MONOTONIC AND OSCILLATORY SOLUTIONS OF A LINEAR NEUTRAL DELAY EQUATION WITH INFINITE LAG*

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Abstract. This paper is devoted to the discussion of monotonic and oscillatory solutions of the linear neutral delay equation

\[ y'(t) = Ay(t) + \sum_{i=1}^{M} B_i y(A_i t) + \sum_{i=1}^{N} C_i y'(\eta_i t), \]

where \( 0 < \lambda_i < 1 \) for \( i = 1, \ldots, M \), and \( 0 < \eta_i < 1 \) for \( i = 1, \ldots, N \). Under one set of conditions, all nontrivial solutions are absolutely monotone. Under a different set of conditions, all nontrivial solutions oscillate unboundedly. This resolves most parts of the conjecture recently made by Feldstein and Jackiewicz. Some existence, uniqueness, and analyticity results are also included.

Key words. monotonic solutions, oscillatory solutions, unbounded oscillations, neutral delay equation, infinite lag, Phragmén–Lindelöf principle

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1. Introduction. In a recent paper, Feldstein and Jackiewicz [6] investigate the neutral functional differential equation (where \( t \) is complex)

\[ y'(t) = Ay(t) + By(At) + Cy'(\eta t) \]

where \( A, B, C, \lambda, \) and \( \eta \) are complex parameters and \( 0 \leq |\lambda| < 1, 0 \leq |\eta| < 1 \). The derivative \( y'(\eta t) \) means \( y'(x) \) evaluated at \( x = \eta t \). Obviously, as \( t \to +\infty \), the lag in (1.1) becomes infinite. At the end of their manuscript, the following conjecture was proposed.

**Conjecture.** In (1.1), suppose that \( A = 0 \), that \( B, C, \lambda, \) and \( \eta \) are all real, and that \( -1 < C < 1 \).

(a) If \( B < 0 \), then every nontrivial solution to (1.1) oscillates (unboundedly).

(b) If \( B > 0 \), then every nontrivial solution to (1.1) is monotonic.

This paper is motivated mainly by their conjecture. The equation discussed here is the following generalized version of (1.1):

\[ y'(t) = Ay(t) + \sum_{i=1}^{M} B_i y(A_i t) + \sum_{i=1}^{N} C_i y'(\eta_i t) \quad \text{for } t \in \mathbb{R}. \]

The analysis presented here indicates that most parts of the above conjecture are indeed true and even true for (1.2).

When \( C = 0 \), equation (1.1) arises as a mathematical idealization and simplification of an industrial problem involving wave motion in the overhead supply line to an electrified railway system (see Fox et al. [8]). In this special case, (1.1) has been considered by Feldstein and Grafton [5], by Kato and McLeod [13], and by Morris, Feldstein, and Bowen [15], as well as second-order variations by Waltman [17] and Bélair [1].

As indicated in Feldstein and Jackiewicz [6], the existence of monotonic or oscillatory solutions to (1.2), while of considerable interest in its own right, is of
particular importance in numerical analysis because of its applicability to the development of stiffly stable numerical methods for neutral equations. See, for example, Dahlquist [4], Gear [9], and Bellen, Jackiewicz, and Zennaro [2].

This paper is organized as follows. Section 2 establishes theorems on existence, uniqueness, and analyticity of solutions to (1.2), followed by a section devoted to a discussion of conditions under which solutions to (1.2) are monotonic. Section 4 contains oscillatory and unboundedness results; Theorem 4.1 is the main result of this investigation. This paper concludes with a brief discussion and a list of some open questions.

