Solutions and Properties of Some Degenerate Systems of Difference Equations

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ABSTRACT
This paper is devoted to obtain the form of the solution and the qualitative properties of the following systems of a rational difference equations of order two

\[
\begin{align*}
x_{n+1} &= \frac{y_n y_{n-1}}{x_n (\pm 1 \pm y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n (\pm 1 \pm x_n x_{n-1})},
\end{align*}
\]

with positive initial conditions \(x_0, x_0, y_0\) and \(y_0\) are nonzero real numbers. If we let \(u_n = x_n x_{n-1}\) and \(v_n = y_n y_{n-1}\), then these systems can be viewed as special cases of the system of the form

\[
\begin{align*}
u_{n+1} &= f(v_n),
\end{align*}
\]

This system has applications in modeling population growth with age structure or the dynamics of plant-herbivore interaction. Let \(w_n = u_2 n\), we have \(w_{n+1} = f(g(w_n)) \equiv h(w_n)\). At a nonzero steady state \(w^*\) of the last difference equation, we have

\[
|h'(w^*)| = |f'(g(w^*))g'(w^*)| = 1,
\]
indicating that the system is degenerate at this steady state.

Keywords: difference equations, recursive sequences, stability, periodic solution, system of difference equations.
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1. INTRODUCTION
Owing to their rich dynamics, interest and scope in studying the solutions and properties of nonlinear difference equation systems is continuously expanding. In particular, there is a growing need of practical methods that explore and discuss a real life matters described by mathematical models. Such applications we find in environment as biology, genetics and economy [1, 2, 14].

There are some well documented and focused studies deal with some specific nonlinear difference equations system. For example, the periodicity of the positive solutions of the rational difference equations systems

\[
\begin{align*}
x_{n+1} &= \frac{1}{x_n},
\end{align*}
\]

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has been obtained by Cinar in [3]. The behavior of positive solutions of the following system
\[ x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}. \]
has been studied by Kurbanli et al. [4]. In [5], Ozban studied the positive solutions of the system of rational difference equations
\[ x_{n+1} = \frac{a}{y_{n-3}}, \quad y_{n+1} = \frac{b_{n-3}}{x_{n-q}y_{n-q}}. \]
Touafek et al. [6] studied the periodicity and gave the form of the solutions of the following systems
\[ x_{n+1} = \frac{y_n}{x_{n-1}(1 + x_n y_{n-1})}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(1 + x_n y_{n-1})}. \]
Other similar difference equations and nonlinear systems of rational difference equations were investigated see [7]-[14].

In this paper, we deal with the existence and properties of solutions and the periodicity character of the following systems of rational difference equations with order two
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( \pm 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( \pm 1 \pm x_n x_{n-1} \right)}, \]
with nonnegative initial conditions \( x_0, y_0 \) and \( y_n \). If we let \( u_n = x_n x_{n-1} \) and \( v_n = y_n y_{n-1} \), then these systems can be viewed as special cases of the system of the form
\[ u_{n+1} = f(v_n), \quad v_{n+1} = g(u_n). \]
This system has applications in modeling population growth with age structure [2] or the dynamics of plant-herbivore interaction [14]. Let \( w_n = u_{2n} \), we have \( w_{n+1} = f(g(w_n)) \equiv h(w_n) \). At a nonzero steady state \( w^* \) of the last difference equation, we have
\[ |h''(w^*)| = |f'(g(w^*))g'(w^*)| = 1, \]
indicating that the system is degenerate at this steady state.

2. SYSTEM \( X_{n+1} = \frac{Y_N Y_{N-1}}{X_N (1 + Y_N Y_{N-1})}, \quad Y_{N+1} = \frac{X_N X_{N-1}}{Y_N (1 + X_N X_{N-1})} \)

In this section, our main goal is to obtain the solutions of the following second order system of difference equations
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( 1 + y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 + x_n x_{n-1} \right)}, \tag{1} \]
where \( n = 0, 1, 2, \ldots \), and the initial conditions are nonnegative real numbers. Before embarking on our lengthy derivation of the solutions, we would like to present some simple but interesting properties of these solutions.

