Contents

Chapter 1 Functions and Graphs .........................................................1
Chapter 2 Differentiation .................................................................17
Chapter 3 Applications of Differentiation ........................................43
Chapter 4 Exponential and Logarithmic Functions .........................103
Chapter 5 Integration .......................................................................125
Chapter 6 Matrices ..........................................................................165
Chapter 7 Functions of Several Variables .....................................175
Chapter 8 First-Order Differential Equations .................................187
Chapter 9 Higher-Order and Systems of Differential Equations ......201
Chapter 10 Probability ....................................................................221
Chapter 1
Functions and Graphs

Exercise Set 1.1

1. Graph \( y = -4 \).

Note that \( y \) is constant and therefore any value of \( x \) we choose will yield the same value for \( y \), which is \(-4\). Thus, we will have a horizontal line at \( y = -4 \).

2. Graph \( x = -4.5 \).

Note that \( x \) is constant and therefore any value of \( y \) we choose will yield the same value for \( x \), which is \(-4.5\). Thus, we will have a vertical line at \( x = -4.5 \).

3. Graph \( y = -2x \).

Compare the equation \( y = -2x \) to the general linear equation form of \( y = mx + b \) to conclude the equation has a slope of \( m = -2 \) and a \( y \)-intercept of \( (0, 0) \).

4. Graph the slope and the \( y \)-intercept of \( y = 0.5x \).

First, we find some points that satisfy the equation, then we plot the ordered pairs and connect the plotted points to get the graph.

When \( x = 0 \), \( y = 0.5(0) = 0 \), ordered pair \((0, 0)\)
When \( x = 6 \), \( y = 0.5(6) = 3 \), ordered pair \((6, 3)\)
When \( x = -2 \), \( y = 0.5(-2) = -1 \), ordered pair \((-2, -1)\)

5. Graph the slope and the \( y \)-intercept of \( y = -3x \).

First, we find some points that satisfy the equation, then we plot the ordered pairs and connect the plotted points to get the graph.

When \( x = 0 \), \( y = -3(0) = 0 \), ordered pair \((0, 0)\)
When \( x = -1 \), \( y = -3(-1) = 3 \), ordered pair \((-1, 3)\)
When \( x = -1 \), \( y = -3(-1) = 3 \), ordered pair \((-1, 3)\)

6. Graph the slope and the \( y \)-intercept of \( y = 0.5x \).

First, we find some points that satisfy the equation, then we plot the ordered pairs and connect the plotted points to get the graph.

When \( x = 0 \), \( y = 0.5(0) = 0 \), ordered pair \((0, 0)\)
When \( x = 6 \), \( y = 0.5(6) = 3 \), ordered pair \((6, 3)\)
When \( x = -2 \), \( y = 0.5(-2) = -1 \), ordered pair \((-2, -1)\)

7. Graph the slope and the \( y \)-intercept of \( y = -2x + 3 \).

First, we find some points that satisfy the equation, then we plot the ordered pairs and connect the plotted points to get the graph.

When \( x = 0 \), \( y = -2(0) + 3 = 3 \), ordered pair \((0, 3)\)
When \( x = -2 \), \( y = -2(-2) + 3 = 7 \), ordered pair \((-2, 7)\)
17. Find the slope and $y$-intercept of $x = 2y + 8$.
Solve the equation for $y$.
\[
x = 2y + 8
\]
\[
x - 8 = 2y
\]
\[
\frac{1}{2}x - 4 = y
\]
Compare to $y = mx + b$ to conclude the equation has a slope of $m = \frac{1}{2}$ and a $y$-intercept of $(0, -4)$.