2. Existence, uniqueness, and differentiability of solutions. This paper is devoted primarily to the discussion of monotonic and oscillatory solutions of the following linear neutral functional differential equation:

\[ y(0) = y_0, \]
\[ y'(t) = Ay(t) + \sum_{i=1}^{M} B_i y(\lambda_i t) + \sum_{i=1}^{N} C_i y'(\eta_i t) \quad \text{for } t \in \mathbb{R}, \]
where \( 0 < \lambda_i < 1, \ 0 < \eta_i < 1, \) and the coefficients \( A, B_i, \) and \( C_i \) are all real constants. First, this section establishes some basic results about the existence, uniqueness, and differentiability of solutions to (2.1).

For \( a < b, \) let \( C[a, b] \) denote the complete metric space consisting of continuous functions on \([a, b],\) with the metric function \( \rho \) defined as

\[ \rho(y_1(t), y_2(t)) = \max_{a \leq t \leq b} |y_1(t) - y_2(t)|, \]

where each \( y_i(t) \) is continuous on \([a, b].\) The following theorem presents conditions for the local existence and uniqueness of solutions to (2.1).

**Theorem 2.1.** Suppose that \( \alpha = \sum_{i=1}^{N} |C_i|^{-1} < 1 \) and that \( 0 < T < (1 - \alpha)(|A| + \sum_{i=1}^{M} |B_i|)^{-1}. \) Then (2.1) has a unique solution on \([0, T].\)

**Proof.** Existence. Since \( \alpha = \sum_{i=1}^{N} |C_i|^{-1} < 1, \) it follows that \( \sum_{i=1}^{N} |C_i| \equiv \alpha < 1. \) Let

\[ z_0 = \left( A + \sum_{i=1}^{M} B_i \right) \left( 1 - \sum_{i=1}^{N} C_i \right)^{-1} y_0, \]
\[ S = \{ z(t) \mid z(0) = z_0, \ z(t) \in C[0, T], \} \]
where \( 0 < T < (1 - \alpha)(|A| + \sum_{i=1}^{M} |B_i|)^{-1}. \) Obviously, \( S \) with the metric defined in (2.2) constitutes a complete metric space.

Consider the mapping \( L : S \to S \) defined as

\[ Lz(t) = z_0 + A \int_{0}^{t} z(s) \, ds + \sum_{i=1}^{M} B_i \int_{0}^{\lambda_i t} z(s) \, ds + \sum_{i=1}^{N} C_i (z(\eta_i t) - z_0). \]

Let \( z_1(t) \in S \) and \( z_2(t) \in S. \) Denote

\[ \rho(z_1, z_2) = \rho(z_1(t), z_2(t)) = \max_{0 \leq t \leq T} |z_1(t) - z_2(t)|. \]

It is easy to see that

\[ Lz_1(t) - Lz_2(t) = A \int_{0}^{t} (z_1(s) - z_2(s)) \, ds + \sum_{i=1}^{M} B_i \int_{0}^{\lambda_i t} (z_1(s) - z_2(s)) \, ds \]
\[ + \sum_{i=1}^{N} C_i (z_1(\eta_i t) - z_2(\eta_i t)). \]
Hence, for \( t \in [0, T] \),

\[
|Lz_1(t) - Lz_2(t)| \leq |A|t\rho(z_1, z_2) + \left( \sum_{i=1}^{M} \lambda_i |B_i| \right) t\rho(z_1, z_2) + \left( \sum_{i=1}^{N} |C_i| \right) \rho(z_1, z_2).
\]

This implies that

\[
\rho(Lz_1, Lz_2) \leq T \left( |A| + \sum_{i=1}^{M} \lambda_i |B_i| \right) + \sum_{i=1}^{N} |C_i| \rho(z_1, z_2).
\]