If we let
\[ u_n = x_n x_{n-1}, \quad v_n = y_n y_{n-1}, \]
then
\[ u_{n+1} = \frac{u_n}{1 + v_n}, \quad v_{n+1} = \frac{v_n}{1 + u_n} \]
and
\[ u_{n+2} = \frac{u_n}{1 + v_n} = \frac{u_n (1 + u_n)}{1 + u_n^2}, \quad \frac{u_{n+1}}{1 + u_{n+1}} = \frac{u_{n+1}}{1 + 2u_n} = f(u_n), \]
\[ v_{n+2} = \frac{v_n}{1 + u_n} = \frac{v_n (1 + v_n)}{1 + v_n^2}, \quad \frac{v_{n+1}}{1 + v_{n+1}} = \frac{v_{n+1}}{1 + 2v_n} = f(v_n). \]
From which we see that if \( u_0 \geq 0 \) (\( u_{-1} \geq 0 \)), then \( u_2n \geq 0 \) (\( u_{2n+1} \geq 0 \)) for nonnegative integers \( n \). This system has \((0, 0)\) as the only steady state. Observe that
\[ u_{n+2} - u_n = -\frac{2u_n^2}{1 + 2u_n}. \]
Figure 1. A solution for the difference system (1) with the initial conditions \( x_{-1} = 5, \ x_0 = 4, \ y_{-1} = 3 \) and \( y_0 = 8 \).

Figure 2. A solution for the difference system (1) with the initial conditions \( x_{-1} = 0.2, \ x_0 = 0.5, \ y_{-1} = 0.6 \) and \( y_0 = 0.3 \).

Hence, if \( u_0 > 0 \ (u_{-1} > 0) \), then \( u_n \ (u_{2n+1}) \) is a strictly decreasing subsequence and hence must approach the only steady state value 0. Similar argument can be made for \( v_n \). Therefore

\[ v_n \to 0 \ 	ext{or} \ x_n x_{n-1} \to 0 \ 	ext{and} \ u_n \to 0 \ 	ext{or} \ y_n y_{n-1} \to 0. \]

In Figures 1 and 2, we present two typical solutions for the difference system (1). Observe that \( x_{n+1} x_n \to 0 \) and \( y_{n+1} y_n \to 0 \).

**Theorem 2.1.** Assume that \( \{x_n, y_n\} \) are solutions of system (1). Then for \( n = 0, 1, 2, \ldots \), we see that all solutions of system (1) are given by the following formula

\[
x_{2n-1} = \frac{x_n y_n}{e^{x_n+y_n}} \prod_{i=0}^{n-1} \frac{(1+(2i)ab)}{(1+(2i+1)cd)}, \quad x_{2n} = \frac{x_n y_n}{e^{x_n+y_n}} \prod_{i=0}^{n-1} \frac{(1+(2i+1)cd)}{(1+(2i+1)ab)}.
\]

and

\[
y_{2n-1} = \frac{a^n b^n}{e^{a^n+b^n}} \prod_{i=0}^{n-1} \frac{(1+(2i)cd)}{(1+(2i+1)ab)}, \quad y_{2n} = \frac{a^n b^n}{e^{a^n+b^n}} \prod_{i=0}^{n-1} \frac{(1+(2i+1)ab)}{(1+(2i+2)cd)}.
\]
where \( \prod_{i=0}^{-1} A_i \equiv 1 \), \( x_{-1} = b, x_0 = a, y_{-1} = d \) and \( y_0 = c \).

**Proof:** We prove it by the method of induction, for \( n = 0 \) the result holds. Assume the theorem is true for \( n - 1 \), that is,

\[
\begin{align*}
x_{2n-3} &= \frac{e^{n-1}a^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+1)c \right)}{e^{n-1}a^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+1)c \right)}, \quad x_{2n-2} = \frac{a^n b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+1)c \right)}{a^n b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+1)c \right)}, \\
y_{2n-3} &= \frac{e^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i)c \right)}{e^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i)c \right)}, \quad y_{2n-2} = \frac{e^n d^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{e^n d^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)},
\end{align*}
\]

are true. We will show that the relations given in the above theorem are true.

From Eq.(1) we see that

\[
\begin{align*}
x_{2n-1} &= \frac{y_{2n-2} y_{2n-3}}{x_{2n-2}} \frac{a^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{a^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)} \\
&= \frac{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)} \\
&= e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right) \\
&= \frac{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)},
\end{align*}
\]

\[
\begin{align*}
y_{2n-1} &= \frac{y_{2n-2} y_{2n-3}}{x_{2n-2}} \frac{a^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{a^{n-1}b^{n-1} \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)} \\
&= \frac{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)} \\
&= e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right) \\
&= \frac{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}{e^n d^n \prod_{i=0}^{n-2} \left( 1 + (2i+2)c \right)}.
\end{align*}
\]
Also, similarly from Eq.(1), we have
\[ x_{2n} = \frac{y_{2n-1} y_{2n-2}}{x_{2n-1} (1+ y_{2n-1} y_{2n-2})} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1+(2i+1)ab}{1+(2i+1)cd} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{(1+2i+1)ab}{(1+2i+1)cd} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{(1+2i+1)ab}{(1+2i+1)cd} \]
\[ y_{2n} = \frac{x_{2n-1} x_{2n-2}}{y_{2n-1} (1+ x_{2n-1} x_{2n-2})} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1+(2i+1)ab}{1+(2i+1)cd} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{(1+2i+1)ab}{(1+2i+1)cd} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{(1+2i+1)ab}{(1+2i+1)cd} \]

The proof is complete.

