Lecture 7, Tu., Sept. 12. Hw 2 is due on Th., Sept. 28

Reading homework: Chapter 2

Derivation of Euler’s equation

We consider the more general scenario that all the vital rates are functions of continuous time. Specifically, let

\( B(t) = \text{total birth rate}, \)

\( l(x) = l_x = \text{fraction of newborn to survive to age } x, \)

\( m(x) = m_x = \text{average birth rate for an individual of age } x. \)

Then

\[
B(t) = \int_0^\infty \text{birth due to parents of age } x \, dx = \int_0^\infty B(t-x)l(x)m(x) \, dx.
\]

Assume that the population is at a stable age distribution and the intrinsic growth rate is \( r \) (the dominating eigenvalue). Then

\[
e^{rt}B(0) = \int_0^\infty e^{r(t-x)}B(0)l(x)m(x) \, dx.
\]

Which gives us the Euler’s equation

\[
1 = \int_0^\infty e^{-rx}l(x)m(x) \, dx.
\]

In discrete time, the Euler’s equation takes the form of

\[
1 = \sum_{x=0}^{\infty} e^{-rx}l_x m_x.
\]

In continuous time, the net reproduction rate is

\[
R_0 = \int_0^\infty l(x)m(x) \, dx.
\]

The stable age distribution is

\[
c(x) = \frac{\text{number of individuals of age } x}{\text{total number of individuals}} = \frac{B(t-x)}{\int_0^\infty B(t-x)l(x) \, dx}.
\]

Chapter 2: Nonlinear Difference Equations

1. Stability of first order nonlinear difference equations. We consider first the scalar equation

\[
x_{n+1} = f(x_n), \quad f(x) \in C^1.
\] (1.1)

We say \( \overline{x} \) is a steady state solution (equilibrium) of (1.1) if \( \overline{x} = f(\overline{x}). \)

**Definition 1.** The steady state solution \( \overline{x} \) is stable if for any positive constant \( \varepsilon, \)

there is a \( \delta \) such that \( |x_0 - \overline{x}| < \delta \) implies that for all \( n > 0, \) \( |x_n - \overline{x}| < \varepsilon. \) If in addition,
\[ \lim_{n \to \infty} x_n = \bar{x}, \text{ then we say that the steady state solution } \bar{x} \text{ is asymptotically stable.} \]

Notice that in the textbook, stable is actually referred as asymptotically stable. The following simple theorem is very useful. We provide its rigorous proof.

**Theorem 1.** The steady state solution \( \bar{x} \) of (1.1) is asymptotically stable if \( |df(\bar{x})/dx| < 1 \).

**Proof.** Since \( f(x) \in C^1 \) and \( |df(\bar{x})/dx| < 1 \), there is a \( \varepsilon_1 > 0 \) such that \( |x_0 - \bar{x}| \leq \varepsilon_1 \) ensures that \( |df(x_0)/dx| < 1 \). Then

\[ \lambda \equiv \max\{|df(x_0)/dx| : |x_0 - \bar{x}| \leq \varepsilon_1\} < 1. \]

Given \( \varepsilon > 0 \), let \( \delta = \min\{\varepsilon/2, \varepsilon_1/2\} \).

Recall that by the mean value theorem, we have

\[ f(x_0) = f(\bar{x}) + f'(\xi)(x_0 - \bar{x}) \]

for some \( \xi \) in between \( x_0 \) and \( \bar{x} \). Since \( \bar{x} = f(\bar{x}) \), we have

\[ |x_1 - \bar{x}| = |f(x_0) - \bar{x}| = |f'(\xi)(x_0 - \bar{x})| \leq \lambda|x_0 - \bar{x}| < \delta. \]

Continue this way, we obtain that

\[ |x_n - \bar{x}| \leq \lambda^n|x_0 - \bar{x}| < \delta. \]

Clearly, \( \lim_{n \to \infty} x_n = \bar{x} \).

A simple application of this theorem to the discrete logistic equation (see example 2 on page 44) \( x_{n+1} = rx_n(1 - x_n) \) yields that \( \bar{x} = 1 - 1/r \) exists and is asymptotically stable if \( 1 < r < 3 \).

2. **Stability of second order nonlinear difference equations.** Read sections 2.7 and 2.8. Make sure that you understand and familiar with the following result.

**Theorem 2.** The roots of \( \lambda^2 - \beta \lambda + \gamma = 0 \) satisfy \( |\lambda| < 1 \) if \( |\beta| < 1 + \gamma < 2 \).

3. **Stability of higher order nonlinear difference equations.** Read section 2.9. Make sure that you are familiar with Jury test in the special case of third order difference equations.

**Theorem 3.** The roots of \( P(\lambda) \equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \) satisfy \( |\lambda| < 1 \) if and only if that

1) \( P(1) > 0 \),
2) \( P(-1) < 0 \),
3) \( |a_3| < 1, |b_3| > |b_1|, |c_3| > |c_2| \),

where \( b_3 = 1 - a_3t^2, b_2 = a_1 - a_3a_2, b_1 = a_2 - a_3a_1, c_3 = b_3 - b_1^2, c_2 = b_3b_2 - b_1b_3 \).

The Jury test is a practical presentation of the so-called Schur-Cohn criterion which can be derived easily from the much more well-known Routh-Hurwitz criterion (see page 233).