Finiteness of limit cycles in planar autonomous systems

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Finiteness of Limit Cycles in Planar Autonomous Systems

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Abstract

Sufficient conditions are given to guarantee the finiteness of limit cycles in planar dissipative autonomous systems. Both analytic and nonanalytic cases are discussed.

KEY WORDS: Planar system, limit cycle, center-focus, graphic

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1. INTRODUCTION

In this paper, we present sufficient conditions which will guarantee that the number of limit cycles in the following planar autonomous system are finite,

\[ \begin{align*}
    \dot{x} &= X(x, y), \\
    \dot{y} &= Y(x, y),
\end{align*} \]

(1.1)

In the case that \( X \) and \( Y \) are polynomials, this work is directly related to the famous Hilbert’s sixteenth problem, where the question as to what is the number of limit cycles in such a system and their relative configurations [7] is posed. Two-dimensional flows of the form (1.1) have been and continue to be investigated intensively, partly because of the many applications in which they arise, and also partly because of their intrinsic interest. In spite of the tremendous

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efforts devoted by generations of mathematicians since the turn of this century, the Hilbert's sixteenth problem remains one of the most challenging problems in nonlinear differential equations. Up to the present time, we are still unable to answer the question about whether a polynomial vector field on the real plane has only finitely many limit cycles $j9j$. A long time ago it was assumed that an affirmative answer could be based on Dulac's memoir [4]. However, a fundamental flaw was recently discovered in Dulac's proof [9]. It was not until 1987 that Bamón has shown that quadratic systems cannot have infinitely many limit cycles, but Dulac's "theorem" remains unproved for $n > 2$ or for the more general analytic system [14].

In recent years, several important papers addressing the problem of accumulation of limit cycles have appeared. The most noteworthy ones are [8], [9], [15] and [16]. In fact, this paper is partially motivated by and based on these works.

In view of the fact that most systems arising from applications (such as physical and biological models) are dissipative and may not be analytic at some points, we will not limit our interest only to polynomial or analytical systems. We expect that the two finiteness results to be presented in section 3 may be of particular interest to applied mathematicians, since it is always of importance to perform a detailed qualitative analysis of the differential equation system model in order to gain a good understanding of the real system being modeled.

2. PRELIMINARIES

A plane autonomous system is *dissipative* if all its solutions eventually enter a bounded set and remain there for all sufficiently large $t$ in the phase plane. A *limit cycle* is a periodic solution whose neighboring solutions are spiraling toward it as $t \to \infty$ or as $t \to -\infty$. A *center-focus* is defined as an equilibrium whose every neighborhood contains both periodic and nonperiodic solutions. A *completely unstable equi-
A **limit** is the one which has a neighborhood such that all its solutions initiating inside the neighborhood (except the equilibrium itself) will leave it eventually. For example, an unstable focus or unstable node is a completely unstable equilibrium.

A **graphic** is a loop formed by equilbria (or a segment consisting of equilibria) and normally oriented separatrices connecting them:

\[ E_1, \Gamma_1, E_2, \Gamma_2, \ldots, E_n, \Gamma_n, E_{n+1} = E_1 \]

such that

1. For \( j = 1, \ldots, m \), \( a(\Gamma_j) \in E_j \), where \( a(\Gamma_j) \) is the s-limit set of \( \Gamma_j \) and \( u(\Gamma_j) \in E_{j+1} \), \( a(\Gamma_j) \) is the s-limit set of \( \Gamma_j \), and
2. For \( j = 1, \ldots, m-1 \), normal orientations \( n_j \) of separatrices are compatible in the sense that if \( \Gamma_j \) has left-hand orientation, then no does \( \Gamma_{j+1} \).

In other words, a graphic of (1.1) can be described as the connected union of a finite number of compatibly oriented separatrix cycles of (1.1) and segments consisting of equilibrium points of (1.1).