The following case can be treated similarly

**Theorem 2.2.** The solutions of the system
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n (1-y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n (1-x_n x_{n-1})}, \]
are given by
\[ x_{2n-1} = \frac{c^n d^n}{a^n b^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)ab}{1-(2i+1)cd}, \quad x_{2n} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)cd}{1-(2i+1)ab}, \]
\[ y_{2n-1} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)cd}{1-(2i+1)ab}, \quad y_{2n} = \frac{c^n d^n}{a^n b^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)ab}{1-(2i+1)cd}. \]

**Theorem 2.3.** Assume that \((x_n, y_n)\) are solutions of the system
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n (1-y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n (1-x_n x_{n-1})}. \]
Then for \(n = 0, 1, 2, ..., \)
\[ x_{2n-1} = \frac{c^n d^n}{a^n b^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)ab}{1-(2i+1)cd}, \quad x_{2n} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)cd}{1-(2i+1)ab}, \]
\[ y_{2n-1} = \frac{a^n b^n}{c^n d^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)cd}{1-(2i+1)ab}, \quad y_{2n} = \frac{c^n d^n}{a^n b^n} \prod_{i=0}^{n-1} \frac{1-(2i+1)ab}{1-(2i+1)cd}. \]

**Theorem 2.4.** The system
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n (1-y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n (1-x_n x_{n-1})}, \]
has the solutions which given by
\[ x_{2n-1} = \frac{e^{a_n}a^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{1-(12i+1)b_1}{1-(12i+1)c_1}, \quad x_{2n} = \frac{e^{a_n}b^n}{a^n} \prod_{i=0}^{n-1} \frac{1-(12i+1)c_1}{1-(12i+1)b_1}, \]
\[ y_{2n-1} = \frac{e^{a_n}b^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{1-(12i+1)d_1}{1-(12i+1)b_1}, \quad y_{2n} = \frac{e^{a_n}b^n}{a^n} \prod_{i=0}^{n-1} \frac{1-(12i+1)b_1}{1-(12i+1)d_1}. \]

3. SYSTEM \( X_{n+1} = \frac{Y_N Y_{n-1}}{Y_N (1 + Y_N Y_{n-1})}, \quad Y_{n+1} = \frac{X_N Y_{n-1}}{Y_N (1 + X_N Y_{n-1})} \)

In this section, our main goal is to obtain the solutions of the following second order system of difference equations
\[ x_{n+1} = \frac{y_n y_{n-1}}{x_n (1 + y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n (1 + x_n x_{n-1})}, \tag{2} \]
where \( n = 0, 1, 2, \ldots \), and the initial conditions are nonnegative real numbers with \( x_{-1} x_0 \neq 1, \neq \frac{1}{2}, \) and \( y_{-1} y_0 \neq 1. \)

If we let
\[ u_n = x_n x_{n-1}, \quad v_n = y_n y_{n-1} \]
then
\[ u_{n+1} = \frac{v_n}{1 + v_n}, \quad v_{n+1} = \frac{u_n}{1 + u_n}, \]
and
\[ u_{n+2} = \frac{v_{n+1}}{1 + v_{n+1}} = \frac{u_n}{1 + u_n} = \frac{u_n}{-1 + u_n} = f(u_n), \]
\[ v_{n+2} = \frac{u_{n+1}}{1 + u_{n+1}} = \frac{v_n}{1 + v_n} = \frac{v_n}{-1 + v_n} = -v_n. \]

This suggests that the \( y_n \) will alternate signs at least every 3 units of time. The system has two steady states \((0, 0)\) and \((1, 1)\). From the fact that \( v_{n+2} = -v_n \), we have
\[ \frac{y_{n+2}}{y_n} = \frac{-y_{n-1}}{y_{n-1}} = \frac{y_n}{y_{n-2}} \]
Let \( r = \frac{y_2}{y_0} \), then we have
\[ y_{2n} = (-1)^{(n-1)} \frac{y_n}{y_{2n-2}} \frac{y_{2n-2}}{y_0} = r^n \]
and hence \( y_{2n} = y_0 r^n \). Similarly, we have and \( y_{2n+1} = (-1)^n y_{-1} a^n \) where \( a = y_1/y_{-1} \). This shows that the values of the highs and lows grow or decay exponentially.

Observe that
\[ \frac{dg(0)}{dv_n} = \left. \frac{-1+2v_n-2v_n}{(-1+2v_n)^2} \right|_{v_n=0} = -1 \]
and
\[ \frac{dg(1)}{dv_n} = \left. \frac{-1+2v_n-2v_n}{(-1+2v_n)^2} \right|_{v_n=0} = -1. \]

This indicates that both are degenerate steady states. Figure 3 depicts a typical solution of the difference equations system (2).

