 0]$, $\psi(s) = e^{-\lambda s}$, $s \in [0, 1]$, $\Phi = (\phi, \overline{\phi})$, $\Psi = (\psi, \overline{\psi})$. The bilinear form of (5.11) is

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \alpha \int_{-1}^{0} \psi(z + 1)\phi(\xi)d\xi. \quad (5.43)$$
We thus have

$$(\psi, \phi) = \frac{1}{1 + \lambda + \beta} \left[ 1 - \alpha \int_{-1}^{0} e^{-\lambda (\xi + 1)} e^{\lambda \xi} d\xi \right]$$

$$= \frac{1}{1 + \lambda + \beta} \left[ 1 - \alpha \int_{-1}^{0} e^{-\lambda} d\xi \right]$$

$$= \frac{1 - \alpha e^{-\lambda}}{1 + \lambda + \beta} = 1,$$

since $\lambda + \beta = -\alpha e^{-\lambda}$. Also,

$$(\overline{\psi}, \phi) = \frac{1}{1 + \lambda + \beta} \left[ 1 - \alpha \int_{-1}^{0} e^{-\overline{\lambda} (\xi + 1)} e^{\lambda \xi} d\xi \right]$$

and

$$1 - \alpha \int_{-1}^{0} e^{-\overline{\lambda} (\xi + 1)} e^{\lambda \xi} d\xi$$

$$= 1 - \alpha \int_{-1}^{0} e^{-\overline{\lambda} \xi} e^{(\lambda - \overline{\lambda}) \xi} d\xi$$

$$= 1 - \frac{\alpha}{\lambda - \overline{\lambda}} e^{-\overline{\lambda}} (1 - e^{(\lambda - \overline{\lambda})}) = \frac{1}{\lambda - \overline{\lambda}} (\lambda - \overline{\lambda} - \alpha e^{-\overline{\lambda}} + \alpha e^{-\lambda})$$

$$= \frac{1}{\lambda - \overline{\lambda}} (-\beta + \beta) = 0.$$

Hence, obviously, $(\overline{\psi}, \phi) = 1$, $(\psi, \overline{\phi}) = 0$. Therefore $(\overline{\psi}, \phi)$ is the identity. Hence, for any $\phi \in C$, $\pi_\lambda \phi = \Phi(\overline{\psi}, \phi)$. To show the conclusion of the lemma is true, it is thus sufficient to show that

$$\inf \{ ||(\overline{\psi}, \phi - 1)||, \phi \in K, ||\phi - 1|| = \delta, \alpha \in J \} > 0.$$

Clearly, it is sufficient to look at $(\psi, \phi - 1)$. If $\phi \in K$, $||\phi - 1|| = \delta$, then $\phi(0) - 1 \geq \delta e^{-1}$ and

$$(\psi, \phi - 1, \alpha) \equiv (\psi, \phi - 1) = \text{Re}(\phi) + i\text{Im}(\phi)$$

$$\text{Re}(\phi) = \phi(0) - 1 - \alpha \int_{-1}^{0} (\phi(\theta) - 1) e^{-\gamma (\theta + 1)} \cos(\sigma(\theta + 1)) d\theta$$

$$\text{Im}(\phi) = \alpha \int_{-1}^{0} (\phi(\theta) - 1) e^{-\gamma (\theta + 1)} \sin(\sigma(\theta + 1)) d\theta$$

where $\lambda = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in (\pi/2, \pi)$. Since $J$ is compact, there is an $\epsilon > 0$ such that $\epsilon < \sigma = \sigma(\alpha) < \pi - \epsilon$ for $\alpha \in J$.

Now, suppose that there exist sequences $\phi_n \in K$, $\phi_n(0)$, $\alpha_n \in J$ such that $(\psi, \phi_n - 1, \alpha_n) \to 0$ as $n \to +\infty$. We may assume that $\alpha_n \to \overline{\alpha}$, $\phi_n(0) - 1 \to \eta \geq \delta e^{-1}$ as $n \to +\infty$, since $J$ is compact and $\delta^{-1} \epsilon \leq \phi_n(0) - 1 \leq \delta$. Since $\text{Im}(\phi_n) \to 0$, 

we must have \( \phi_n(\theta) - 1 \rightarrow 0, \quad -1 \leq \theta < 0. \) Thus, \( \text{Re}(\phi_n) \rightarrow \eta \) as \( n \rightarrow +\infty. \) This is a contradiction to the fact that \( \text{Re}(\phi_n) \rightarrow 0 \) as \( n \rightarrow +\infty \) and the lemma is proved. \( \square \)

Finally, we are ready to state the proof of Theorem 5.1.