Analytic functions \( X(x, y) \) and \( Y(x, y) \) are said to have a **common analytic factor** \( F(x, y) \) in \( \mathbb{C} \) if

\[ X(x, y) = F(x, y)G_1(x, y) \quad \text{and} \quad Y(x, y) = F(x, y)G_2(x, y), \]

where \( F, G_1, \) and \( G_2 \) are analytic in \( \Omega \), and \( F(x, y) = 0 \) for some \((x, y) \in \Omega \). If analytic functions \( X(x, y) \) and \( Y(x, y) \) have a common analytic factor in \( \Omega \), then system (1.1) is a relatively prime analytic system in \( \Omega \).

The proofs of our results in the next section will be based on the following lemmas. Throughout this section, system (1.1) is assumed analytic.

**Lemma 2.1.** Limit cycles cannot accumulate on a cycle of (1.1).

This is a well known result which follows from the fact that the Poincaré map in the neighborhood of a cycle of (1.1) is analytic.
Lemma 2.2 (Andronov, see the Theorem 1 in [16]). Let $E(x', y')$ be an equilibrium of (1.1) and assume
\[
\begin{pmatrix}
X_0(x', y') & X_1(x', y') \\
Y_0(x', y') & Y_1(x', y')
\end{pmatrix}
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
then $E(x', y')$ is not a center-focus for (1.1).

Proof. See the proof of Theorem 1 in [16].

In fact, Lemma 2.2 is simply the combination of the remark at the top of p. 254 in [2] and Theorem 65 (p. 360), Theorem 66 (p. 357) and Theorem 67 (p. 362) in [1].

Lemma 2.3 (Theorem 3 in [6]). Limit cycles of an analytic vector field on a real two-dimensional surface cannot accumulate on a graphic of the field which contains only hyperbolic saddle points.

The proof of this lemma is far from trivial. It is based on the fact that the so-called “monodromy transformation” (which is similar to the one-sided Poincaré map) associated with an analytic semitrajectory to a graphic containing only hyperbolic saddle points is very close to the identity transformation.

Lemma 2.4 (Theorem 1 in [15]). A relatively prime analytic system \((1.1)\) in a compact region of the plane has only a finite number of equilibria if

\[ L_n \]

In the next section, we will a sequence of limit cycles $L_n$ monotonically if it is either expanding (i.e., $L_n \subset \text{int} L_{n+1}$) or contracting (i.e., $L_{n+1} \subset \text{int} L_n$). For a monotone sequence of limit cycles $L_n$, the limit set is equal to the boundary of the set

\[ \bigcup_{n=1}^{\infty} \text{int} L_n \text{ or } \bigcap_{n=1}^{\infty} \text{int} L_n, \]

where $L_n$ is an expanding or contracting sequence of limit cycles respectively.
3. RESULTS

In this section, we consider the following system

\[ \begin{align*}
\dot{x} &= X(x, y), \\
\dot{y} &= Y(x, y).
\end{align*} \tag{3.1} \]

The first theorem to be presented provides us a set of sufficient conditions for the finiteness of limit cycles in system (3.1) when both \( X(x, y) \) and \( Y(x, y) \) are analytic. The second theorem addresses the same problem in the case that \( X(x, y) \) and \( Y(x, y) \) are not analytic at finitely many points. In order to illustrate why some of the conditions are also necessary, we furnish several examples where some of the sufficient conditions are violated and therefore invalidate the conclusions stated in the two theorems.

**Theorem 3.1.** Assume system (3.1) is dissipative and

(i) \( X(x, y) \), \( Y(x, y) \) are analytic,

(ii) The equilibria contained in any graphic of (3.1) are hyperbolic saddle points,

(iii) for any equilibrium \((x^*, y^*)\) of (3.1),

\[
\Delta = \begin{pmatrix}
X_x(x^*, y^*) & X_y(x^*, y^*) \\
Y_x(x^*, y^*) & Y_y(x^*, y^*)
\end{pmatrix} \neq \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

Then system (3.1) has only a finite number of limit cycles. Moreover, if (3.1) has a completely unstable equilibrium, then (3.1) has at least one asymptotically stable limit cycle or one semi-asymptotically stable graphic.

