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On neutral delay logistic Gause-type predator–prey systems

Yang Kuang
Department of Mathematics, Arizona State University, Tempe, Arizona 85287–1804, USA

Abstract
The qualitative behaviour of solutions of the neutral delay logistic Gause-type predator–prey system

\[
\begin{align*}
\dot{x}(t) &= rx(t)[1 - (x(t - \mu) + \rho \dot{x}(t - \tau))/K] - y(t)p(x(t)), \\
y(t) &= y(t)[\alpha + \beta p(x(t - \sigma))]
\end{align*}
\]

(*)
is investigated; sufficient conditions are obtained for the local asymptotic stability of the positive steady state of (*). In fact, some of these sufficient conditions are also necessary except at those critical values. Results on the oscillatory and non-oscillatory characteristics of the positive solutions of (*) are also included.

1. Introduction
The autonomous logistic delay differential equation

\[
\dot{x}(t) = rx(t)[1 - x(t - \tau)/K],
\]

(1.1)
where the dot means \(d/dt\), and where \(r, K, \tau\) are positive constants, has been widely used as a model equation capable of showing oscillations of single-species population sizes in constant environments closed to both immigration and emigration (see (Cushing, 1977; Gopalsamy and Zhang, 1988; Hale, 1977; Kuang and Feldstein, 1991; Pielou, 1977)). It has been the object of intensive analysis by numerous authors (see the references cited in (Gopalsamy and Zhang, 1988)). Indeed, it is a natural generalization of the following well-known logistic single-species population equation:

\[
\dot{x}(t) = rx(t)[1 - x(t)/K].
\]

(1.2)
Here \(r\) is called the intrinsic growth rate of the species \(x\), \(K\) is interpreted as the environment capacity for \(x\), and \(r[1 - x(t)/K]\) is the per capita growth rate of \(x\) at time \(t\). Based on his investigation on laboratory populations of \(Daphnia magna\), F. E. Smith (1963) argued that the per capita growth rate in (1.2) should be replaced by \(r[1 - (x(t) + \rho \dot{x}(t))/K]\) (for details see (Pielou, 1977)). This leads to the following equation:

\[
\dot{x}(t) = rx(t)[1 - (x(t) + \rho \dot{x}(t))/K].
\]

(1.3)
We may think of \(x\) as a species grazing upon vegetation, which takes time \(\tau\) to recover. In this case, it will be even more realistic to incorporate a single discrete
delay \( \tau \) in the per capita growth rate, which results in the following neutral delay logistic equation:

\[
\frac{dx(t)}{dt} = r x(t) \left[ 1 - \left( x(t - \tau) + \rho \dot{x}(t - \tau) \right) / K \right].
\]  \hspace{1cm} (1.4)

This equation was introduced and investigated by Gopalsamy and Zhang (1988). Subsequently, it was studied by Freedman and Kuang (1991), Kuang and Feldstein (1991). The focus of these works is the qualitative behaviour of the solutions, such as boundedness, asymptotic stability and oscillation.

Assume that the population \( x(t) \) described by (1.4) is a prey species, and suppose there exists a predatory species \( y(t) \) preying on species \( x(t) \); then it is natural to propose the following mathematical model to describe their interaction:

\[
\begin{align*}
\frac{dx(t)}{dt} &= r x(t) \left[ 1 - \left( x(t - \tau) + \rho \dot{x}(t - \tau) / K \right) - y(t)p(x(t)) \right], \\
\frac{dy(t)}{dt} &= y(t) \left[ -\alpha + \beta p(x(t - \sigma)) \right].
\end{align*}
\]  \hspace{1cm} (1.5)

Here \( \alpha, \beta, \sigma \) are all positive constants, and \( p(x) \) is the predator response function for the predator species \( y \) with respect to the prey species \( x \). The first equation of system (1.5) states that the prey growth is enhanced by its own presence in a manner analogous to single-species growth and is diminished by an amount proportional to the number of predators present multiplied by the predator response to the prey. Equation (1.5) describes the growth of the predator population. In the absence of prey, the predator population declines. The growth is enhanced by the presence of prey by an amount proportional to the number of prey. In some sense, \( \beta p(x(t - \sigma)) \) may be interpreted as spelling out what proportion of prey eaten \( \sigma \) units of time ago become predators. The reader is referred to Freedman (1980) and Pielou (1977) for more details on both the mathematical and ecological grounds of system (1.5), when there are no delays and no neutral term \( \rho \dot{x}(t - \tau) / K \); in this case, (1.5) is called a logistic Gause-type predator–prey model.

The focus of our present study is the qualitative behaviour of positive solutions of system (1.5). This includes the asymptotic stability of its positive steady state, and oscillatory behaviour of its solution about it.

This paper is organized as follows. Section 2 presents some preliminary analyses of the system. Section 3 is devoted to the asymptotic stability analysis of the unique positive steady state. The technique utilized here is a generalized version of the one developed by Cooke and Grossmann (1982) mainly for equations of linear constant coefficients with single delay. It is further developed by Cooke and van den Driessche (1986) for several other situations. Section 4 contains a detailed oscillation analysis for both the single-species equation (1.4) and the predator–prey system (1.5). In fact, our results are established for systems slightly more general than that of (1.4) and (1.5). The paper concludes with a brief discussion and a list of open questions.

2. Preliminaries

In section 4, we shall establish some oscillatory results for the following slightly more general version of system (1.5):

\[
\begin{align*}
\frac{dx(t)}{dt} &= r x(t) \left[ 1 - \left( x(t - \mu) + \rho \dot{x}(t - \tau) / K \right) - y(t)p(x(t)) \right], \\
\frac{dy(t)}{dt} &= y(t) \left[ -\alpha + \beta p(x(t - \sigma)) \right],
\end{align*}
\]  \hspace{1cm} (2.1)
where $\mu$ is a positive constant and $\rho$ is a real number. Throughout the rest of this paper, the following standard properties for the predator response function $p(x)$ are assumed:

(H1) $p(x)$ is continuously differentiable, $p(0) = 0$ and $p'(x) > 0$ for $x > 0$,
(H2) $\lim_{x \to \infty} p(x) > \alpha/\beta$.

Let $\delta = \max \{\mu, \tau, \sigma\}$. We always assume that the initial conditions for (2.1) are of the type

$$
\begin{align*}
\phi_1(s) &> 0, \quad s \in [-\delta, 0], \quad \phi_1(0) > 0 \\
\phi_2(s) &> 0, \quad s \in [-\delta, 0], \quad \phi_2(0) > 0
\end{align*}
$$

We say $(x(t), y(t))$ is a solution of (2.1) on $[-\delta, \infty)$, if both $x(t)$ and $y(t)$ are positive continuously differentiable functions and satisfy both the above initial conditions and system (2.1). It is not difficult to see that solutions of (2.1) corresponding to the initial conditions of the above type exist and are unique, and they are always positive on $[0, \infty)$ and are defined on $[0, \infty)$.