**Proof of Theorem 5.1.** From the definition of \( K \), we see that it is a closed, bounded and convex set of infinite dimension in the Banach space \( C. \) \( A \) as defined in (5.39) is completely continuous by Lemma 5.5 and 1 is an ejective fixed point of \( A \) by Lemmas 5.6 and 5.7. Therefore, by Lemma 5.3, we conclude that \( A \) has a fixed point \( \phi \) in \( K_1 \) which clearly corresponds to a nonconstant periodic solution \( x(t, \phi) \) of period greater than 2. This completes the proof. \( \square \)

**Remark 5.1.** By making arguments similar to that of Kuang and Smith [20], and modifying the previous discussion accordingly, one can establish the global existence of periodic solutions in the following state-dependent delay equation

\[
x'(t) = -g(x(t)) + f(x(t - \tau(x_t))),
\]

where \( \tau \) is lipschitzian on each bounded subset of \( C. \) The key observation that can be exploited is that if a solution \( x(t) \) of (5.44) satisfies \( x'(t_0) = 0, \) then for \( t > t_0, \) the solution “forgets” its history prior to \( t_0 - \tau(x_{t_0}) \) in the sense that \( t - \tau(x_t) > t_0 - \tau(x_{t_0}) \) for \( t > t_0. \) Interesting examples of \( \tau(x_t) \) include the bell-shaped function \( \tau(x_t) = \tau(x) = 1 - \alpha + \alpha \exp[-x^2(t)], \) \( 0 \leq \alpha \leq 1, \) and a special class of the generalized threshold delay \( \tau = \tau(x_t) \) defined implicitly by

\[
\int_{t-\tau}^{t} k(x(t), x(s))ds = m
\]

where \( k : R^2 \rightarrow (0, \infty) \) is locally lipschitzian and \( m > 0. \) The detail of such generalization is rather technical and tedious; we choose to omit it here.

**Remark 5.2.** By careful modification, one can also use the method adopted by Hadeler and Tamsiuk [14] to show that 1 is ejective. However, the procedure is slightly more involved and needs further adjustment if one would like to consider the more general state-dependent delay equation (5.44). The key observation that should be used in such a modification is that for small \( \delta > 0, \) we can choose small \( \epsilon(\delta) > 0, \) such that if \( |x(t) - 1| < \delta \) for large \( t, \) then \( y(t) = e^{\omega}x(t) \) is monotone on \( (z_i, z_i + 1), \) where \( i \) is large, \( \omega = \beta + \epsilon, \) and \( x(z_i) = 1. \)

6. **Discussion**

In this paper we have established various criteria for the global and absolute global asymptotic stability of the unique positive steady state of (with appropriate assumptions)

\[
x'(t) = f\left(\int_{-\tau}^{-\sigma} x(t + s)d\mu(s)\right) - g(x(t))
\]

or
\[ x'(t) = x(t) \left[ f \left( \int_{-\tau}^{-\sigma} x(t+s) d\mu(s) \right) - g(x(t)) \right]. \]

These two equations are more general than the three first-order nonlinear differential delay equations (1.1)–(1.3) proposed by Mackey and Glass [25] in order to describe some physiological control systems.

Our criteria for the absolute global asymptotic stability have a strong geometrical background, which makes them practically easy to use. Assume first that \( f(x) \) is strictly decreasing, then the steps to apply the criterion are:

(i) Graph functions \( f(x) \) and \( g(x) \), locate the positive intersection \( x^* \).

(ii) Graph function \( h(x) = g(2x^* - x) \), which is symmetric to \( g(x) \) with respect to line \( x = x^* \).

(iii) Compare \( f(x) \) with \( h(x) \). If, for \( x \in [0, x^*] \), \( h(x) > f(x) \) and for \( x > x^* \), \( f(x) > h(x) \), then \( x^* \) is absolutely globally asymptotically stable (see Fig. 3). Otherwise, the criterion is inconclusive.

![Graph showing comparison of functions](image)

Fig. 3. Here \( h(x) \geq f(x) \) for \( 0 \leq x \leq x^* \), and \( h(x) \leq f(x) \) for \( x \geq x^* \), thus \( x^* \) is A.G.A.S.

If \( f(x) \) is strictly increasing and satisfies (H1b), then, by Theorem 4.1, we know...
$x^*$ is absolutely globally asymptotically stable. If $f(x)$ satisfies (H1c) and its only hump lies at the right side of line $x = x^*$, then the same conclusion as above is true. Otherwise, graph the function $h(x) = f(2x^* - x)$, the symmetric function of $f(x)$ with respect to line $x = x^*$. If we have $h(x) > g(x)$ for $x \in (0, x^*)$, then, again, $x^*$ is absolutely globally asymptotically stable (see Fig. 4). Otherwise, it is inconclusive.

![Graph](image)

**Fig. 4.** $h(x) \geq g(x)$ for $0 \leq x \leq x^*$ implies that $x^*$ is A.G.A.S.

It is easy to see from these criteria that, in order to have $x^*$ the global attractor, roughly speaking, it is sufficient that the slope of $g(x)$ be steeper than that of $f(x)$ for $x > x^*$, i.e., $|g'(x)| > |f'(x)|$. Clearly, this requires that, after passing the steady state $x^*$, the self-crowding effect becomes stronger than the growth mechanism in the described systems.