The next theorem generalizes the above results to the situation that \( X(x, y) \) and \( Y(x, y) \) may not be analytic at some points.

**Theorem 3.2.** Assume system (3.1) is continuously differentiable, dissipative and

(i) \( X(x, y) \), \( Y(x, y) \) are analytic except at \( P_i, i \in I = \{1, 2, \ldots, N\} \),

(ii) graphics of (3.1) contain only hyperbolic saddle points and do not contain any \( P_i, i \in I \),
(iii) for any equilibrium \((x^*, y^*)\) of (3.1), if \((x^*, y^*) \notin P = \{P_i | i \in I\}\), then
\[
\Delta = \begin{pmatrix}
X_x(x^*, y^*) & X_y(x^*, y^*) \\
Y_x(x^*, y^*) & Y_y(x^*, y^*)
\end{pmatrix} \neq \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\]
and if \((x^*, y^*) \in P\), then either
(a) \(\delta = X_x(x^*, y^*) + Y_y(x^*, y^*) \neq 0\) and \(\det \Delta \neq 0\), or
(b) \(\delta = 0\) and \(\det \Delta < 0\).

Then system (3.1) has only a finite number of limit cycles. Moreover, if (3.1) has a completely unstable equilibrium, then (3.1) either has at least one asymptotically stable limit cycle, or has at least one semi-asymptotically stable graphic.

In order to prove the above two theorems, we need the following two lemmas.

Lemma 3.1 (Generalized version of Theorem 2 in [15]). If \(L_n\) is a monotone sequence of limit cycles of the analytic system (3.1), contained in a bounded region of the plane, then the limit \(S = \lim_{n \to \infty} L_n\) is either an equilibrium or a graphic of (3.1) if \(L_n\) is a contracting sequence, and \(S\) is a graphic of (3.1) if \(L_n\) is an expanding sequence.

Proof of Lemma 3.1. The proof is essentially the same as the proof of Theorem 2 in [15]. We need only notice that the concept of graphic in this paper is a generalized version of the one in [15].

Lemma 3.2. Under the assumption of Theorems 3.1 or 3.2, each equilibrium of (3.1) is in the interior of at most a finite number of limit cycles of (3.1).

Proof of Lemma 3.2. Suppose the conclusion is not true. Then there is a monotone sequence of limit cycles \(L_n\) surrounding an equilibrium \(E(x^*, y^*)\) in system (3.1) in a bounded region of the plane, since we assume system (3.1) is dissipative.

We consider first the case of Theorem 3.1. According to Lemma 3.1 above, we know that \(S = \lim_{n \to \infty} L_n\) is either an equilibrium or
We note that $S$ cannot be an equilibrium by Lemma 2.2 and the assumption (iii). Also, it is clear that $S$ cannot be a graphic of (3.1), since otherwise we would have a sequence of limit cycles which accumulates on a graphic having only saddle points by assumption (ii), which is contrary to Lemma 2.3. This contradiction proves Lemma 3.2 in the case of Theorem 3.1.

Now, we consider the case of Theorem 3.2. We observe first that $S$ cannot be a graphic of (3.1), since all the graphics of (3.1) are assumed to contain only hyperbolic saddle points which do not belong to the nonanalytical points set $P$. This implies that for every graphic of (3.1), there is a neighborhood of it which does not contain any $E_i$, $i \in P$. Therefore, by Lemma 2.3, limit cycles cannot accumulate in such a neighborhood. Hence, $S = E$ by Lemma 3.1.

