Lectures 14, Th., Oct. 5

Reading homework: chapter 4

1. A chemostat model. We covered sections 4.6-4.7. In addition, we have shown that the solutions of the model (19a)-(19b) with positive initial values are positive and bounded. Standard existence and uniqueness theorems of MAT 475 (or MAT 574) ensure that such solution exist and unique for \( t > 0 \).

**Theorem 1.** (Positivity) The solutions of the model (19a)-(19b) with positive initial values are positive for \( t > 0 \).

**Proof.** If not, then there is a \( t_1 > 0 \), such that \( N(t_1)C(t_1) = 0 \) and \( N(t)C(t) > 0 \) for \( t \in [0, t_1) \). Assume first that \( C(t_1) = 0 \). Then \( C'(t_1) \leq 0 \). However, (19b) implies that \( C'(t_1) = \alpha_2 > 0 \), a contradiction. In the rest of this proof, we show that it is impossible that \( N(t_1) = 0 \). From (19a) and the fact that it is impossible that \( C(t_1) = 0 \), we see that for \( t \in [0, t_1) \), we have that

\[
N'(t) \geq -N(t)
\]

which yields that for \( t \in [0, t_1) \),

\[
N(t) \geq N(0)e^{-t} > N(0)e^{-t_1}.
\]

Since \( N(t) \) is continuous on \([0, t_1]\), we see that \( N(t_1) \geq N(0)e^{-t_1} > 0 \). This is a contradiction.

**Theorem 2.** (Boundedness) The solutions of the model (19a)-(19b) with positive initial values are bounded for \( t > 0 \).

**Proof.** Let \( Z(t) = N(t) + \alpha_1C(t) \). Then

\[
Z'(t) = \alpha_1\alpha_2 - Z(t).
\]

Hence \( Z(t) = \alpha_1\alpha_2 + (Z(0) - \alpha_1\alpha_2)e^{-t} \leq \max\{\alpha_1\alpha_2, Z(0)\} \). This shows that both \( N(t) \) and \( C(t) \) are bounded for \( t > 0 \).