Let \( \beta = T(|A| + \sum_{i=1}^{M} \lambda_i |B_i|) + \sum_{i=1}^{N} |C_i| \). Then the second hypothesis implies that

\[
\beta \leq T(|A| + \sum_{i=1}^{M} |B_i|) + \sum_{i=1}^{N} |C_i| < 1 - \alpha + \sum_{i=1}^{N} |C_i|
\]

\[
\leq 1 - \sum_{i=1}^{N} |C_i| + \sum_{i=1}^{N} |C_i| = 1,
\]

i.e., \( \beta < 1 \). Therefore, \( L \) is a contraction mapping. Hence, there exists a unique \( z(t) \in S \) (Waltman [18, p. 170]) such that \( Lz(t) = z(t) \). This is equivalent to

\[
z(t) = z_0 + \int_0^t z(s) \, ds + \sum_{i=1}^{M} B_i \int_0^t z(s) \, ds + \sum_{i=1}^{N} C_i (z(\eta_i t) - z_0).
\]

Now, let \( y(t) = y_0 + \int_0^t z(s) \, ds \). It is easy to see that \( y(0) = y_0 \), and

\[
y'(t) = Ay(t) + \sum_{i=1}^{M} B_i y(\lambda_i t) + \sum_{i=1}^{N} C_i y'(\eta_i t).
\]

In other words, \( y(t) = y_0 + \int_0^t z(s) \, ds \) is a solution of (2.1) on \([0, T]\).

**Uniqueness.** Assume \( y(t) \) is a solution of (2.1) on some interval \([0, T]\), where \( 0 < T < (1 - \alpha)(|A| + \sum_{i=1}^{M} |B_i|)^{-1} \). Then \( y(t) \) must be a solution of

\[
y(t) = y_0 + A \int_0^t y(s) \, ds + \sum_{i=1}^{M} B_i \lambda_i^{-1} \int_0^t y(s) \, ds + \sum_{i=1}^{N} C_i \eta_i^{-1}(y(\eta_i t) - y_0),
\]

which is obtained by integrating both sides of (2.1).

Let

\[
\bar{S} = \{ y(t) \mid y(0) = y_0, y(t) \in C[0, T] \}.
\]

\( \bar{S} \), with the metric defined in (2.2), constitutes a complete metric space. Consider the mapping \( \bar{L} : \bar{S} \to \bar{S} \) defined by

\[
\bar{L}y(t) = y_0 + A \int_0^t y(s) \, ds + \sum_{i=1}^{M} B_i \lambda_i^{-1} \int_0^t y(s) \, ds
\]

\[
+ \sum_{i=1}^{N} C_i \eta_i^{-1}(y(\eta_i t) - y_0).
\]

By an argument similar to that in the proof for the existence part, it is easy to see that \( \bar{L} \) is a contraction mapping. Hence, the solution of the integral equation (2.9) is unique. This implies that there is a unique solution \( y(t) \) of (2.1) and so completes the proof. \(\square\)
**Theorem 2.2.** Under the same assumptions as in Theorem 2.1, the solution of (2.1) is analytic.

*Proof.* For any given positive integer $l$, consider the differential equation

$$X'(0) = y_1,$$

(2.12)

$$X'(t) = AX(t) + \sum_{i=1}^{M} B_i \lambda_i^j X(\lambda_i t) + \sum_{i=1}^{N} C_i \eta_i^j X'(\eta_i t),$$

where

$$y_{j+1} = \frac{A + \sum_{i=1}^{M} b_i \lambda_i^j}{1 - \sum_{i=1}^{N} C_i \eta_i^{j+1}} y_j \quad \text{for } j = 0, 1, \ldots, l - 1.$$  

It is easy to see that the conditions in Theorem 2.1 are satisfied for (2.12). Hence, the existence and uniqueness of its solution is guaranteed. Let $X(t)$ be such a solution. Then it can be shown by induction that $y(t)$ given by

$$y(t) = \int_0^t \left( \cdots \int_0^{s_2} \left( \int_0^{s_1} X(s) \, ds \right) \, ds_1 \cdots \right) ds_{l-1} + \sum_{j=0}^{l-1} y_j t^j$$

(2.14)

is a solution of (2.1). Theorem 2.1 implies that $y(t)$ is the unique solution of (2.1). Since this is true for all $l$, then $y(t)$ is infinitely many times differentiable.

