Reading homework: Chapter 6

1. Applications of Liapunov-LaSalle Theorem in a Food Chain, II. We first recall the definition of Liapunov function. Let

\[ x' = f(x), \quad x \in \mathbb{R}^n. \]

be an n-dimensional system of differential equations. Let \( f(x) \) be defined on \( G^* \), an open set in \( \mathbb{R}^n \), and let \( G \) be a subset of \( G^* \). A function \( V(x) : G \rightarrow \mathbb{R} \) is said to be a Liapunov function for (1.1) on \( G \) if
1. \( V \) is continuously differentiable at each point \( x \in G \),
2. \( \dot{V} = \frac{dV}{dt}|_{(1.1)} = \nabla \cdot V \leq 0 \) on \( G \).

The following Liapunov-LaSalle theorem will be the key in our effort in seeking global stability results in some population models.

**Theorem 1.1. (Liapunov-LaSalle)** Let \( V \) be a Liapunov function for (1.1) on a region \( G \). Let \( E = \{ x \mid \dot{V}(x) = 0, x \in G \bigcap G^* \} \) and let \( M \) be the largest invariant set in \( E \). Then every bounded (for \( t \geq 0 \)) trajectory of (1.1) that remains in \( G \) tends to the set \( M \) as \( t \to \infty \).

Consider now the following Lotka-Volterra food chain model

\[ x' = x[r(1 - x/K) - by], \quad y' = y(-m + cx - dz), \quad z' = z(ey - f). \]  

(1.2)

We assume that (1.2) has a positive steady state \( E = (x^*, y^*, z^*) \). This is equivalent to say that \( re > bf \) and \( \frac{x}{a}(r - b f) > 0 \).

**Theorem 1.2.** In (1.2), if \( re > bf \) and \( \frac{x}{a}(r - b f) > 0 \), then all positive solutions tend to \( E = \left( \frac{1}{a}(r - b f), \frac{r}{e}, \frac{f}{d} \right), \text{ and } b - \frac{f}{d} \).

Again, we will proof the above theorem by applying Liapunov-LaSalle theorem. As in the previous example of the Lotka-Volterra predator-prey model, the key step is to construct an appropriate Liapunov function. To this end, we would like to try our luck for a Liapunov function that separate the variables \( x, y \) and \( z \). In other word, we assume that

\[ V(x, y, z) = V_1(x) + V_2(y) + V_3(z). \]

Let

\[ X = x - x^*, \quad Y = y - y^*, \quad Z = z - z^*. \]

Then (1.2) can be rewritten as

\[ x' = X(-aX - bY), \quad y' = Y(cX - dZ), \quad z' = z(eY). \]  

(1.3)

The derivative of this function along a solution of (1.2) takes the form of

\[ \dot{V} = V'_1(x)(-aX - bY) + V'_2(y)(cX - dZ) + V'_3(z)zeY \]

\[ = -axXV'_1(x) - bV'_1(x)X + V'_2(y)cX - V'_2(y)ydZ + V'_3(z)zeY. \]
We would like to have all the mixed terms in the expression of $\dot{V}$ to cancel each other. In other words, we want
\[ bV_1'(x) xY = V_2'(y) ycX, \ V_2'(y) ydZ = V_3'(z) zeY. \]
This is equivalent to say that
\[ f(x) \equiv bV_1'(x) x/X = V_2'(y) yc/Y \equiv g(y) \]
and
\[ V_2'(y) yd/Y \equiv \frac{d}{c}g(y) = V_3'(z) ze/Z \equiv h(z). \]
This says a function of $x$ is identical to a different function of another independent variable $y$ and a function of $y$ is identical to a different function of another independent variable $z$. This can only be true if these functions are constants. For our purpose, we may simply assume that the first constant is 1 ($f(x) = g(y)$). This yields
\[ bV_1'(x) = X/x = 1 - x^*/x, \ cV_2'(y) = Y/y = 1 - y^*/y. \]
and
\[ \frac{cd}{d} V_3'(z) = Z/z = 1 - z^*/z. \]
This in turn suggest that
\[ V_1(x) = \frac{1}{b}(x - x^* \ln x), \ V_2(y) = \frac{1}{c}(y - y^* \ln y), \ V_3(z) = \frac{d}{cd}(z - z^* \ln z). \]
With these functions, we have
\[ \dot{V} = -ax XV_1'(x) = -\frac{a}{b}(x - x^*)^2. \]
Hence, $V = \frac{1}{b}(x - x^* \ln x) + \frac{1}{c}(y - y^* \ln y) + \frac{d}{cd}(z - z^* \ln z)$ is indeed a Liapunov function for (1.2). We have $E = \{(x, y, z) : x = x^*, y, z \geq 0\}$. We shall show that the largest invariant set $M$ in $E$ for (1.2) is $\{E\}$. To this end, we assume that $(x(0), y(0), z(0)) \in M$. Since $(x(t), y(t), z(t)) \in E$, we have $x(t) \equiv x^*$ which implies that
\[ x'(t) \equiv 0. \]
This in turn yields
\[ 0 = x'(t) = x^*(r - ax^* - by(t)) \]
and hence $y(t) = y^*$. In particular, we must have $y(0) = y^*$. Similarly, we must have $z(0) = z^*$, proving that $M = \{E\}$. This shows that all positive solutions of (1.2) tend to the unique positive steady state $E$. 

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