**Theorem 3.1.** Let \( \{x_n, y_n\}_{n=1}^{\infty} \) be solutions of system (2). Then \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) are given by the following formula for \( n = 0, 1, 2, \ldots, \)
\[ x_{2n} = \frac{e^{a_n}a^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{1-(12i+1)b_1}{1-(12i+1)c_1}, \quad x_{2n+1} = \frac{e^{a_n}b^n}{a^n} \prod_{i=0}^{n-1} \frac{1-(12i+1)c_1}{1-(12i+1)b_1}, \]
\[ x_{4n} = \frac{e^{a_n}a^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{1-(12i+1)b_1}{1-(12i+1)c_1}, \quad x_{4n+1} = \frac{e^{a_n}b^n}{a^n} \prod_{i=0}^{n-1} \frac{1-(12i+1)c_1}{1-(12i+1)b_1}, \]
\[ x_{4n+2} = \frac{e^{a_n}a^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{1-(12i+1)b_1}{1-(12i+1)c_1}, \quad x_{4n+3} = \frac{e^{a_n}b^n}{a^n} \prod_{i=0}^{n-1} \frac{1-(12i+1)c_1}{1-(12i+1)b_1}. \]
Also, we can prove the other relations. This completes the proof.

and

\[ y_{4n-1} = \frac{a^{2n}b^{2n}}{a^{2n}+b^{2n}(-1+ab)^{2n}}, \quad y_{4n} = \frac{a^{2n+1}d^{2n}(-1+ab)^{2n}}{a^{2n}+b^{2n}}, \]

\[ y_{4n+1} = \frac{a^{2n+1}d^{2n+1}(-1+ab)^{2n+1}}{a^{2n}+b^{2n}}, \quad y_{4n+2} = \frac{a^{2n+2}d^{2n+1}(-1+ab)^{2n+1}}{a^{2n}+b^{2n}}. \]

**Proof:** For \( n = 0 \), the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \). That is,

\[ x_{4n-3} = \frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}, \quad x_{4n-2} = \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}, \]

\[ x_{4n-3} = \frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}, \quad x_{4n-2} = \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}. \]

Now it follows from Eq.(2) that

\[ x_{4n-1} = \frac{y_{4n-2}(x_{4n-4}+y_{4n-3})}{x_{4n-2}(1+y_{4n-3})}, \]

\[ x_{4n-1} = \frac{\frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}} \cdot \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}}{\frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}} \cdot \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}}, \]

\[ x_{4n-1} = \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}} \cdot \frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}. \]

and

\[ y_{4n-1} = \frac{y_{4n-2}(x_{4n-4}+y_{4n-3})}{y_{4n-2}(1+y_{4n-3})}, \]

\[ y_{4n-1} = \frac{\frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}} \cdot \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}}{\frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}} \cdot \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}}, \]

\[ y_{4n-1} = \frac{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}}{a^{2n}b^{2n-1}(1+cd)(-1+c^2d^2)^{n-1}} \cdot \frac{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}{a^{2n-1}b^{2n-1}(-1+2ab)^{n-1}}. \]

Also, we can prove the other relations. This completes the proof.

**Figure 3.** A typical solution of the difference equations system (2). The initial conditions are \( x_{-1} = 4.2, \ x_0 = 0.5, \ y_{-1} = -1.6 \) and \( y_0 = 1.3 \). Notice that the \( y_n \) alternates signs at least every 3 units of time, and the values of the highs and lows grow or decay exponentially.
We consider the following systems and the proof of the theorems are similar to above theorem and so, left to the reader.

\begin{align*}
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{3} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{4} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{5} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{6} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{7} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{8} \\
X_{n+1} &= \frac{y_n y_{n-1}}{x_n (1 - y_n y_{n-1})}, \quad Y_{n+1} = \frac{x_n y_{n-1}}{y_n (1 - x_n y_{n-1})} \tag{9}
\end{align*}

We will devote, for example, in the following theorems the form of the solutions of systems (3) and (8).

**Theorem 3.2.** Let \( \{x_n, y_n\}_{n=0}^{\infty} \) be solutions of system (3) and \( x_{-1} \neq -1, \neq -\frac{1}{2}, y_{-1} \neq 1 \). Then for \( n = 0, 1, 2, \ldots \),