Consider the possible power series expression of the unique solution $y(t)$. Assume $y(t) = \sum_{i=0}^{\infty} a_i t^i$. Then

$$y(t) = y_0 + \sum_{i=1}^{\infty} a_i t^i.$$  

(2.15)

$$(l+1)a_{l+1} = \left( A + \sum_{i=1}^{M} B_i \lambda_i^l \right) a_l + \left( \sum_{i=1}^{N} C_i \eta_i^{l+1} \right) (l+1)a_{l+1}.$$

Hence

$$a_{l+1} = \frac{A + \sum_{i=1}^{M} B_i \lambda_i^l}{1 - \sum_{i=1}^{N} C_i \eta_i^{l+1}} a_l,$$

(2.16)

which leads to

$$a_l = \frac{a_0}{l!} \prod_{j=0}^{l-1} \left( \frac{A + \sum_{i=1}^{M} B_i \lambda_i^j}{1 - \sum_{i=1}^{N} C_i \eta_i^{j+1}} \right).$$

By the ratio test, it is easy to see that the series $\sum_{i=0}^{\infty} a_i t^i$ converges everywhere. Indeed, $y(t) = \sum_{i=0}^{\infty} a_i t^i$ is the unique solution of (2.1) provided that $a_0 = y_0$. Obviously, $\sum_{i=0}^{\infty} a_i t^i$ represents an analytic function. This proves the theorem. □

**Remark 2.1.** Since (2.1) is autonomous, it is easy to see that the local existence result in Theorem 2.1 is indeed a global one in the sense that the solution, in fact, exists for all $t \geq 0$. This can be shown rigorously by a simple induction argument.

### 3. Monotonicity results

This section presents conditions under which solutions of (2.1) are monotone, or nonoscillatory. By virtue of the proof of Theorem 2.2, the following lemma is true.

**Lemma 3.1.** Suppose $\sum_{i=1}^{N} |C_i \eta_i^{l+1}| < 1$; then the unique solution of (2.1) is given by

$$y(t) = y_0 + \sum_{i=0}^{\infty} \left( \prod_{j=0}^{i-1} \left( \frac{A + \sum_{i=1}^{M} B_i \lambda_i^j}{1 - \sum_{i=1}^{N} C_i \eta_i^{j+1}} \right) \right) \frac{t^i}{i!},$$

(3.1)

where empty products are equal to 1.
DEFINITION 3.1. Let \( y(t) = \sum_{i=0}^{\infty} a_i t^i \). If \( d^k y(t)/dt^k > 0 \) (\(< 0\)) for all nonnegative integers \( k \) and for all \( t > 0 \), then \( y(t) \) is said to be strictly absolutely monotone increasing (decreasing), or simply absolutely monotone. (Compare with Widder [19, p. 144], who does not require the strict inequality for the derivatives.) If \( d^k y(t)/dt^k \geq 0 \) (\( \leq 0 \)) for all nonnegative integers \( k \) and for all \( t \geq T > 0 \), then \( y(t) \) is said to be eventually absolutely monotone.

The following lemma is obviously true.

**LEMMA 3.2.** If \( y(t) = \sum_{i=0}^{\infty} a_i t^i \) is absolutely monotone or eventually absolutely monotone, then it is nonoscillatory.

**LEMMA 3.3.** If \( y_0 \neq 0 \), \( A \geq \min \{0, B_1\} \), and \( \sum_{i=1}^{N} |C_i| \eta_i < 1 \), then the power series in (3.1) is absolutely monotone.