It is easy to see that if we let $\bar{x}(t) = x(t)/K$ and change other notation accordingly, we shall obtain a system similar to (2.1) with $K = 1$. Therefore, without loss of generality, throughout the rest of this paper we shall assume that $K = 1$. With this assumption, system (2.1) reduces to

$$
\begin{align*}
\dot{x}(t) &= r x(t)[1 - (x(t) - \mu + \rho \bar{y}(t - \tau)) - y(t)p(x(t))] \\
\dot{y}(t) &= y(t)[1 - x(t) - \sigma])
\end{align*}
$$

(2.2)

System (2.2) has three steady states. They are $(0, 0)$, $(1, 0)$ and $(x^*, y^*)$, where

$$
x^* = p^{-1}(\alpha/\beta), \quad y^* = r x^*(1 - x^*)/p(x^*).
$$

It is easy to see that the positive cone

$$
\mathbb{R}^{2+} = \{(x, y) : x \geq 0, \ y \geq 0\}
$$

is invariant in the sense that if $(\phi_1(s), \phi_2(s))$, $s \in [-\delta, 0]$, is in $\mathbb{R}^{2+}$, then $(x(t), y(t))$ will in $\mathbb{R}^{2+}$ for all $t \geq 0$. In a similar sense, we see that both the $x$-axis and the $y$-axis are invariant also. Write

$$
\|\phi\| = \max \{|\phi(s)| : s \in [-\delta, 0]\} + \max \{|\phi'(s)| : s \in [-\delta, 0]\};
$$

then $\|\phi\|$ defines a norm for functions in $C^1([-\delta, 0], \mathbb{R}^+)$. It is easy to see that if both $\|\phi_1\|$ and $\|\phi_2\|$ are very small, we shall witness an increase in the prey population and a decrease in the predator population. This indicates that the origin is a saddle point in the conventional sense. Suppose that $(x_0, y_0)$ is a steady state of (2.2), we say it is stable if, for any $\varepsilon > 0$, there is a $\delta_0 = \delta_0(\varepsilon)$ such that if

$$
\max \{|\phi_1(s) - x_0| : s \in [-\delta, 0]\} + \max \{|\phi_2(s) - y_0| : s \in [-\delta, 0]\} < \delta_0,
$$

then

$$
\sup \{|x(t) - x_0| : t \in \mathbb{R}^+\} + \sup \{|y(t) - y_0| : t \in \mathbb{R}^+\} < \varepsilon.
$$

We say that $(x_0, y_0)$ is unstable if $(x_0, y_0)$ is not stable. In this sense, we see that both $(0, 0)$ and $(1, 0)$ are not stable since, in the first case, the steady state $(0, 0)$ is unstable along the $x$-axis; in the second case, the steady state is not stable along the $y$-direction.

Now it is easy to see that the properties associated with the unique positive equilibrium $(x^*, y^*)$ play a central role in determining the qualitative behaviour
of solutions to system (2.2). Consequently, this becomes the focus of our analysis in the subsequent sections.

3. Local stability of the positive equilibrium

In this section, we consider the local stability of the unique positive equilibrium \((x^*, y^*)\) of the neutral delay logistic Gause-type predator–prey system:

\[
\begin{align*}
\dot{x}(t) &= rx(t)[1-x(t-\tau)-\rho x(t-\tau)] - y(t)p(x(t)), \\
\dot{y}(t) &= y(t)[-\alpha + \beta p(x(t-\sigma))].
\end{align*}
\]

Equation (3.1) is equivalent to

\[
\frac{d}{dt} [\ln (x(t)/x^*) + rpx(t-\tau)] = r(1-x(t-\tau)) - y(t)p(x(t))/x(t).
\]

Let \(u(t) = \ln (x(t)/x^*),\)

\[
v(t) = y(t) - y^*;
\]

then \(x(t) = x^*e^{u(t)},\ y(t) = v(t) + y^*.\) System (3.1) is reduced to

\[
\begin{align*}
\frac{d}{dt} [u(t) + rpx^*u(t-\tau) + G_1(u(t-\tau))] &= -rx^*u(t-\tau) - y^*p'(x^*)u(t) \\
&\quad + (p(x^*)y^*/x^*)u(t) \\
&\quad -(p(x^*)/x^*)v(t) \\
&\quad + F_1(u(t), u(t-\tau), v(t)),
\end{align*}
\]

\[
\frac{d}{dt} u(t) = \beta y^*p'(x^*)x^*u(t-\sigma) + F_2[u(t-\sigma), v(t)],
\]

where \(G_1, F_1, F_2\) satisfy

\[
\begin{align*}
G_1(0) &= G_{1u(0)}(0) = 0, \\
F_1(0, 0, 0) &= F_{1u(0)}(0, 0, 0) = F_{1u(\tau)}(0, 0, 0) = 0, \\
F_2(0, 0) &= F_{2u(\tau)}(0, 0) = 0.
\end{align*}
\]

In the notation of Hale (1977), (3.3) can be written as

\[
\frac{d}{dt} [DX_t + G(X_t)] = LX_t + F(X_t),
\]

where \(X(t) = (u(t), v(t)),\ X_t = X(t + \theta),\ \theta \in [-\delta, 0],\ \delta = \max \{\tau, \sigma\},\ X_t \in C([-\delta, 0], \mathbb{R}^2),\\) and \(D, G, L, F\) are maps of \(C([-\delta, 0], \mathbb{R}^2)\) into \(\mathbb{R}^2\) defined by