\begin{align*}
X_{4n-1} &= \frac{c^{2n} d^{2n} (-1 - 2ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad X_{4n} = \frac{a^{2n+4} b^{2n} (-1 + c^{2d} d^{2n})}{c^{2n} d^{2n} (1 - 2ab)}, \\
X_{4n+1} &= \frac{c^{2n+4} d^{2n+4} (1 - 2ab)^n}{a^{2n+4} b^{2n+4} (1 - 1 + c^{2d} d^{2n+4})}, \quad X_{4n+2} = \frac{a^{2n+2} b^{2n+2} (1 - c^{2d} d^{2n+4})}{c^{2n+2} d^{2n+2} (1 - 2ab)}, \\
Y_{4n-1} &= \frac{c^{2n} b^{2n} (1 + ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad Y_{4n} = \frac{a^{2n+2} b^{2n} (1 + c^{2d} d^{2n})}{c^{2n+2} b^{2n} (1 + 2ab)^n}, \\
Y_{4n+1} &= \frac{c^{2n+4} b^{2n+4} (1 + 2ab)^n}{a^{2n+4} b^{2n+4} (1 - 1 + c^{2d} d^{2n+4})}, \quad Y_{4n+2} = \frac{a^{2n+4} b^{2n+4} (1 + c^{2d} d^{2n+4})}{c^{2n+4} b^{2n+4} (1 + 2ab)^n}.
\end{align*}

**Theorem 3.3.** Assume that \( \{x_n, y_n\} \) are solutions of system (4) with \( x_{-1} \neq -1, \neq -\frac{1}{2}, \) and \( y_{-1} \neq 1 \). Then for \( n = 0, 1, 2, \ldots \),

\begin{align*}
X_{4n-1} &= \frac{c^{2n} d^{2n} (-1 - 2ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad X_{4n} = \frac{a^{2n+1} b^{2n} (-1 + c^{2d} d^{2n})}{c^{2n} d^{2n} (1 - 2ab)}, \\
X_{4n+1} &= \frac{c^{2n+1} d^{2n+1} (1 - 2ab)^n}{a^{2n+1} b^{2n+1} (1 - 1 + c^{2d} d^{2n+1})}, \quad X_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 - c^{2d} d^{2n+1})}{c^{2n+3} d^{2n+3} (1 - 2ab)}, \\
Y_{4n-1} &= \frac{c^{2n} b^{2n} (1 + ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad Y_{4n} = \frac{a^{2n+2} b^{2n} (1 + c^{2d} d^{2n})}{c^{2n+2} b^{2n} (1 + 2ab)^n}, \\
Y_{4n+1} &= \frac{c^{2n+1} b^{2n+1} (1 + 2ab)^n}{a^{2n+1} b^{2n+1} (1 - 1 + c^{2d} d^{2n+1})}, \quad Y_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 + c^{2d} d^{2n+1})}{c^{2n+3} b^{2n+3} (1 + 2ab)^n}.
\end{align*}

**Theorem 3.4.** Suppose that \( \{x_n, y_n\} \) are solutions of system (5) such that \( x_{-1} \neq -1, \neq -\frac{1}{2}, \) and \( y_{-1} \neq 1 \). Then for \( n = 0, 1, 2, \ldots \),

\begin{align*}
X_{4n-1} &= \frac{c^{2n} d^{2n} (-1 - 2ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad X_{4n} = \frac{a^{2n+1} b^{2n} (-1 + c^{2d} d^{2n})}{c^{2n} d^{2n} (1 - 2ab)}, \\
X_{4n+1} &= \frac{c^{2n+1} d^{2n+1} (1 - 2ab)^n}{a^{2n+1} b^{2n+1} (1 - 1 + c^{2d} d^{2n+1})}, \quad X_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 - c^{2d} d^{2n+1})}{c^{2n+3} d^{2n+3} (1 - 2ab)}, \\
Y_{4n-1} &= \frac{c^{2n} b^{2n} (1 + ab)^n}{a^{2n} b^{2n} (1 - 1 + c^{2d} d^{2n})}, \quad Y_{4n} = \frac{a^{2n+2} b^{2n} (1 + c^{2d} d^{2n})}{c^{2n+2} b^{2n} (1 + 2ab)^n}, \\
Y_{4n+1} &= \frac{c^{2n+1} b^{2n+1} (1 + 2ab)^n}{a^{2n+1} b^{2n+1} (1 - 1 + c^{2d} d^{2n+1})}, \quad Y_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 + c^{2d} d^{2n+1})}{c^{2n+3} b^{2n+3} (1 + 2ab)^n}.
\end{align*}