19. Find the equation of the line with $m = -5$, containing $(1, -5)$
Plug the given information into equation $y - y_1 = m(x - x_1)$ and solve for $y$
\[
y - y_1 = m(x - x_1)
\]
\[
y - (-5) = -5(x - 1)
\]
\[
y + 5 = -5x + 5
\]
\[
y = -5x
\]

21. Find the equation of line with $m = -2$, containing $(2, 3)$
Plug the given information into the equation $y - y_1 = m(x - x_1)$ and solve for $y$
\[
y - 3 = -2(x - 2)
\]
\[
y - 3 = -2x + 4
\]
\[
y = -2x + 1 + 3
\]
\[
y = -2x + 7
\]

23. Find the equation of line with $m = 2$, containing $(3, 0)$
Plug the given information into the equation $y - y_1 = m(x - x_1)$ and solve for $y$
\[
y - 0 = 2(x - 3)
\]
\[
y = 2x - 6
\]

25. Find the equation of line with $y$-intercept $(0, -6)$ and $m = \frac{1}{2}$
Plug the given information into the equation $y = mx + b$
\[
y = \frac{1}{2}x + b
\]
\[
y = \frac{1}{2}x + (-6)
\]
\[
y = \frac{1}{2}x - 6
\]

27. Find the equation of line with $m = 0$, containing $(2, 3)$
Plug the given information into the equation $y - y_1 = m(x - x_1)$ and solve for $y$
\[
y - 3 = 0(x - 2)
\]
\[
y - 3 = 0
\]
\[
y = 3
\]
29. Find the slope given \((-1, -2)\) and \((-2, 1)\)
Use the slope equation \(m = \frac{y_2 - y_1}{x_2 - x_1}\). NOTE: It does not matter which point is chosen as \((x_1, y_1)\) and which is chosen as \((x_2, y_2)\) as long as the order the point coordinates are subtracted in the same order as illustrated below.

\[
m = \frac{1 - (-2)}{-2 - (-1)} = \frac{3}{-1} = -3 \quad \text{or} \quad \frac{-2 - 1}{-1 - (-2)} = \frac{-3}{2}
\]

31. Find the slope given \((\frac{3}{2}, \frac{1}{2})\) and \((-3, \frac{1}{2})\)

\[
m = \frac{\frac{1}{2} - \frac{1}{2}}{-3 - \frac{3}{2}} = \frac{0}{-\frac{9}{2}} = 0 \quad \text{or} \quad \frac{-\frac{1}{2} - \frac{1}{2}}{-3 - \frac{3}{2}} = \frac{-\frac{3}{2}}{-\frac{9}{2}} = \frac{1}{3}
\]

33. Find the slope given \((3, -7)\) and \((3, -9)\)

\[
m = \frac{-9 - (-7)}{3 - 3} = \frac{-2}{0} \quad \text{undefined quantity}
\]
This line has no slope.

35. Find the slope given \((2, 3)\) and \((-1, 3)\)

\[
m = \frac{3 - 3}{-1 - 2} = \frac{0}{-3} = 0
\]

37. Find the slope given \((x, 3x)\) and \((x + h, 3(x + h))\)

\[
m = \frac{3(x + h) - 3x}{x + h - x} = \frac{3xh}{h} = 3
\]

39. Find the slope given \((x, 2x + 3)\) and \((x + h, 2(x + h) + 3)\)

\[
m = \frac{2(x + h) + 3 - (2x + 3)}{x + h - x} = \frac{2h}{h} = 2
\]

41. Find equation of line containing \((-4, -2)\) and \((-2, 1)\)
From Exercise 29, we know that the slope of the line is \(\frac{3}{2}\). Using the point \((-2, 1)\) and the value of the slope in the point-slope formula \(y - y_1 = m(x - x_1)\) and solving for \(y\) we get:

\[
y - 1 = \frac{3}{2}(x - (-2))
\]

\[
g - 1 = \frac{3}{2}x + 3
\]

\[
y = \frac{3}{2}x + 3 + 1
\]

\[
y = \frac{3}{2}x + 4
\]

NOTE: You could use either of the given points and you would reach the final equation.