**Proof.** Let \( a_i = \prod_{j=0}^{i-1} ((A + \sum_{j=1}^{M} B_j \lambda_j^i)/\left(1 - \sum_{i=1}^{N} C_i \eta_i^{i+1}\right)) \). Then it is easy to see that the assumptions made in this lemma ensure that \( a_i > 0 \) for integer \( l \geq 0 \). Note that \( y(t) = y_0 \sum_{i=0}^{\infty} a_i t^i/\sigma_i! \), and that \( d^k y(t)/dt^k = y_0 \sum_{i=0}^{\infty} (a_i t^i)/\sigma_i! \). Obviously, \( d^k y(t)/dt^k > 0 \) (\( < 0 \)) for all \( k \geq 0 \) and for all \( t > 0 \), if \( y_0 > 0 \) (\( < 0 \)). This completes the proof.

**LEMMA 3.4.** If \( A + \sum_{j=1}^{M} B_j \lambda_j^i \neq 0 \) for all integers \( j \geq 0 \), \( \sum_{i=1}^{N} |C_i| \eta_i < 1 \), and \( A > 0 \), then the power series in (3.1) is eventually absolutely monotone.

**Proof.** Let \( a_j = (A + \sum_{j=1}^{M} B_j \lambda_j^i)/\left(1 - \sum_{i=1}^{N} C_i \eta_i^{i+1}\right) \) for integers \( j > 0 \). The second hypothesis together with \( 0 < \eta_i < 1 \) implies that \( 1 - \sum_{i=1}^{N} C_i \eta_i^{i+1} > 0 \). Since \( 0 < \lambda_i < 1 \), there exists a nonnegative integer \( J \), such that \( A + \sum_{j=1}^{M} B_j \lambda_j^i < 0 \) for \( j < J \) and such that \( A + \sum_{j=1}^{M} B_j \lambda_j^i > 0 \) for \( j \geq J \). Without loss of generality, assume \( y_0 = 1 \). Then

\[
y(t) = \sum_{i=0}^{J-1} \left( \frac{t^i}{i!} + \sum_{j=0}^{\infty} \left( \frac{t^i}{i!} \right) \frac{t^i}{i!} \right).
\]

Obviously, the second summation term on the right-hand side of (3.2) is absolutely monotone and the first summation term is a polynomial of degree \( \leq J - 1 \). Therefore, there exists a sufficiently large \( T > 0 \) such that the dominant terms in \( d^k y(t)/dt^k \) will be given by \( d^k /dt^k \left( \sum_{i=0}^{\infty} \left( \frac{t^i}{i!} \right) \frac{t^i}{i!} \right) \) for \( k \geq 0 \) and \( t \geq T \). This implies that \( y(t) \) is eventually absolutely monotone.

The combination of the above lemmas results in the following theorem.

**THEOREM 3.1.** Assume that \( \sum_{i=1}^{N} |C_i \eta_i^{i+1}| < 1 \) and that \( y_0 \neq 0 \).

(i) If \( A > -\sum_{i=1}^{M} \min \{0, B_i\} \), then the unique solution of (2.1) is absolutely monotone.

(ii) If \( A > 0 \) and \( A + \sum_{j=1}^{M} B_j \lambda_j^i \neq 0 \), for \( j \geq 0 \), then the unique solution of (2.1) is eventually absolutely monotone.

**Proof.** The proof follows immediately from the theorems in § 2 and the lemmas in this section.

**Remark 3.1.** Suppose that \( y_0 \neq 0, \ v > 0, A > 0, A + \sum_{j=1}^{M} B_j \lambda_j^i \neq 0, \) and \( \sum_{i=1}^{N} C_i \eta_i^{i+1} \neq 1 \). From the power series expression of \( y(t) \) in (3.1), it is then easy to see that \( \lim_{t \to \infty} y(t) e^{-(A+\varepsilon)t} = 0 \) and that \( \lim_{t \to \infty} |y(t)| e^{-(A-\varepsilon)t} = 0 \). This indicates that \( y(t) \) is eventually absolutely monotone and grows like the function \( e^{\varepsilon t} \).

4. Oscillatory and unboundedness results. This section presents conditions under which solutions of (2.1) are oscillatory, or unbounded, or both.