We note that the absolute global asymptotic stability is independent of the delay length $\tau$ and the distribution function $\mu(s)$. In these situations, the effects caused by the time delay can be ignored. We also note that whether or not $f(x)$ has only one hump is not critical. What is really important here are the locations and shapes of these humps.

When all the above-mentioned criteria fail, it generally implies that the delay effects play important roles in the dynamics of the considered systems. As the delay length $\tau$ increases from zero, periodic and aperiodic ("chaotic") solutions may appear. In these cases, our criteria (including Theorems 3.2–3.4, Corollaries
require that the delay length \( \tau \) be small in order to have \( x^* \) continue to be the global attractor. These results are largely in agreement with the findings of Gopalsamy et al. [6, 7]. However, it may be difficult to judge which result is sharper mathematically, since the method used here is entirely different from the one used by Gopalsamy et al. Nevertheless, it is easy to see that our results are relatively easy to use, since one needs only to estimate the values \( \bar{x}, \bar{f} \) and \( \bar{g} \) as defined in (3.13)-(3.15) or in (4.29)-(4.31). Corollaries 3.2 and 3.3 are just two simple examples of their applications.

It should be pointed out here that our methods for the global attractivity can be applied to the so-called state dependent delay differential equations

\[
x'(t) = f(x(t - \tau(x_t))) - g(x(t))
\]  

and

\[
x'(t) = x(t)[f(x(t - \tau(x_t))) - g(x(t))],
\]

where \( \tau(x_t) \) is as the one that was mentioned in Remark 5.1. Existence and uniqueness of solutions for these equations are established in Bélaïr [1]. If \( \tau(x_t) \) is replaced by \( r(t) \), a bounded nonnegative continuously differentiable function, our methods still apply.

It should be mentioned here that our approaches seem to fail to work for the following more general equation

\[
x'(t) = \int_{-\tau}^{-\sigma} x(t + s) d\mu_1(s) - h\left(\int_{-\omega}^{-\sigma} x(t + s) d\mu_2(s)\right) - g(x(t)),
\]

where \( 0 < \sigma < \tau < \omega \) are constants, \( \mu_1(s) \) and \( \mu_2(s) \) are nondecreasing, \( \int_{-\tau}^{-\sigma} d\mu_1(s) = \int_{-\omega}^{-\sigma} d\mu_2(s) = 1 \), \( h(x) \) has similar properties as that of \( f(x) \) which are stated in the previous sections. This equation is more general than a simple model proposed by Bélaïr and Mackey [2] for the regulation of mammalian platelet production:

\[
P'(t) = -\gamma P(t) + \beta(P(t - \tau)) - \beta(P(t - \tau - \sigma))e^{-\gamma \sigma},
\]

where \( \beta(P) = \beta_0 \theta^n P / (\theta^n + P^n) \), and \( \beta_0, \theta, n, \gamma, \tau \) and \( \sigma \) are positive constants. More realistically, one should probably replace \( g(x(t)) \) in (6.3) by \( g(\int_{-\delta}^{0} x(t + s) d\mu_3(s)) \), where \( \delta < \sigma \), \( \mu_3(s) \) is nondecreasing and \( \mu(0) - \mu(-\delta) = 1 \). One of the main difficulties involved in the analysis of (6.3) is that solutions of positive initial values may fail to be positive. However, in these cases, partial results for the convergence of positive solutions can be obtained by arguments similar to that of Kuang and Smith [21].

Generally speaking, the global asymptotic stability results for equations of the forms

\[
x'(t) = f\left(\int_{-\tau}^{-\sigma} x(t + s) d\mu_1(s)\right) - g\left(\int_{-\delta}^{0} x(t + s) d\mu_2(s)\right)
\]

and
\[ x'(t) = x(t) \left[ f \left( \int_{-\tau}^{-\sigma} x(t + s) d\mu_1(s) \right) - g \left( \int_{-\delta}^{0} x(t + s) d\mu_2(s) \right) \right] \] (6.6)

are rare (if there are any). Even if we assume that \( \delta \) is very small, the difficulty still remains. However, instinct tells us that a realistic model consisting of delay equations should allow delay, no matter how small, to appear in every single term, since almost no change in this world is absolutely instantaneous. In other words, there is an urgent need to investigate the global qualitative behavior of solutions of these equations. Several attempts along this line are documented in [5, 9, 16, 22] for differential delay models in population dynamics. Related works for systems of nonlinear (autonomous and nonautonomous) differential delay equations can be found in [17–19, 21].

When the functions \( f(x) \) and \( g(x) \) in (1.6) and (1.7) are replaced by nonautonomous ones \( f(t, x) \) and \( g(t, x) \), our methods may still apply, provided that we assume that there is a unique \( x^* > 0 \), such that \( f(t, x^*) = g(t, x^*) \) for all \( t \geq 0 \). In this case, we will need an argument that, roughly speaking, combines the one presented in [19] and a perturbation argument presented in [18] for systems of delay equations in population growth models.

References


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