43. Find equation of line containing \((\frac{3}{4}, \frac{1}{2})\) and \((-3, \frac{1}{2})\)
From Exercise 31, we know that the slope of the line is \(-\frac{1}{3}\). Using the point \((-3, \frac{1}{2})\)

\[
y - \frac{1}{2} = \frac{-1}{3}(x - (-3))
\]

\[
y - \frac{1}{2} = \frac{-1}{3}x + 1
\]

\[
y = \frac{-1}{3}x + 1 + \frac{1}{2}
\]

\[
y = \frac{-1}{3}x + \frac{17}{6}
\]

45. Find equation of line containing \((3, -7)\) and \((3, -9)\)
From Exercise 33, we found that the line containing \((3, -7)\) and \((3, -9)\) has no slope. We notice that the \(x\)-coordinate does not change regardless of the \(y\)-value. Therefore, the line is vertical and has the equation \(x = 3\).

47. Find equation of line containing \((2, 3)\) and \((-1, 3)\)
From Exercise 35, we found that the line containing \((2, 3)\) and \((-1, 3)\) has a slope of \(m = 0\). We notice that the \(y\)-coordinate does not change regardless of the \(x\)-value. Therefore, the line is horizontal and has the equation \(y = 3\).
49. Find equation of line containing \((x, 3x)\) and \((x + h, 3(x + h))\.

From Exercise 37, we found that the line containing \((x, 3x)\) and \((x + h, 3(x + h))\) had a slope of \(m = 3\). Using the point \((x, 3x)\) and the value of the slope in the point-slope formula

\[
\begin{align*}
y - 3x & = 3(x - x) \\
y - 3x & = 3(0) \\
y - 3x & = 0 \\
y & = 3x
\end{align*}
\]

51. Find equation of line containing \((x, 2x + 3)\) and \((x + h, 2(x + h) + 3)\).

From Exercise 37, we found that the line containing \((x, 2x + 3)\) and \((x + h, 2(x + h) + 3)\) had a slope of \(m = 2\). Using the point \((x, 2x + 3)\) and the value of the slope in the point-slope formula

\[
\begin{align*}
y - (2x + 3) & = 2(x - x) \\
y - (2x + 3) & = 2(0) \\
y - (2x + 3) & = 0 \\
y & = 2x + 3
\end{align*}
\]

53. Slope = \(\frac{3.4}{11} = 0.08\). This means the treadmill has a grade of 8%.

55. The slope (or head) of the river is \(\frac{\Delta h}{\Delta x} = \frac{0.035}{3} = 0.035\) ft/m.

57. The average rate of change of life expectancy at birth is computed by finding the slope of the line containing the two points \((1990, 73.7)\) and \((2000, 76.0)\), which is given by

\[
\text{Rate} = \frac{\text{Change in Life expectancy}}{\text{Change in Time}}
\]

\[
\begin{align*}
&= \frac{76.0 - 73.7}{2000 - 1990} \\
&= \frac{3.2}{10} \\
&= 0.32 \text{ per year}
\end{align*}
\]

59. a) Since \(R\) and \(T\) are directly proportional we can write that \(R = kT\), where \(k\) is a constant of proportionality. Using \(R = 12.51\) when \(T = 3\) we can find \(k\).

\[
\begin{align*}
R & = kT \\
12.51 & = k(3) \\
\frac{12.51}{3} & = k \\
4.17 & = k
\end{align*}
\]

Thus, we can write the equation of variation as \(R = 4.17T\).

b) This is the same as asking: find \(R\) when \(T = 6\). So, we use the variation equation

\[
R = 4.17T
\]

\[
= 4.17(6)
\]

\[
= 25.02
\]

61. a) Since \(B\) is directly proportional to \(W\) we can write \(B = kW\).

b) When \(W = 200\) lb, \(B = kW\)

\[
\begin{align*}
B & = kW \\
5 & = k(200) \\
\frac{5}{200} & = k \\
0.025 & = k
\end{align*}
\]

This means that the weight of the brain is 2.5% the weight of the person.

c) Find \(B\) when \(W = 120\)

\[
\begin{align*}
B & = 0.025W \\
& = 0.025(120 \text{ lbs}) \\
& = 3 \text{ lbs}
\end{align*}
\]