However, this is impossible if $F \not\in P$ by the assumption (iii) and Lemma 2.4. Therefore, the only possibility is $S = E_i$ for some $i \in I$, since the monotone sequence of limit cycles $\{C_i\}$ cannot exist. If $S = E_i$, then we see that $E(z^*, y^*)$ can only be one of the following types: (1) focus (stable or unstable); (2) node (stable or unstable); (3) hyperbolic saddle point. If $\delta = 0$ and $\det \Delta_0 \neq 0$, then $E(z^*, y^*)$ is clearly a hyperbolic saddle point. Therefore, we have shown that $S = E_i$, $i \in I$, and each $E_i$ cannot be a center-focus, which contradicts the existence of $\Delta_0$. This completes the proof for Lemma 3.2.

Proof of Theorem 3.1. First, we assume all solutions of (3.1) enter a compact and connected set $\Omega$. The claim system (4.1) is in fact a relatively prime analytic system. Since, otherwise, we have

$$X(x, y) = F(x, y)G(y, z) \quad \text{and} \quad Y(x, y) = F(x, y)P(z, y)$$

for some analytic function $F, G, H$ in $\Omega$, and $F(x, y) \sim 0$ for some points in $\Omega$. Therefore, there is an analytic function $f(x)$ such that $F(x, y) = 0$ and the curve described by $(x, f(x))$ lies in $\Omega$. and
hence it must be a loop, i.e., a graphic of system (3.1). However, this possibility is excluded by assumption (ii). Therefore, system (3.1) is a relatively prime analytic system, and by Lemma 2.4, we conclude that (3.1) only has a finite number of critical points in $\Omega$. By Lemma 3.2, we see immediately that (3.1) has at most finitely many limit cycles.

If (3.1) has a completely unstable equilibrium, then by the Poincaré-Bendixson Theorem there is a finite number of limit cycles or graphics surrounding it. By a simple induction, one can easily show that $$ has at least one asymptotically stable limit cycle or one semi-asymptotically stable graphic, proving the theorem.

**Proof of Theorem 3.2.** As in the proof of the previous theorem, once the finiteness of limit cycles is established, the second conclusion of the theorem follows naturally by a simple induction.

Let $\Omega$ have the same meaning as in the proof of Theorem 3.1. Assume that system (3.1) has infinitely many limit cycles. Then they all must lie inside $\Omega$. By Lemma 3.2, we see there is a sequence of limit cycles $\{L_n\}_{n=1}^{\infty}$ such that no two of these intersect and no one encircles the other. Since every limit cycle contains at least one equilibrium, this implies that there is a sequence of equilibria $\{E_{ln} = (x_{ln}, y_{ln})\}_{n=1}^{\infty}$, such that $E_1 \neq E_2 \neq \cdots$ and an equilibrium $E_{ln} = (x_{ln}, y_{ln})$. Since the variational system of (3.1) is integrable, we see that $E_{ln} \in \mathbb{R}^2$.

In other words, $\mathbb{R}^2$ is a compact region of $\mathbb{R}^2$. By Lemma 4.4, in a neighborhood of $E_{ln}$, $X(z, y)$ and $Y(z, y)$ have a common analytic factor, i.e., we have $X(z, y) = F(z, y)G(z, y)$ and $Y(z, y) = G(z, y)H(z, y)$ for some analytic functions $F, G, H$. If in a neighborhood of $E_{ln}$, such that $F(z, y) = 0$ and $G(z, y) = 0$ and $H(z, y) \neq 0$. Since $F(z, y)$ is analytic and $F(z, y) = 0$, we see the equation $F(z, y) = 0$ has at most a finite number of solutions $\ln$. Therefore, there are at least two of the equilibria from the subsequence $\{E_{ln}\}$ which lie in
the same branch. Assume they are $E_a$ and $E_b$, and $L_a, L_b$ are the
limit cycles that encircle $E_a, E_b$ respectively, such that $L_a$ does not
intersect with $L_b$. However, this is impossible, since otherwise both
$L_a$ and $L_b$ will have to intersect the solution branch (of $F(x, y) = 0$)
which join $E_a$ and $E_b$. This contradiction completes our proof of
Theorem 3.2.