\[
\begin{align*}
DX &= \begin{bmatrix} u(0) + rpx^*u(\tau) \\ v(0) \end{bmatrix}, \\
LX &= \begin{bmatrix} y^*[p(x^*)/x^*]u(0) - rx^*u(\tau) - (p(x^*)v(0)) \\ \beta y^*p'(x^*)x^*u(\sigma) \end{bmatrix}, \\
G(X) &= \begin{bmatrix} G_1(u(\tau)) \\ 0 \end{bmatrix}, \\
F(X) &= \begin{bmatrix} F_1(u(0), u(\tau), v(0)) \\ F_2(u(\sigma), v(0)) \end{bmatrix}.
\end{align*}
\]
It is well known that if the trivial solution of the linear system
\[ \frac{d}{dt} DX_t = LX_t \] (3.6)
is uniformly asymptotically stable, then, with \( G(X) \) and \( F(X) \) defined as in (3.5),
the trivial solution of the nonlinear system (3.3) is (locally) exponentially
asymptotically stable (for details, see (Hale, 1977, Theorem 9.1, pp. 304–305)).
Therefore, in the rest of this section we shall consider the stability of the trivial
solution of (3.6). Write
\[
\begin{align*}
\epsilon &= rpx^* , \\
a &= -y^* \left[ (p(x^*)/x^*) - p'(x^*) \right] , \\
b &= rx^* , \\
c &= p(x^*)/x^* , \\
d &= \beta y^* p'(x^*) x^*. 
\end{align*}
\] (3.7)
System (3.6) becomes
\[
\begin{align*}
\dot{u}(t) + \epsilon \dot{u}(t - \tau) &= -au(t) - bu(t - \tau) - cv(t) , \\
\dot{v}(t) &= du(t - \sigma).
\end{align*}
\] (3.8)
Differentiating both sides of (3.8)_1 and using (3.8)_2 leads to
\[ \ddot{u}(t) + \epsilon \ddot{u}(t - \tau) + a\dot{u}(t) + b\dot{u}(t - \tau) + cd\dot{u}(t - \sigma) = 0. \] (3.9)
Equation (3.8)_2 implies that
\[ u(t) = \dot{v}(t + \sigma)/d. \] (3.10)
By substituting (3.10) into (3.8)_1, we have
\[ \frac{1}{d} (\ddot{u}(t + \sigma) + \epsilon \ddot{u}(t + \sigma - \tau) + a\dot{u}(t + \sigma) + b\dot{u}(t + \sigma - \tau)) + cv(t - \sigma) = 0. \] (3.11)
Obviously, (3.11) is equivalent to
\[ \ddot{u}(t) + \epsilon \ddot{u}(t - \tau) + a\dot{u}(t) + b\dot{u}(t - \tau) + cd\dot{u}(t - \sigma) = 0. \] (3.12)
Therefore, \( u(t) \) and \( v(t) \) have the same stability. From the geometric point
of view, this implies somehow that the unique positive steady state \((x^*, y^*)\) is either
a 'source' or a 'sink.' It cannot be in hyperbolic-type equilibrium. This indeed
coincides with the behaviour of the positive steady state in the case when there
are no delays.

It is well known that the stability of the trivial solutions of an autonomous
differential difference equation depends on the location of the roots of its
characteristic equation. If the characteristic equation associated with a linear
neutral equation has roots only with negative real parts and if all the roots are
uniformly bounded away from the imaginary axis, then the trivial solution of the
linear neutral equation is uniformly asymptotically stable in the sense of Hale
(1977) (for details, see (Hale, 1977, Corollary 10.1, p. 310)). The characteristic
equation of (3.9) is
\[ \lambda^2 + \epsilon \lambda^2 e^{-\lambda \tau} + a \lambda + b \lambda e^{-\lambda \tau} + cd e^{-\lambda \sigma} = 0. \] (3.13)
Before we state and prove our results, we need the following theorem which is
essentially the same as (Freedman and Kuang, 1991, Theorem 4.1). In the rest of
this section, stability means uniform asymptotic stability, unless stated otherwise. Therefore, the stability of the trivial solution of (3.8) implies the local stability of the trivial solution of system (3.3), which is equivalent to the local stability of the positive equilibrium of the original system (3.1).

**Theorem 3.1** (Freedman and Kuang) Consider the following second-order real scalar linear neutral delay equation:

\[
\ddot{u}(t) + \varepsilon \dot{u}(t - \tau) + a \dot{u}(t) + b \ddot{u}(t - \tau) + \eta u(t) + \xi u(t - \tau) = 0. \tag{3.14}
\]

Assume that \(|\varepsilon| > 1\), \(\eta + \xi \neq 0\) and \(a^2 + b^2 + (\xi - \varepsilon \eta)^2 \neq 0\). Then the number of different positive (negative) imaginary roots of the characteristic equation of (3.14)

\[
\lambda^2 + \varepsilon \lambda e^{-\lambda \tau} + a \lambda + b \lambda e^{-\lambda \tau} + \eta + \xi e^{-\lambda \tau} = 0 \tag{3.15}
\]

can only be zero, one or two. For (3.14), the following statements are true.

1. If there are no such roots, then the stability of the trivial solution does not change for any \(\tau \geq 0\).
2. If there is one imaginary root, then an unstable trivial solution at \(\tau = 0\) never becomes stable for any \(\tau \geq 0\). If the trivial solution is uniformly asymptotically stable for \(\tau = 0\), then it is uniformly asymptotically stable up to the delay time \(\tau_{0,1}\), and becomes unstable afterwards.
3. If there are two imaginary roots, \(i \omega_+\) and \(i \omega_-\), such that \(\omega_+ > \omega_- > 0\), then the stability of the zero trivial solution can change at most a finite number of times as \(\tau\) is increased, and eventually \(\dot{u}\) becomes unstable.

In the above statements, \(\omega_+\), \(\omega_-\), \(\tau_{0,1}\) (if they exist) are defined as

\[
\omega_{\pm}^2 = \frac{1}{2}(1 - \varepsilon^2)^{-1} \left( (b^2 + 2\eta - a^2 - 2\xi \varepsilon) \pm \sqrt{(b^2 + 2\eta - a^2 - 2\xi \varepsilon)^2 - 4(1 - \varepsilon^2)(\eta^2 - \xi^2)} \right), \tag{3.16}
\]

\[
\tau_{0,1} = \frac{\theta_1}{\omega_+}, \tag{3.17}
\]

where \(0 \leq \theta_1 < 2\pi\), and

\[
\cos \theta_1 = -\frac{ab \omega_+^2 + (\eta - \omega_+^2)(\xi - \varepsilon \omega_+^2)}{b^2 \omega_+^2 + (\xi - \varepsilon \omega_+^2)^2}, \tag{3.18}
\]

\[
\sin \theta_1 = \frac{(\xi - \varepsilon \omega_+^2)\omega_+ - b\omega_+(\eta - \omega_+^2)}{b^2 \omega_+^2 + (\xi - \varepsilon \omega_+^2)^2}. \tag{3.19}
\]

Now we are ready to formulate our results. We shall discuss only two cases, namely (i) \(\sigma = 0\); (ii) \(\sigma = \tau\). Our first theorem deals with case (i).

**Theorem 3.2** In system (3.8), assume that \(\sigma = 0\), \(|\varepsilon| < 1\). Then the following statements are true.