**DEFINITION 4.1.** Let \( f(t) \) be defined on \((-\infty, \infty)\), the order \( p \) of \( f(t) \) is defined as

\[
p = \inf \{\omega: f(t) = 0(\exp(\omega t)), |t| \to \infty\}.
\]


**LEMMA 4.1.** If \( A = 0 \) and \( \sum_{i=1}^{N} C_i \eta_i^{i+1} \neq 1 \) for all integers \( j \geq 0 \), then the power series \( y(t) \) defined by (3.1) has order zero.
Proof. Let $B = \sum_{i=1}^{M} |B_i|$, $\lambda = \max \{ \lambda_i \mid i = 1, 2, \ldots, M \}$. Then

$$
|y(t)| \leq |y_0| \sum_{k=0}^{\infty} \left( \frac{k!}{\prod_{j=0}^{k-1} \left( 1 - \sum_{i=1}^{N} C_i \eta_i^{j+1} \right)} \right) \frac{B^k \lambda^{(k/2)(k-1)} \left( \prod_{j=0}^{k-1} \left( 1 - \sum_{i=1}^{N} C_i \eta_i^{j+1} \right) \right)^{-1}}{k!}.
$$

Obviously,

$$
\lim_{j \to \infty} \sum_{i=1}^{N} C_i \eta_{i+1} = 0.
$$

Hence, for any $1 > \varepsilon > 0$, there exists a $K \geq 0$, such that

$$
(1 - \varepsilon)^k < \left( \prod_{j=0}^{k-1} \left( 1 - \sum_{i=1}^{N} C_i \eta_i^{j+1} \right) \right) < (1 + \varepsilon)^k \quad \text{for} \quad k \geq K.
$$

It is known (see Titchmarsh [16, p. 253]) that the power series $\sum_{k=0}^{\infty} a_k t^k$ has a finite order $p$ if and only if

$$
\liminf_{k \to \infty} \frac{\ln (1/|a_k|)}{k \ln k} = \frac{1}{p}.
$$

In the case of $y(t)$ from (3.1), $k \geq K$ implies that

$$
\lim_{k \to \infty} \frac{\ln (1/|a_k|)}{k \ln k} = \lim_{k \to \infty} \frac{k(\ln (1 - \varepsilon) - \ln B) - (k/2)(k-1) \ln \lambda - \ln (y_0)}{k \ln k}
= \lim_{k \to \infty} \frac{-\frac{1}{2}(k-1) \ln \lambda}{\ln k} = \infty;
$$

hence, $p = 0$. \square

The following so-called Phragmén–Lindelöf principle can be found in § 8.73 of Titchmarsh [1958, p. 274].

PHRAGMÉN–LINDELÖF PRINCIPLE. Let $f(z)$ be a complex function given by $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$. Let $m(r)$ denote the minimum of $|f(z)|$ on the circle $|z| = r$. If $f(z)$ has order less than $1/2$, then there is a sequence of values of $r$ tending to infinity through which $m(r) \to \infty$.

Applying this principle to the solution of (2.1) yields the lemma below. It generalizes Corollary 2 of Feldstein and Jackiewicz [6] and Theorem 5 of Morris, Feldstein, and Bowen [15].

Lemma 4.2. If $A = 0$, $\sum_{i=1}^{N} C_i \eta_{i+1} \neq 1$, and $\sum_{i=1}^{M} B_i \lambda_i^{j} \neq 0$ for $j \geq 0$, then the solution of (2.1), and all of its derivatives, is unbounded.