**Theorem 3.5.** If \( \{x_n, y_n\} \) are solutions of system (6) and \( x_{-1} \neq -1, \) and \( y_{-1} \neq -1, \neq -\frac{1}{2} \). Then the solutions are given by

\begin{align*}
X_{4n-1} &= \frac{c^{2n} d^{2n}}{a^{2n} b^{2n} (1 + ab)^n}, \quad X_{4n} = \frac{a^{2n+1} b^{2n} (1 + cd)^{2n}}{c^{2n} d^{2n}}, \\
X_{4n+1} &= \frac{c^{2n+1} d^{2n+1}}{a^{2n+1} b^{2n+1} (1 + ab)^n}, \quad X_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 + cd)^{2n+1}}{c^{2n+3} d^{2n+3}}, \\
Y_{4n-1} &= \frac{c^{2n} b^{2n}}{a^{2n} b^{2n} (1 - 2cd)^n}, \quad Y_{4n} = \frac{a^{2n+2} b^{2n} (1 + cd)^n}{c^{2n+2} b^{2n}}, \\
Y_{4n+1} &= \frac{c^{2n+1} b^{2n+1}}{a^{2n+1} b^{2n+1} (1 - 2cd)^n}, \quad Y_{4n+2} = \frac{a^{2n+3} b^{2n+3} (1 + cd)^n}{c^{2n+3} b^{2n+3} (1 - 2cd)^{n+1}}.
\end{align*}
Theorem 3.6. The solutions of the system (7) with non zero initial conditions real numbers with \( x_{-1}x_0 \neq \pm 1, \) and \( y_{-1}y_0 \neq \pm \frac{1}{2} \) are given by

\[
\begin{align*}
x_{4n-1} &= \frac{a^{2n/3}a^n}{2^{a^{2n+1}}}, \quad x_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
-x_{4n+1} &= \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad x_{4n+2} = \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n-1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n+1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n+2} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}.
\end{align*}
\]

Theorem 3.7. Assume that \( \{x_n, y_n\} \) are solutions of system (8) and \( x_{-1}x_0 \neq \pm 1, y_{-1}y_0 \neq \pm \frac{1}{2}, \) then

\[
\begin{align*}
x_{4n-1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad x_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
-x_{4n+1} &= \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad x_{4n+2} = \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n-1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n+1} &= \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n+2} = \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}.
\end{align*}
\]

Theorem 3.8. Suppose that \( \{x_n, y_n\} \) are solutions of system (9) with \( x_{-1}x_0 \neq \pm 1, \) and \( y_{-1}y_0 \neq -1, \neq -\frac{1}{2}, \) then the solutions of system (9) are given by

\[
\begin{align*}
x_{4n-1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad x_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
-x_{4n+1} &= \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad x_{4n+2} = \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n-1} &= \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n} = \frac{a^{2n+1}2^{a^n}(-1+2d)2^n}{c_{x,a,d}2^n}, \\
y_{4n+1} &= \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}, \quad y_{4n+2} = \frac{a^{2n+1}2^{a^n+1}(-1+2d)2^n}{c_{x,a,d}2^n}.
\end{align*}
\]

4. SYSTEM \( X_{N+1} = \frac{Y_NY_{N-1}}{X_N(1+Y_NY_{N-1})}, \quad Y_{N+1} = \frac{X_NX_{N-1}}{Y_N(1+X_NX_{N-1})} \)

In this section, our main goal is to obtain the solutions of the following second order system of difference equations

\[
x_{n+1} = \frac{y_nx_{n-1}}{x_n(1+y_ny_{n-1})}, \quad y_{n+1} = \frac{x_nx_{n-1}}{y_n(1-x_nx_{n-1})},
\]

where \( n = 0, 1, 2, \ldots \) and the initial conditions \( x_{-1}, x_0, \) \( y_{-1} \) and \( y_0 \) are arbitrary nonzero real numbers with \( x_{-1}x_0 \neq 1, \) and \( y_{-1}y_0 \neq -1. \)

From (10), if we take

\[
\begin{align*}
u_n &= x_nx_{n-1}, \quad v_n = y_ny_{n-1}, \\
\end{align*}
\]

then \( u_n \) for \( u_n \) and

\[
\begin{align*}
u_{n+2} &= \frac{u_{n+1}}{1+v_{n+1}} = \frac{u_{n+1}}{1+u_n/(1-u_n)^{1/2}} = u_n, \\
v_{n+2} &= \frac{u_{n+1}}{1-v_{n+1}} = \frac{u_{n+1}}{1-v_n/(1+v_n)^{1/2}} = v_n.
\end{align*}
\]
It is easy to see that if initial values are positive and such that \( u_0 < 1 \), then \( u_n < 1 \) for all positive integers and hence such a solution stays positive. In addition, from the property \( u_{n+2} = u_n \), we see that

\[
\frac{x_{n+2}}{x_n} = \frac{x_{n-1}}{x_{n-2}}
\]

Let \( r = \frac{x_2}{x_0} \), then we have \( x_{2n} = x_2 x_{2n-2} \) and \( r^n = r^n \) and hence \( x_{2n} = x_0 r^n \). Similarly, we have and \( x_{2n-1} = x_{-1} a^n \) where \( a = x_1/x_{-1} \). This shows that the values of the highs and lows grow and decay exponentially. Figure 4 shows the behavior of a typical solution of the difference equation system (10).