1. Suppose that either \(b^2 + 2cd - a^2 \leq 0\), or \((b^2 + 2cd - a^2)^2 < 4(1 - \varepsilon^2)c^2d^2\).
   (a) If \(a > b\), then the trivial solution of system (3.8) is uniformly asymptotically stable for all delay times \(\tau \geq 0\).
   (b) If \(a < -b\), then the trivial solution of system (3.8) is unstable for all delay times \(\tau \geq 0\).

2. Suppose that \(b^2 + 2cd - a^2 > 0\), and \((b^2 + 2cd - a^2)^2 > 4(1 - \varepsilon^2)c^2d^2\).
   (a) If \(a + b > 0\), then the stability of the trivial solution of (3.8) will change from stability to instability a finite number of times as \(\tau\) is increased, eventually becoming unstable.
(b) If $a + b < 0$, then the stability of the trivial solution of (3.8) can change from instability to stability a finite number of times as $\tau$ is increased, eventually becoming unstable.

**Proof** When $\sigma = 0$, the characteristic equation of system (3.8) becomes

$$\lambda^2 + \varepsilon \lambda^2 e^{-\lambda \tau} + a \lambda + b \lambda e^{-\lambda \tau} + cd = 0. \quad (3.20)$$

Comparing (3.14), we have $\eta = cd$, $\xi = 0$.

When $\tau = \sigma = 0$, (3.8) reduces to

$$\begin{cases} (1 + \varepsilon) \dot{u}(t) = -(a + b)u(t) - cv(t), \\ \dot{u}(t) = du(t). \end{cases} \quad (3.21)$$

From (3.7), we see that $b > 0$, $c > 0$ and $d > 0$. Hence the trivial solution of (3.21) is asymptotically stable if and only if $a + b > 0$, and is unstable if and only if $a + b < 0$.

Suppose that either $b^2 + 2cd - a^2 \leq 0$, or $(b^2 + 2cd - a^2)^2 < 4(1 - \varepsilon^2)c^2d^2$; then equation (3.16) indicates that $\omega_\pm$ do not exist. Therefore, the first part of Theorem 3.1 can be applied to this situation. Since $a \geq b$ implies that $a + b > 0$, the trivial solution of (3.21) is stable when $\tau = 0$, and $a \leq -b$ implies that $a + b < 0$, resulting in the instability of the trivial solution of (3.21). Combining all this information, we arrive at the conclusions of part (1) of the theorem immediately.

Suppose that $b^2 + 2cd - a^2 > 0$, and $(b^2 + 2cd - a^2)^2 > 4(1 - \varepsilon^2)c^2d^2$. Then, from equation (3.16), we see that both $\omega_+$ and $\omega_-$ exist. Hence, part (3) of Theorem 3.1 applies. A similar argument as in the proof of part (1) of the theorem can be repeated, thus proving the statements of part (2).

In equation (3.14), we write

$$\tau_{n,1} = \frac{\theta_1}{\omega_+} + \frac{2n\pi}{\omega_+}, \quad (3.22)$$

where $\theta_1$ is defined by (3.18) and (3.19). We also write

$$\tau_{n,2} = \frac{\theta_2}{\omega_-} + \frac{2n\pi}{\omega_-}, \quad (3.23)$$

where $0 \leq \theta_2 < 2\pi$, and

$$\cos \theta_2 = -\frac{ab\omega_-^2 + (\eta - \omega_-^2)(\xi - \varepsilon\omega_-^2)}{b^2\omega_-^2 + (\xi - \varepsilon\omega_-^2)^2}, \quad (3.24)$$

$$\sin \theta_2 = \frac{(\xi - \varepsilon\omega_-^2)a\omega_- - b\omega_-(\eta - \omega_-^2)}{b^2\omega_-^2 + (\xi - \varepsilon\omega_-^2)^2}. \quad (3.25)$$

When the trivial solution of (3.14) is stable for $\tau = 0$, there occur $k$ switches from stability to instability to stability when the parameters are such that

$$\tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \ldots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \tau_{k,2} \ldots,$$

or $k$ switches from instability to stability to instability may occur when

$$\tau_{0,2} < \tau_{0,1} < \tau_{1,2} < \ldots < \tau_{k-1,2} < \tau_{k-1,1} < \tau_{k,1} < \tau_{k,2} \ldots,$$
when the trivial solution of (3.14) is unstable for \( \tau = 0 \) (for details, see (Freedman and Kuang, 1989)).

The following theorem deals with case (ii).

**Theorem 3.3** In system (3.8), assume that \( \sigma = \tau, |\varepsilon| < 1 \). Then the following statements are true.

1. If \( a + b > 0 \), then for \( \tau < \tau_{0,1} \), the trivial solution of (3.8) is uniformly asymptotically stable, and for \( \tau > \tau_{0,1} \), the trivial solution of (3.8) is unstable. Here \( \tau_{0,1} \) is defined as in (3.17), (3.18), (3.19), with \( \eta = 0, \xi = cd \).
2. If \( a + b < 0 \), then for any \( \tau \geq 0 \), the trivial solution of (3.8) is always unstable.

**Proof** When \( \sigma = \tau \), the characteristic equation of (3.8) becomes

\[
\lambda^2 + 2\lambda e^{-\lambda \tau} + a\lambda + b\lambda e^{-\lambda \tau} + cd e^{-\lambda \tau} = 0. \tag{3.26}
\]

Comparing this with (3.14), we have the correspondences \( \eta = 0, \xi = cd \). In this case, (3.16) indicates that only \( \omega_c \) exists. Therefore, part (2) of Theorem 3.1 can be applied to this situation. Combining it with the fact that \( a + b > 0 \) implies that the trivial solution of (3.8) is stable while \( a + b < 0 \) implies the converse, we see that the conclusions of the theorem are indeed true.

It is well known that when \( |\varepsilon| > 1 \) in equation (3.15), then for all \( \tau > 0 \), there is an infinite number of roots of equation (3.15) whose real parts are positive. In other words, this asserts that the trivial solution of (3.14) is unstable (for details, see (Freedman and Kuang, 1989, Theorem 2.1) or (Hale, 1977, Corollary 7.1, p. 29)). This fact leads to the following theorem.

**Theorem 3.4** In system (3.8), if \( |\varepsilon| > 1 \), then for all \( \tau > 0 \), the trivial solution is always unstable.

One of the results obtained from the critical case analysis for the second-order neutral delay equation made in (Freedman and Kuang, 1989) is equivalent to the following theorem.

**Theorem 3.5** Assume that \( |\varepsilon| = 1, \sigma = 0 \) in system (3.8). If \( a \geq (b^2 + 2cd)^{\frac{1}{3}} \), then the trivial solution of (3.8) is always uniformly asymptotically stable.