We now construct examples to illustrate the necessity of some of
our hypotheses.

Let us consider the following system which is in polar coordinates
\[
\begin{align*}
\dot{r} &= \alpha e^{-(r-r_0)^2} \left( \sin \frac{\pi}{r-r_0} \right)^\beta = f(r, r_0), \\
\dot{\theta} &= a^2 + b^2 \sin \theta + (r - r_0)^2 = g(\theta, r, r_0).
\end{align*}
\tag{3.2}
\]
where $\alpha \neq 0, \beta > 0$ and $f(r_0, r_0) = \lim_{r \to r_0} f(r, r_0) = 0$. This system
is analytic except on the circle $r = r_0$. Nevertheless, system (3.2) is
infinitely smooth, since both $f$ and $g$ are in $C^\infty$.

(i) If we let $r_0 = b = 0, \alpha = \beta = a = 1$ then (3.2) gives us an
example where $(0, 0)$ (in the $x - y$ coordinates) is a center-focus.

(ii) If we let $r_0 = a = \beta = a = 1$ and $b = 0$, then (3.2) describes
a system having limit cycle $r = 1$ as the limit of a sequence of limit
cycles in the system.

(iii) If we let $r_0 = a = \beta = 1, a = 0, b = 1$, then $r = 1$ is a graphic
of (3.2), where $(1, 0)$ and $(1, \pi)$ (correspondingly $(1, 0), (-1, 0)$ in the
$x - y$ coordinates) are two equilibria of the system. This graphic is
the limit of the limit cycles $r = 1 \pm \frac{1}{n}, n = 2, 3, \ldots$.

(iv) If we let $a = -1, r_0 = 0, \beta = 2, a = b = 0$, then system
(3.2) is dissipative and analytic except at the origin $(0, 0)$. It can
be shown that $\delta = \det \Delta = 0$, which violates the assumption (iii) of
Theorem 3.2. Obviously, (3.2) has infinitely many limit cycles $r = \frac{1}{n}$
accumulate at the origin $(0, 0)$.

Examples can also be constructed which satisfy all the assump-
tions of the two theorems but have no complete unstable equilibrium,
and therefore have no stable limit cycle or semi-asymptotically stable
graphic.
Remarks.

1. In [16], Perko has shown that if the minimum degree of $X$ and $Y$ is even, then (1.1) does not have a center-focus at the origin; also, if $X$ and $Y$ are fourth degree polynomials, then (1.1) does not have a center-focus. Thus we can replace the assumption (iii) in Theorem 3.1 and the corresponding part of (iii) in Theorem 3.2 by either one of the two assumptions just mentioned.

2. We would like to comment on the fact that our results are readily applicable to many realistic mathematical models. For example, many systems of two interacting populations are modeled by (see [5] and [10])

$$\begin{align*}
\dot{x} &= xf(x, y), \\
\dot{y} &= yg(x, y).
\end{align*}$$

(4.1)

Generally, one can expect system (4.1) is always dissipative and either analytic or analytic except at a finite number of points. Also, (4.1) tends to have only several equilibria in the domain of interest. In these situations, if (4.1) does have graphic, it usually contains only hyperbolic saddle points. In particular, the system analyzed in [12] is analytic, and a trivial application of Theorem 3.1 shows that the number of the limit cycles in that system must be finite.

3. In a remarkable development, Ecalle et al. [17] have announced that they have proved Dulac’s theorem that no polynomial system can have infinitely many limit cycles. Indeed, their results are also valid for some analytic systems. However, the proofs of these results have not yet been published. Their manuscript runs to well over a hundred pages. A full understanding of it will itself present a formidable challenge [14]. Their method exploits the properties of certain formal series which, though not convergent, are quasi-analytic in a defined sense. This idea may also work in the system discussed in Theorem 3.1. If this is the case, then assumption (ii) or (iii) in Theorem 3.1 may be removed without affecting the
conclusion. Thus arises an interesting question remaining to be answered.
References