**Theorem 4.1.** If \( \{x_n, y_n\} \) are solutions of difference equation system (10), then for \( n = 0, 1, 2, \ldots \),

\[
\begin{align*}
x_{4n-1} &= \frac{a^{2n-1}d^{n-1}}{a^{2n-1}d^{n-1} + (1+cd)^n}, & x_{4n} &= \frac{a^{2n+1}d^{n}(1+cd)^{2n}}{e^n d^{2n}}, \\
x_{4n+1} &= \frac{a^{2n+1}d^{n+1}}{a^{2n+1}d^{n+1} + (1+cd)^{2n+1}}, & x_{4n+2} &= \frac{a^{2n+2}d^{2n+1}(1+cd)^{2n+1}}{e^n d^{2n+2}}, \\
y_{4n-1} &= \frac{e^n d^{2n}}{e^n d^{2n} + (1-ab)^n}, & y_{4n} &= \frac{e^n d^{2n} + (1-ab)^n}{a^n d^{2n}}, \\
y_{4n+1} &= \frac{a^{2n+1}d^{n+1} + (1-ab)^n}{e^n d^{2n+2}}, & y_{4n+2} &= \frac{a^{2n+2}d^{2n+1} + (1-ab)^{2n+1}}{a^n d^{2n+2}}.
\end{align*}
\]

**Proof:** For \( n = 0 \), the result holds. Now, assume that \( n > 1 \) and that our assumption holds for \( n - 1 \). That is,

\[
\begin{align*}
x_{4n-3} &= \frac{e^{n-1}}{e^{n-1} + (1+cd)^{n-1}}, & x_{4n-2} &= \frac{a^{2n}d^{2n-1} + (1+cd)^{n-1}}{e^{n-1}d^{2n}}, \\
y_{4n-3} &= \frac{e^{n-1}d^{2n}}{e^{n-1}d^{2n} + (1-ab)^{n-1}}, & y_{4n-2} &= \frac{a^{2n}d^{2n-1} + (1-ab)^{n-1}}{e^{n-1}d^{2n}}.
\end{align*}
\]

It follows from Eq.(10) that

\[
\begin{align*}
x_{4n-1} &= \frac{y_{4n-2}x_{4n-3}}{x_{4n-2} + y_{4n-2}x_{4n-3}} \\
&= \frac{a^{2n-1}d^{2n-1} + (1-ab)^{n-1}}{e^{n-1}d^{2n}} \left( \frac{a^{2n-1}d^{2n-1} + (1-ab)^{n-1}}{e^{n-1}d^{2n}} \right)^{-1} \\
&= \frac{a^{2n}d^{2n} + (1+cd)^{n-1}}{a^{2n}d^{2n} + (1+cd)^{n-1}}.
\end{align*}
\]
Thus we have a periodic solution of period two and the proof is complete.

Theorem 4.2. System (10) has a periodic solution of period two if and only if \( x_{-1}x_0 = 2 \), \( y_0y_{-1} = -2 \), and will be taken the form \( \{x_n\} = \{b, a, b, a, \ldots \} \), \( \{y_n\} = \{d, c, d, c, \ldots \} \).

Proof: First suppose that there exists a prime period two solution

\[
\{x_n\} = \{b, a, b, a, \ldots \}, \quad \{y_n\} = \{d, c, d, c, \ldots \},
\]

of system (10), we see from the form of the solution of system (10) that

\[
b = \frac{a^{2n}b^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}}, \quad a = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}}.
\]

Then we get

\[
(ab)^{2n} = (cd)^{2n} \quad \text{and} \quad 1 - ab = -1, \quad 1 + cd = -1.
\]

Thus

\[
ab = 2, \quad cd = -2.
\]

Second assume that \( ab = 2, \ cd = -2 \). Then we see from the form of the solution of system (10) that

\[
x_{4n-1} = b, \quad x_{4n} = a, \quad x_{4n+1} = b, \quad x_{4n+2} = a,
\]

\[
y_{4n-1} = d, \quad y_{4n} = c, \quad y_{4n+1} = d, \quad y_{4n+2} = c.
\]

Thus we have a periodic solution of period two and the proof is complete.

In a similar fashion, we can obtain the following theorems.