**Example 3.1** In system (3.1), let \( \sigma = 0, p(x) = \frac{x^2}{s^2 + x^2} \), where \( s \) is a positive constant. Then

\[
p'(x) = \frac{2xs^2}{(x^2 + s^2)^2}, \tag{3.27}
\]
\[
p''(x) = \frac{2s^2(s^2 - 3x^2)}{(x^2 + s^2)^3}. \tag{3.28}
\]

In this case,

\[x^* = s(\alpha/(\beta - \alpha))^{\frac{1}{3}}.\]

It is easy to see that if \( \alpha \) is chosen to be very small, then \( x^* \) will be very small as well. Indeed, \( x^* = o(\sqrt{\alpha}), y^* x^* = r(1 - x^*)(s^2 + x^2) \). So if \( x^* \) is very small, then \( y^* \) will be very big.

We have \( \varepsilon = rpx^*, b = r^2 x^*, c = p(x^*)/x^*, a = x^* y^*(s^2 - x^2)/(s^2 + x^2)^2, \) \( d = \beta y^* p'(x^*)x^* \). If we choose \( \alpha \) as the control parameter, then we see that as \( \alpha \) increases from zero, \( b^2 + 2cd - a^2 \) changes from negative to positive. So there exists \( \alpha_0 > 0 \) such that when \( 0 < \alpha < \alpha_0 \), then \( a \geq b \) and \( b^2 + 2cd - a^2 \geq 0 \). Hence part (1a) of Theorem 3.2 applies, which asserts that for \( 0 < \alpha < \alpha_0 \), and
with other parameters fixed, the unique positive steady state of system (3.1) is always uniformly asymptotically stable for all delay times \( \tau \geq 0 \). In other words, the delay does not destabilize the stable steady state.

When \( \alpha \) passes \( \alpha_0 \), we have \( b^2 + 2cd - a^2 > 0 \). There is \( \alpha_1 > \alpha_0 \) such that for \( \alpha_0 < \alpha < \alpha_1 \), \( b^2 + 2cd - a^2 > 0 \), \( a > b \) and \( (b^2 + 2cd - a^2)^2 < 4c^2d^2 \). Now, for any such \( \alpha \), we let \( \rho \) vary from zero to \( (r x^*)^{-1} \), or vary from zero to \(- (r x^*)^{-1}\). We see there exists \( \rho_0 > 0 \) in the first case (and \(- \rho_0 \) in the second case), such that when \( \rho < \rho < (r x^*)^{-1} \), then \( (b^2 + 2cd - a^2)^2 > 4(1 - \varepsilon^2)c^2d^2 \). Part (2a) of Theorem 3.2 indicates that as \( \tau \) increases, the stability of the steady state may be lost, and eventually will be lost forever. This somehow may indicate that the neutral term is destabilizing rather than stabilizing as pointed out by Gopalsamy and Zhang (1988) for the case when the predator \( y \) is absent from the system. This also shows that the role played by the neutral term is rather complicated.

4. Oscillatory results

We first consider the single-prey-species equation

\[
\dot{x}(t) = r x(t) [1 - x(t - \mu) - \rho x(t - \tau)],
\]

(4.1)

where \( r, \mu \) and \( \tau \) are positive constants.

In (Gopalsamy and Zhang, 1988, section 3), several oscillatory and non-oscillatory results of equation (4.1) are stated. Unfortunately, there is a serious flaw associated with their proofs. The flaw appears in the proof of Theorem 3.2. In order to obtain equation (3.12), the following wrong argument was used: suppose for \( t \geq T \), where \( T \) is a constant, that \( f'(t) < 0 \), and \( f(t) \) is bounded below; then \( \lim_{t \to \infty} f'(t) = 0 \). The consequence of this flaw is a disaster. First of all, their Theorem 3.2 can be wrong. Since their Corollary 3.3, Theorem 3.3, Theorem 3.4 and Corollary 3.4 all depend on the validity of the conclusion of Theorem 3.2, their whole theory may collapse.

Our first two theorems in this section will provide a partial remedy to this disaster. Recall that a positive solution of (4.1) defined on \([-\bar{\tau}, \infty) \) (where \( \bar{\tau} = \max \{\mu, \tau\} \)) is said to be ‘oscillatory about the positive steady state’ \( x(t) = 1 \) if there exists a sequence \( \{t_n\} \to \infty \) as \( n \to \infty \) for which \( x(t_n) = 1, \; n = 1, 2, \ldots \).

**Theorem 4.1** In (4.1), assume that \( \rho \) is a negative constant such that \(|\rho| > 1\), and \( x(t) \) is a bounded solution of (4.1). Then one of the three statements is true:

1. \( x(t) \) is oscillatory,
2. \( x(t) \) is eventually smaller than 1, and \( \lim_{t \to \infty} \int_0^t [1 - x(s - \mu)] \, ds \) exists and is finite,
3. \( x(t) \) is eventually bigger than 1, and \( \lim_{t \to \infty} x(t) = 1 \).

**Proof** Obviously, if \( x(t) \) is not oscillatory, then, it is either eventually bigger than 1 or eventually smaller than 1. We first suppose it is eventually smaller than 1. In this case there exists a positive constant \( t_0 \), such that when \( t \geq t_0 \), \( x(t) < 1 \). Note that equation (4.1) is equivalent to

\[
x(t) = x(t^*) \exp \left( r \int_{t^*}^t [1 - x(s - \mu)] \, ds \right) \exp (r |\rho| [x(t - r) - x(t^* - r)]),
\]

(4.2)
for any \( t^* \geq 0 \). Write
\[
f(t) = r \int_{t'}^{t} [1 - x(s - \mu)] \, ds.
\] (4.3)

Obviously, for \( t \geq t_0 \), \( f(t) \) is an increasing function. With \( t^* \) fixed and assuming that \( x(t) \) is bounded, it is easy to see that if \( \lim_{t \to \infty} f(t) = \infty \), then
\[
\lim_{t \to \infty} x(t) = +\infty.
\]

This clearly contradicts the assumption that \( x(t) \) is bounded. Hence
\[
\lim_{t \to \infty} \int_{t'}^{t} [1 - x(s - \mu)] \, ds
\]

must exist and be finite. This indeed implies somehow that for any \( 0 < \varepsilon < 1 \), and any \( t \geq 0 \), it is almost certain that \( 1 - \varepsilon < x(t) < 1 \).