63. a) \(D(0) = 2(0) + 115 - 0 + 115 = 230\) ft

\(D(20) = 2(-20) + 115 = -40 + 115 = 75\) ft

\(D(10) = 2(10) + 115 = 20 + 115 = 135\) ft

\(D(32) = 2(32) + 115 = 64 + 115 = 179\) ft

b) The stopping distance has to be a non-negative value. Therefore we need to solve the inequality

\[
\begin{align*}
0 & \leq 2F + 115 \\
-115 & \leq 2F \\
-57.5 & \leq F
\end{align*}
\]

The 32" limit comes from the fact that for any temperature above that there would be no ice. Thus, the domain of the function is restricted in the interval \([-57.5, 32]\).

65. a) \(M'(x) = 2.89x + 70.64\)

\(M'(26) = 2.89(26) + 70.64 = 75.14 + 70.64 = 145.78\)

The male was 145.78 cm tall.

b) \(F'(x) = 2.75x + 71.48\)

\(F'(26) = 2.75(26) + 71.48 = 71.5 + 71.48 = 142.98\)

The female was 142.98 cm tall.

67. a) \(A(0) = 0.08(0)+19.7 = 0+19.7 = 19.7\)

\(A(1) = 0.08(1)+19.7 = 0.08+19.7 = 19.78\)

\(A(10) = 0.08(10)+19.7 = 0.8+19.7 = 20.5\)

\(A(30) = 0.08(30)+19.7 = 2.4+19.7 = 22.1\)

\(A(50) = 0.08(50)+19.7 = 1+19.7 = 20.7\)
b) First we find the value of \( t \), which is \( 2003 - 1950 = 53 \). So, we have to find \( A(53) \).

\[
A(53) = 0.68(53) + 19.7 = 4.24 + 19.8 = 23.91
\]

The median age of women at first marriage in the year 2003 is 23.91 years.

c) \( A(t) = 0.08t + 19.7 \)

---

Exercise Set 1.2

1. \( y = \frac{1}{2}x^2 \) and \( y = -\frac{1}{2}x^3 \)

3. \( y = x^2 \) and \( y = -(x - 1)^2 \)

5. \( y = x^2 \) and \( y = (x + 1)^2 \)

7. \( y \quad x^3 \) and \( y = x^4 \)

9. Since the equation has the form \( ax^2 + bx + c \) with \( a \neq 0 \), the graph of the function is a parabola. The \( x \)-value of the vertex is given by

\[
x = -\frac{b}{2a} = \frac{-1}{2(1)} = -1
\]

The \( y \)-value of the vertex is given by

\[
y = \frac{(-2)^2 + 4(-2) - 7}{2} = \frac{4 - 8 - 7}{2} = -11
\]

Therefore, the vertex is \((-2, 11)\).

11. Since the equation is not in the form of \( ax^2 + bx + c \), the graph of the function is not a parabola.

13. \( y \quad x^2 - 4x + 3 \)
15. \( y = -x^2 + 2x - 1 \)

\[ y = 1 \pm \sqrt{3} \]

The solutions are \( 1 + \sqrt{3} \) and \( 1 - \sqrt{3} \)

23. Solve \( 3y^2 + 8y + 2 = 0 \)

Use the quadratic formula, with \( a = 3 \), \( b = 8 \), and \( c = 2 \), to solve for \( y \).

\[ y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ y = \frac{-8 \pm \sqrt{(8)^2 - 4(3)(2)}}{2(3)} \]

\[ y = \frac{-8 \pm \sqrt{64 - 24}}{6} \]

\[ y = \frac{-8 \pm \sqrt{40}}{6} \]

\[ y = \frac{-8 \pm 2\sqrt{10}}{6} \]

\[ y = \frac{2(-1 \pm \sqrt{10})}{3} \]

The solutions are \(-\frac{2}{3} \pm \sqrt{10} \) and \(-\frac{2}{3} - \sqrt{10} \)

25. Solve \( x^2 - 2x + 10 = 0 \)

Using the quadratic formula with \( a = 1 \), \( b = -2 \), and \( c = 10 \).