Theorem 4.3. If \( \{x_n, y_n\} \) are solutions of the following difference equation system

\[
x_{n+1} = \frac{y_ny_{n-1}}{x_n(1-y_ny_{n-1})}, \quad y_{n+1} = \frac{x_nx_{n-1}}{y_n(1+x_nx_{n-1})},
\]

(11)

where the initial conditions \( x_{-1}, x_0, y_{-1} \) and \( y_0 \) are arbitrary nonzero real numbers with \( x_{-1}x_0 \neq -1, \ y_0y_{-1} \neq 1 \). Then for \( n = 0, 1, 2, \ldots \),

\[
x_{4n-1} = \frac{a^{2n}b^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}}, \quad x_{4n} = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}},
\]

\[
x_{4n+1} = \frac{a^{2n+1}b^{2n+1}}{a^{2n+1}b^{2n+1} - (1+cd)^{2n+1}}, \quad x_{4n+2} = \frac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{a^{2n+1}b^{2n+1} - (1+cd)^{2n+1}},
\]

\[
y_{4n-1} = \frac{a^{2n}b^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}}, \quad y_{4n} = \frac{a^{2n+1}b^{2n}(1+cd)^{2n}}{a^{2n}b^{2n} - (1+cd)^{2n}},
\]

\[
y_{4n+1} = \frac{a^{2n+1}b^{2n+1}}{a^{2n+1}b^{2n+1} - (1+cd)^{2n+1}}, \quad y_{4n+2} = \frac{a^{2n+2}b^{2n+1}(1+cd)^{2n+1}}{a^{2n+1}b^{2n+1} - (1+cd)^{2n+1}},
\]

and all these solutions are unbounded except if \( x_{-1}x_0 = -2, \ y_0y_{-1} = 2 \), then the system (11) has a periodic solution of period two in the form \( \{x_n\} = \{b, a, b, a, \ldots \} \), \( \{y_n\} = \{d, c, d, c, \ldots \} \).
Theorem 4.4. The solutions of the following two systems of difference equations

\[
\begin{align*}
x_{n+1} &= \frac{y_n y_{n-1}}{x_n \left( 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 \pm x_n x_{n-1} \right)}, \\
x_{4n-1} &= a^{2n} d^{2n} e^{-\left( 1 \pm cd \right) 2n}, \quad x_{4n} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm cd \right)(2n+1)}, \\
x_{4n+1} &= a^{2n+1} d^{2n+1} e^{-\left( 1 \pm cd \right)(2n+1)}, \quad x_{4n+2} = a^{2n+2} d^{2n+2} e^{-\left( 1 \pm cd \right)(2n+1)}, \\
y_{4n-1} &= a^{2n} d^{2n} e^{-\left( 1 \pm ab \right) 2n}, \quad y_{4n} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm ab \right)(2n+1)}, \\
y_{4n+1} &= a^{2n+1} d^{2n+1} e^{-\left( 1 \pm ab \right)(2n+1)}, \quad y_{4n+2} = a^{2n+2} d^{2n+2} e^{-\left( 1 \pm ab \right)(2n+1)},
\end{align*}
\]

where the initial conditions are arbitrary nonzero real numbers with \(x_{-1} x_0, y_0 y_{-1} \neq \pm 1\). Then for \(n = 0, 1, 2, \ldots\),

\[
x_{4n-1} = a^{2n} d^{2n} e^{-\left( 1 \pm cd \right) 2n}, \quad x_{4n} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm cd \right)(2n+1)},
\]

\[
x_{4n+1} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm cd \right)(2n+1)}, \quad x_{4n+2} = a^{2n+2} d^{2n+2} e^{-\left( 1 \pm cd \right)(2n+1)},
\]

\[
y_{4n-1} = a^{2n} d^{2n} e^{-\left( 1 \pm ab \right) 2n}, \quad y_{4n} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm ab \right)(2n+1)},
\]

\[
y_{4n+1} = a^{2n+1} d^{2n+1} e^{-\left( 1 \pm ab \right)(2n+1)}, \quad y_{4n+2} = a^{2n+2} d^{2n+2} e^{-\left( 1 \pm ab \right)(2n+1)}.
\]

Theorem 4.5. Systems (12) have a periodic solutions of period two if and only if \(x_{-1} x_0 = y_0 y_{-1} = \pm 2\), and will be in the form \(\{x_n\} = \{b, a, b, a, \ldots\}, \{y_n\} = \{d, c, d, c, \ldots\}\).

5. CONCLUSION

This paper discussed the existence of solutions and periodicity of all cases of the systems of difference equations

\[
x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 \pm x_n x_{n-1} \right)},
\]

In Section 2, we obtained the form of the solution of the system

\[
x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 \pm x_n x_{n-1} \right)}
\]

and other similar cases. In Section 3, we have got the expressions of the solutions of some cases of the systems especially \(x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 \pm x_n x_{n-1} \right)}\). In Section 4, we proved that the solution of the system \(x_{n+1} = \frac{y_n y_{n-1}}{x_n \left( 1 \pm y_n y_{n-1} \right)}, \quad y_{n+1} = \frac{x_n x_{n-1}}{y_n \left( 1 \pm x_n x_{n-1} \right)}\) is unbounded and has a periodic solution of period two under some conditions and we have written the specific solutions of this system, other systems studied. Finally, using Matlab we gave numerical examples of some cases and drew them to support our results.

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