Now, suppose that \( x(t) \) is eventually bigger than 1, that is, there exists a positive constant \( t_1 \) such that \( t \geq t_1 \) implies that \( x(t) > 1 \). In this situation, \( f(t) \) defined as above is decreasing for \( t \geq t_1 \). A similar argument as above shows that \( \lim_{t \to \infty} f(t) \) also exists and is finite. Clearly, this indicates that
\[
\lim_{t \to \infty} \inf x(t) = 1.
\] (4.4)

Write
\[
\lim_{t \to \infty} \sup x(t) = \bar{x};
\] (4.5)

the boundedness of \( x(t) \) implies that \( \bar{x} \) is a finite number. In the following, we shall try to show that \( \bar{x} = 1 \). Otherwise, \( \bar{x} > 1 \) and \( r |\rho| > 1 \) implies that there exists \( \varepsilon_0 \), with \( 0 < \varepsilon_0 < 1 \), such that
\[
(1 - \varepsilon_0) e^{-\varepsilon_0} e^{r t (1 - 2\varepsilon_0)} > \bar{x} + 2 \varepsilon_0.
\] (4.6)

Since we have \( \lim_{t \to \infty} x(t) = \bar{x} \), there exists \( t_2 > 0 \) such that \( t \geq t_2 \) implies that \( x(t) < \bar{x} + \varepsilon_0 \). Since \( f(t) \) is eventually decreasing, and \( \lim_{t \to \infty} f(t) \) exists and is finite, there exists \( t_3 > 0 \) such that \( t > t^* > t_3 \) implies that \( -\varepsilon_0 < f(t) - f(t^*) < 0 \).

Let \( T \) be a big positive number. Write
\[
I_1(T) = \{ t \mid t \in [T, 2T], x(t) - 1 < \frac{1}{2} \varepsilon_0 \},
\] (4.7)
\[
I_2(T) = \{ t \mid t \in [T, 2T], x(t) - \tau - 1 < \frac{1}{2} \varepsilon_0 \},
\] (4.8)

and let \( m(I_i(T)) \), \( i = 1, 2 \), be the measures of the sets \( I_1(T) \) and \( I_2(T) \). Then we have
\[
\lim_{T \to \infty} \frac{m(I_1(T))}{T} = 1.
\] (4.9)

Hence, there exists \( T_0 > 0 \) such that for any \( T > T_0 \),
\[
I_1(T) \cap I_2(T) \neq \emptyset.
\] (4.10)

In other words, for any \( T > 0 \) there always exists \( t > T \) such that \( x(t) - 1 < \frac{1}{2} \varepsilon_0 \),
and \( x(t - \tau) - 1 < \frac{1}{2} \epsilon_0 \). Hence, there exists \( \bar{t} \) such that

\[
\bar{t} \geq \max \{ t_1, t_2, t_3, T_0 \},
\]

\[
1 < x(\bar{t}) > 1 + \frac{1}{2} \epsilon_0, \quad 1 < x(\bar{t} - \tau) < 1 + \frac{1}{2} \epsilon_0.
\]

From (4.2), we have

\[
x(t) = x(\bar{t})e^{\rho (t - \bar{t})}e^{\gamma |x(t - \tau) - x(\bar{t} - \tau)|}.
\]

Let \( \tilde{t} > \bar{t} \), such that

\[
x(\tilde{t} - \tau) > \bar{x} - \epsilon_0.
\]

Then

\[
x(\tilde{t}) \geq (1 - \epsilon_0)e^{-\epsilon_0}e^{\gamma \rho (\tilde{t} - 1 + 2\epsilon_0)} > \bar{x} + \epsilon_0.
\]

This contradicts the fact that

\[
\limsup_{t \to \infty} x(t) = \bar{x}.
\]

Hence \( \lim_{t \to \infty} x(t) = 1 \).

The statement of our next theorem is the same as (Gopalsamy and Zhang, 1988, Corollary 3.3). As pointed out earlier, their proof contains a flaw.

**Theorem 4.2** In (4.1), assume that \( \rho \) is a positive number. Then every non-oscillatory solution of (4.1) satisfies \( \lim_{t \to \infty} x(t) = 1 \).

**Proof** Assume first that \( x(t) \) is a non-oscillatory solution of (4.1) which is eventually bigger than 1. Therefore, there exists \( t_1 \), such that when \( t \geq t_1 \), \( x(t) > 1 \). As in the proof of Theorem 4.1, we write

\[
f(t) = r \int_{t_1}^{t} [1 - x(s - \mu)] \, ds
\]

For \( t \geq t_1 \), \( f(t) \) is negative and decreasing. We claim that \( x(t) \) must be bounded. For otherwise there exists \( t^* > t_1 \) such that

\[
x(t^* - \tau) > x(t_1 - \tau) + (r \rho)^{-1} [\ln (x(t_1)) + 1].
\]

This implies that

\[
x(t^*) = x(t_1) \exp \left( r \int_{t_1}^{t^*} [1 - x(s - \mu)] \, ds \right) \exp \left( -r \rho [x^*(t - \tau) - x(t_1 - \tau)] \right)
\]

\[
< x(t_1) \exp \left( - [1 + \ln (x(t_1))] \right) = e^{-1} < 1,
\]

which contradicts the assumption that \( x(t) > 1 \) for \( t \geq t_1 \).

Obviously, the boundedness of \( x(t) \) implies that \( \lim_{t \to \infty} f(t) \) exists and is finite. Hence

\[
\liminf_{t \to \infty} x(t) = 1.
\]

Assume that

\[
\limsup_{t \to \infty} x(t) = \bar{x}.
\]
If $\bar{x} > 1$, then there exists $\varepsilon_0$, with $0 < 2\varepsilon_0 < \bar{x} - 1$, such that

$$\left(1 + \varepsilon_0\right)e^{-r\rho(\bar{x} - 1 - 2\varepsilon_0)} < 1.$$ \hspace{1cm} (4.20)

A similar argument to the one in the proof of Theorem 4.1 indicates that there exist $T_1$, $T_2$ such that $T_2 > T_1 > t_1$,

$$1 < x(T_1) < 1 + \varepsilon_0, \quad 1 < x(T_1 - \tau) < 1 + \varepsilon_0,$$ \hspace{1cm} (4.21)

and

$$x(T_2 - \tau) > \bar{x} - \varepsilon_0.$$ \hspace{1cm} (4.22)

Hence, we have

$$x(T_2) = x(T_1) \exp \left( \int_{T_1}^{T_2} [1 - x(s - \mu)] ds \right) \exp \left( -r\rho [x(T_2 - \tau) - x(T_1 - \tau)] \right)$$

$$< x(T_1) \exp \left( -r\rho [\bar{x} - \varepsilon_0 - x(T_1 - \tau)] \right)$$

$$< (1 + \varepsilon_0) \exp \left( -r\rho (\bar{x} - 1 - 2\varepsilon_0) \right) < 1.$$ \hspace{1cm} (4.23)