Proof. Let $y(t) = \sum_{i=0}^{\infty} a_i t^i$ be the solution of (2.1). It follows from Lemma 4.1 that the order of $y(t)$ is zero. Apply the Phragmén–Lindelöf principle to $y(z)$, here $z$ is a complex variable. Then $y(z)$ is unbounded on any ray. In particular, $y(z)$ is unbounded on the real line; that is, $y(t)$ is unbounded for the real variable $t$. Let

$$
y_i(t) = \frac{d^i y(t)}{dt^i} = \sum_{l=0}^{\infty} \frac{(i+l)!}{i!} a_{i+l} t^l = \sum_{l=0}^{\infty} a_{i+l} t^l
$$

where

$$
a_i = \frac{(i+l)!}{i!} a_{i+l}.
$$
The same argument as in the proof of Lemma 4.1 yields
\[
\lim_{k \to \infty} \frac{\ln \left( \frac{1}{|a_k|} \right)}{k \ln k} \geq \lim_{k \to \infty} \frac{-\left( \ln \left| a_{i+k} \right| \right) - i \ln (i+k)}{k \ln k} = \lim_{k \to \infty} \frac{-\ln |a_{i+k}|}{k \ln k} = \lim_{k \to \infty} -\frac{1}{2} (k+i)(k+i-1) \ln \lambda \cdot k \ln k
\]

Hence the order of \( y_i(t) \) is zero for \( i = 1, 2, \ldots \). Thus, the Phragmèn–Lindelöf principle implies that the \( y_i(t) \) are all unbounded for \( i = 1, 2, \ldots \). This proves the lemma. \( \square \)

**Theorem 4.1.** In (2.1), assume that \( \sum_{i=1}^{N} \eta_i^{-1} \max \{0, C_i\} < 1 \), that \( A = 0 \), and that \( B_i < 0 \) for \( i = 1, 2, \ldots, M \). Then every nontrivial solution of (2.1) oscillates unboundedly.

**Proof.** Assume \( y(t) \neq 0 \) is a solution of (2.1) which is not oscillatory. By the linearity of (2.1), \( -y(t) \) is also a nonoscillatory solution. Therefore, without loss of generality, assume that \( y(t) \) is eventually positive. That is, assume that there exists a \( \bar{t} > 0 \) such that \( y(t) > 0 \) for \( t \geq \bar{t} \). Let
\[
r = \min_{i,j} (\lambda_i, \eta_j),
\]
\[
t_0 = \bar{t}/r.
\]
If \( t \geq t_0 \), then \( y(\lambda_i t) \) for \( i = 1, 2, \ldots, M \) and \( y(\eta_j t) \) for \( j = 1, 2, \ldots, N \) are all positive. Integrating both sides of (2.1) results in
\[
y(t) - \sum_{i=1}^{N} C_i \eta_i^{-1} y(\eta_i t) = y(t_0) - \sum_{i=1}^{N} C_i \eta_i^{-1} y(\eta_i t_0) + \sum_{i=1}^{M} B_i \lambda_i^{-1} \int_{\lambda_i t_0}^{\lambda_i t} y(s) \, ds.
\]
Since \( B_i < 0 \) and \( y(s) > 0 \) for \( s \in [\lambda_i t_0, \lambda_i t] \) and for \( i = 1, 2, \ldots, M \), it follows that
\[
y(t) - \sum_{i=1}^{N} C_i \eta_i^{-1} y(\eta_i t) < y(t_0) - \sum_{i=1}^{N} C_i \eta_i^{-1} y(\eta_i t_0) \quad \text{for all} \quad t \geq t_0.
\]
Denote
\[
g(t) = \min_{s \in [t_0, t]} y(s) \quad \text{and} \quad \eta(t) = \max \{y(\tau) | \tau \in [t_0, t] \}.
\]
Clearly, \( g(t) \) and \( y(g(t)) \) are nondecreasing, and, by Lemma 4.2,
\[
\lim_{t \to \infty} g(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} y(g(t)) = \infty.
\]
Choose \( t^* \) large enough so that \( rg(t^*) > t_0 \), and so that
\[
y(g(t^*)) > \left\{ y(t_0) + \sum_{i=1}^{M} |C_i\eta_i^{-1}|y(\eta_i t_0) \right\}\left(1 - \sum_{i=1}^{N} \eta_i^{-1} \max \{0, C_i\} \right)^{-1}.
\]
Then for this \( t^* \),
\[
y(g(t^*)) - \sum_{i=1}^{N} C_i \eta_i^{-1} y(\eta_i g(t^*)) \geq y(g(t^*)) - \sum_{i=1}^{N} \max \{0, C_i\} \eta_i^{-1} y(g(t^*))
\]
\[
y(g(t^*)) \left(1 - \sum_{i=1}^{N} \eta_i^{-1} \max \{0, C_i\} \right)
\]
\[
y(t_0) + \sum_{i=1}^{N} |C_i\eta_i^{-1}| y(\eta_i t_0).
\]
Obviously, (4.9) contradicts (4.7); this contradiction implies that \( y(t) \) must be oscillatory. By Lemma 4.2, \( y(t) \) in fact oscillates unboundedly as \( t \to +\infty \). □