This contradicts the assumption that $x(t) > 1$ for all $t \geq t_1$. Hence

$$\lim_{t \to \infty} x(t) = 1.$$ \hspace{1cm} (4.24)

Now, suppose that $x(t)$ is a non-oscillatory solution of (4.1) which is eventually smaller than 1. In this case, $x(t)$ is always bounded, hence $\lim_{t \to \infty} f(t)$ always exists and is finite. This implies that

$$\lim sup_{t \to \infty} x(t) = 1.$$ \hspace{1cm} (4.24)

Assume that

$$\lim inf_{t \to \infty} x(t) = \bar{x}.$$ \hspace{1cm} (4.25)

If $\bar{x} < 1$, then there exists $\varepsilon_0$, with $0 < 2\varepsilon_0 < 1 - \bar{x}$, and

$$\left(1 - \varepsilon_0\right)e^{-r\rho(\bar{x} + 1 + 2\varepsilon_0)} > 1.$$ \hspace{1cm} (4.26)

An argument similar to the one for the previous case will result in a similar inequality which contradicts the assumption that $x(t) < 1$, for $t$ big enough. This completes the proof of the theorem.

The rest of this section is devoted to the discussion of the oscillatory behaviour of positive solutions of the following neutral delay logistic Gause-type predator–prey system with three delays:

$$\begin{align*}
\dot{x}(t) &= r x(t) \left[ 1 - (x(t - \mu) + \rho \dot{x}(t - \tau)) \right] - y(t) p(x(t)), \\
\dot{y}(t) &= y(t) \left[ -\alpha + \beta p(x(t - \sigma)) \right].
\end{align*}$$ \hspace{1cm} (4.27)

where $\mu$, $\tau$ and $\sigma$ are negative constants.

Let $(x(t), y(t))$ be a positive solution of system (4.27). An interesting problem is to investigate what is the oscillatory behaviour of $x(t)$ about $x^*$ and $y(t)$ about $y^*$, where $(x^*, y^*)$ is the unique positive steady state of (4.27). Our next two theorems give some partial answers to this question.
Theorem 4.3  In system (4.27), assume that \( p > 0 \), and \((x(t), y(t))\) is a positive solution. If \( x(t) \) is bounded and non-oscillatory about \( x(t) = x^* \), then

\[
\lim_{t \to \infty} (x(t), y(t)) = (x^*, y^*).
\]

Proof  Since \( x(t) \) is not oscillatory about \( x^* \), we have two possibilities:

(i) \( x(t) > x^* \) eventually;
(ii) \( x(t) < x^* \) eventually.

Assume first that \( x(t) > x^* \) eventually. Then

\[
y(t) = y(t_0) \exp \left[ \int_{t_0}^{t} \beta(p(x(s - \sigma)) - p(x^*)) \, ds \right]. \tag{4.28}
\]

Let

\[
\hat{x} = \sup \{ x(t) \mid t \geq 0 \}, \tag{4.29}
\]

\[
p_1 = \min \{ p'(x) \mid x \in [x^*, \hat{x}] \}, \tag{4.30}
\]

and

\[
p_2 = \max \{ p'(x) \mid x \in [x^*, \hat{x}] \}. \tag{4.31}
\]

We have \( p_1 > 0, \ p_2 > 0 \) and

\[
p_1 \cdot [x(s - \sigma) - x^*] \leq p(x(s - \sigma)) - p(x^*) \leq p_2 \cdot [x(s - \sigma) - x^*]. \tag{4.32}
\]

Equation (4.28) indicates that \( y(t) \) is increasing for large \( t \). If

\[
\lim_{t \to \infty} \int_{t_0}^{t} [x(s - \sigma) - x^*] \, ds = \infty,
\]

then

\[
\lim_{t \to \infty} y(t) = \infty. \tag{4.33}
\]

Equation (3.1) is equivalent to

\[
x(t) = x(t_0) \exp \left\{ \int_{t_0}^{t} \left[ 1 - x(s - \mu) \right] \, ds - rp[x(t - \tau) - x(t_0 - \tau)] 
- \int_{t_0}^{t} \left[ \frac{y(s)p(x(s)/x(s))}{y(s)p(x^*)} \right] \, ds \right\}
\]

\[
= x(t_0) \exp \left( r \int_{t_0}^{t} \left[ x^* - x(s - \mu) \right] \, ds \right) \cdot \exp \left( -rp[x(t - \tau) - x(t_0 - \tau)] \right)
\]

\[
\cdot \exp \left( -\int_{t_0}^{t} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y^*p(x^*)}{x^*} \right] \, ds \right).
\]

By (4.33) we can choose \( t_0 \) so large that if \( s > t_0 \) then

\[
[y(s)p(x(s))] / x(s) > [y^*p(x^*) / x^*] + 1 \tag{4.35}
\]
and \(x(s - \mu) > x^*\). Hence, if we let \(t \to \infty\) in (4.34), we obtain
\[
\lim_{t \to \infty} x(t) \leq x(t_0) \lim_{t \to \infty} \exp \left\{ - \int_{t_0}^{t} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y^*p(x^*)}{x^*} \right] ds - r \rho [x(t - \tau) - x(t_0 - \tau)] \right\}
\]
\[
\leq x(t_0) \lim_{t \to \infty} \exp \left\{ -(t - t_0) - r \rho [x(t - \tau) - x(t_0 - \tau)] \right\} = 0.
\]
Obviously, this contradicts the assumption that \(x(t) > x^*\) for \(t \geq t_0\). This proves that \(\lim_{t \to \infty} y(t)\) exists and is finite.

With \(\lim_{t \to \infty} y(t)\) finite, we see from (4.29) and (4.32) that
\[
\lim_{t \to \infty} \int_{t_0}^{t} [x(s - \sigma) - x^*] \, ds
\]

is also a finite number. Hence, for any \(T > 0\),
\[
\lim_{T \to \infty} \int_{t_0}^{t + T} [x(s - \sigma) - x^*] \, ds = 0. \tag{4.36}
\]

Now, assume that
\[
\lim_{t \to \infty} y(t) = \bar{y}. \tag{4.37}
\]

Assume also that \(\bar{y} > y^*\). Then
\[
- \int_{t_0}^{t} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y^*p(x^*)}{x^*} \right] ds = - \int_{t_0}^{t} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y(s)p(x^*)}{x^*} \right] ds
\]
\[
- \int_{t_0}^{t} \left[ y(s) - y^* \right] \frac{p(x^*)}{x^*} ds. \tag{4.38}
\]

Note that (4.36) implies that for any Lipschitz continuous function \(f(x)\) defined for all positive \(x\), the following is true:
\[
\lim_{T \to \infty} \int_{t_0}^{t + T} [f(x(s - \sigma) - f(x^*))] \, ds = 0. \tag{4.39}
\]