Theorem 4.1 clearly implies the following corollary.

**Corollary 4.1.** If the coefficients in (2.1) satisfy \( A = 0, B_i < 0, \) and \( C_i < 0 \), then every nontrivial solution oscillates unboundedly.

**Remark 4.1.** Note here that \( |C| \) need not necessarily be less than 1. This indicates that Theorem 4.1 exceeds the expectation of Feldstein and Jackiewicz [6] in their conjecture. The proof of Theorem 4.1 for the case \( N = M = 1, A = 0, \) and \( B < 0 \) shows that a smaller upper bound for \( C \) may be required in order to make nontrivial solutions of (2.1) oscillate. In Theorem 4.1, this bound is \( \eta \), which is smaller than the one suggested in the conjecture in Feldstein and Jackiewicz [6]. This is also in agreement with the assumptions in Theorems 2.1 and 2.2.

5. Discussion. This paper responds to the conjecture recently proposed by Feldstein and Jackiewicz [6] based on their numerical experiments. It turns out that most parts of their conjecture are indeed true, and in fact, much more has been proved in the last two sections. These results are relevant to the work of Morris, Feldstein, and Bowen [15], and those in Fox et al. [8] and Kato and McLeod [13].

When \( A \neq 0 \), the conclusion to Lemma 4.2 may no longer be true. That is why Theorem 4.1 assumes \( A = 0 \), although the proof of its conclusion seems very promising when \( A < 0 \). This certainly raises an interesting question to be answered. The other problems remaining to be investigated are:

1. Can an existence and uniqueness theorem similar to Theorem 2.1 be established in the case \( \sum_{i=1}^{N} |C_i \eta_i^{-1}| \geq 1 \)? If this can be resolved affirmatively, then the power series solution in (3.1) is the unique solution of (2.1). In that case, the assumption \( \sum_{i=1}^{N} |C_i \eta_i^{-1}| < 1 \) could be deleted from Theorem 3.1.

2. If \( A < 0 \) and \( \sum_{i=1}^{M} B_i > 0 \), then what are the conditions for which every nontrivial solution of (2.1) oscillates?

3. What can be said if the delay functions \( \lambda_i t \) and \( \eta_i t \) are replaced by \( \lambda_i (t - \tau_i) \) and by \( \eta_i(t - \sigma_j) \), where \( \tau_i \geq 0 \) and \( \sigma_j \geq 0 \)? When \( \lambda_i = \eta_i = 1 \), various results have been recently obtained by Grammatikopoulos, Grove, Ladas, and Meimaridou [10], [11] and Freedman and Kuang [7]. The method developed in Cooke and Grossman [3] may contribute to the discussion of this problem.

4. It may be interesting to consider the case where the coefficients in (2.1) are matrix rather than scalar quantities. Aspects of such problems have been considered by Waltman [17] and Bélair [1] for the case \( C_j = 0, M = 1, \) and the order of the matrix is 2.

5. Nonautonomous and nonlinear versions of (2.1) can also be investigated.

**References**