This indeed implies that
\[
\lim_{T \to \infty} \int_{t_0}^{t + T} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y^*p(x^*)}{x^*} \right] ds = +\infty. \tag{4.40}
\]

Since \(x^* < x(t) < \hat{x}\) for \(t \geq t_0\), we have from (4.34) that
\[
x(t + T) \leq \hat{x} \exp \left\{ - \int_{t}^{t + T} \left[ \frac{y(s)p(x(s))}{x(s)} - \frac{y^*p(x^*)}{x^*} \right] ds + r \rho (\hat{x} - x^*) \right\}
\]

Obviously, this leads to
\[
\lim_{t \to \infty} x(t + T) = 0,
\]

a contradiction to \(x^* < x(t) < \hat{x}\) for \(t \geq t_0\). This proves that
\[
\lim_{t \to \infty} y(t) = y^*. \tag{4.41}
\]
It is easy to see that (4.36) implies that
\[
\liminf_{t \to \infty} x(t) = x^*.
\]

(4.42)

Suppose that
\[
\limsup_{t \to \infty} x(t) = \bar{x}.
\]

(4.43)

If \(\bar{x} > x^*\), then there exists \(\varepsilon_0\) such that \(0 < 2\varepsilon_0 < \bar{x} - x^*\) and
\[
(x^* + \varepsilon_0)e^{+2\varepsilon_0e^{-\tau \rho (\bar{x} - x^* - 2\varepsilon_0)}} < x^*.
\]

(4.44)

By an argument similar to that in the proofs of Theorems 4.1 and 4.2, we see that there exist \(t_1, t_2\), with \(t_2 > t_1 > t_0\), such that
\[
x^* < x(t_1) < x^* + \varepsilon_0, \quad x^* < x(t_1 - \tau) < x^* + \varepsilon_0,
\]
\[
r \int_{t_1}^{t_2} (x(s - \mu) - x^*) \, ds < \varepsilon_0, \quad x(t_2 - \tau) > \bar{x} - \varepsilon_0
\]

and
\[
\int_{t_1}^{t_2} \left( \left| \frac{y(s)p(x(s))}{x(s)} - \frac{y*p(x(s))}{x(s)} \right| + \left| \frac{y*p(x(s))}{x(s)} - \frac{y*p(x^*)}{x^*} \right| \right) \, ds < \varepsilon_0.
\]

With this choice of \(t_1\) and \(t_2\), it follows from (4.34) that
\[
x(t_2) \leq (x^* + \varepsilon_0)e^{+2\varepsilon_0e^{-\tau \rho (\bar{x} - x^* - 2\varepsilon_0)}} < x^*.
\]

This contradicts the fact that \(x(t) \geq x^*\) for \(t > t_0\), hence proving that
\[
\lim_{t \to \infty} x(t) = x^*.
\]

(4.45)

Now we consider the case that \(x(t)\) is eventually smaller than \(x^*\). It is easy to see that \(\lim_{t \to \infty} y(t)\) must exist. We claim that this limit cannot be zero since if it is zero, then (4.34) will imply that
\[
\lim_{t \to \infty} x(t) = \infty.
\]

(4.46)

Indeed, (4.46) follows from the fact that
\[
\lim_{t \to \infty} \int_{t_0}^{t} [1 - x(s - \mu)] \, ds - \int_{t_0}^{t} \left[ y(s)p(x(s))/x(s) \right] \, ds
\]
\[
\geq \lim_{t \to \infty} \int_{t_0}^{t} (1 - x^*) \, ds - \int_{t}^{t} \left[ y(s)p(x(s))/x(s) \right] \, ds = +\infty.
\]

By a tedious argument similar to the one for case (i), we can show again that
\[
\lim_{t \to \infty} (x(t), y(t)) = (x^*, y^*).
\]

We omit the details to avoid repetition.

An immediate consequence of the above theorem is the following corollary.
Corollary 4.1 Let \((x(t), y(t))\) be a positive solution of system (4.27). If \(y(t)\) is oscillatory about \(y^*\), or \(\lim_{t \to \infty} y(t) \neq y^*\), then \(x(t)\) must be oscillatory about \(x^*\), where \((x^*, y^*)\) is the unique positive steady state of (4.27).

5. Discussion

In order to apply the recent results obtained by Freedman and Kuang (1991), we have purposely chosen \(\mu = \tau\) and \(\sigma = 0\) or \(\tau\). Although the method utilized by Freedman and Kuang can be extended to establish more general results to be applied to system (2.1), the computation required for this process is expected to be very tedious.

As far as only asymptotic stability and oscillation analysis are concerned, we do not have to choose the logistic per capita growth rate for the prey species in order to obtain results such as we have established so far. We can use instead some more general growth-rate functions. We can also allow the per capita growth rate for predator species to be replaced by \([-\alpha + q(x(t - \sigma))]\). However, this will only add complexity to an already tedious computation or analysis.

As indicated by Example 3.1, the role played by the neutral term \(\rho x(t - \tau)\) in the per capita growth rate for prey species is very complex. It may serve as both a stabilizing and a destabilizing factor, depending on the specific situation. At the very least, we know its presence causes various difficulties for our analysis.

It is well known and easy to prove that solutions of the equation (1.1) and equation (1.3) (see (Gopalsamy et al., 1988)) are eventually uniformly bounded. With the neutral term introduced into the per capita growth rate, this is no longer true. As pointed out by Kuang and Feldstein (1991), for some negative number \(\rho\), \(x(t)\) may indeed become unbounded. In system (2.1), if \(\rho = 0\), then it is known and also easy to show that the system is dissipative. However, at this moment, we do not know whether the same conclusion is true for system (2.1) when \(\rho \neq 0\). One thing we know is that it will be very difficult to prove even if the conclusion is true.

One of the important questions left untouched in this paper is the criterion for the existence of periodic solutions in system (2.1). In fact, so far, we do not even know what is the condition for a Hopf bifurcation to occur for (2.1). Although our numerical results indicate that a similar Hopf bifurcation theorem may be true for the nonlinear neutral delay system (2.1), we cannot provide any rigorous proof for that yet. It seems that the integral manifold technique developed by Hale (1971, 1974, 1977) may contribute to the solution of this general problem.

Another interesting question remaining to be investigated is to find the conditions for the global stability of the unique positive steady state in system (2.1). The method introduced in (Haddock and Terjék, 1983) may be useful in this regard.

Finally, we would like to admit that the way we have introduced the delays in system (2.1) is not the most realistic one. A better way to do so is probably to replace those discrete delays by continuously distributed delays.

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