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} \]

\[ x = 2 \pm \sqrt{1 - 40} \]

\[ x = \frac{2 \pm \sqrt{-39}}{2} \]

\[ x = \frac{2 \pm 6i}{2} \]

\[ x = 1 \pm 3i \]

The solutions are \( 1 + 3i \) and \( 1 - 3i \)

27. Solve \( x^2 + 6x - 1 = 0 \)

Write the equation so that one side equals zero, that is \( x^2 + 6x - 1 = 0 \), then use the quadratic formula, with \( a = 1 \), \( b = 6 \), and \( c = -1 \), to solve for \( x \).

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(-1)}}{2(1)} \]

\[ x = \frac{-6 \pm \sqrt{36 + 4}}{2} \]

\[ x = \frac{-6 \pm 2\sqrt{10}}{2} \]

\[ x = -3 \pm \sqrt{10} \]
29. Solve \( x^2 + 4x + 8 = 0 \)

Using the quadratic formula with \( a = 1, b = 4, \) and \( c = 8 \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(8)}}{2(1)}
\]

\[
x = \frac{-4 \pm \sqrt{16 - 32}}{2}
\]

\[
x = \frac{-4 \pm \sqrt{-16}}{2}
\]

\[
x = -2 \pm 2i
\]

The solutions are \( -2 + 2i \) and \( -2 - 2i \)

31. Solve \( 4x^2 - 4x - 1 = 0 \)

Write the equation so that one side equals zero, that is \( 1x^2 - 4x - 1 = 0 \), then use the quadratic formula, with \( a = 1, b = -4, c = -1, \) to solve for \( x \).

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{4 \pm \sqrt{16 + 16}}{2}
\]

\[
x = \frac{4 \pm 4i}{2}
\]

\[
x = 2 \pm 2i
\]

The solutions are \( 2 + 2i \) and \( 2 - 2i \)

33. Find \( f(7), f(10), \) and \( f(12) \)

\[
f(7) = \frac{1}{6}(7)^3 + \frac{1}{2}(7)^2 + \frac{1}{2}(7)
\]

\[
\begin{aligned}
&= \frac{1}{6}(343) + \frac{1}{2}(49) + \frac{1}{2}(7) \\
&= 56 + 24.5 + 3.5 \\
&= 84.0 \\
&\approx 85, 165 \approx 85 \text{ oranges}
\end{aligned}
\]

\[
f(10) = \frac{1}{6}(10)^3 + \frac{1}{2}(10)^2 + \frac{1}{2}(10)
\]

\[
\begin{aligned}
&= \frac{1}{6}(1000) + \frac{1}{2}(100) + \frac{1}{2}(10) \\
&= 166.66 \approx 167, 100 \approx 167 \text{ oranges}
\end{aligned}
\]

35. Solve \( 50 - 9.11 - 0.19x - 0.09x^2 \). First, let us rewrite the equation as \( 0 = -40.59 - 0.19x - 0.09x^2 \) then we can use the quadratic formula to solve for \( x \)

\[
x = \frac{-(-0.19) \pm \sqrt{(-0.19)^2 - 4(-0.09)(-40.59)}}{2(0.09)}
\]

\[
x = \frac{0.19 \pm \sqrt{0.0361 + 14.6121}}{0.18}
\]

\[
x = \frac{0.19 \pm 3.8273}{0.18}
\]

\[
x = 22.3183
\]

Therefore, the average price of a ticket will be $50 will happen during the 1990-1991 season.

37. \( f(x) = x^3 - x^2 \)

a) For large values of \( x, x^3 \) would be larger than \( x^2 \).

\( x^3 - x^2 \) is only positive for very large values of \( x \) since there is a extra factor of \( x \) in \( x^3 \) which causes \( x^3 \) to be larger than \( x^2 \).

b) As \( x \) gets very large the values of \( x^3 \) become much larger than those of \( x^2 \) and therefore we can "ignore" the effect of \( x^2 \) in the expression \( x^3 - x^2 \). Thus, we can approximate the function to look like \( x^3 \) for very large values of \( x \).

c) Below is a graph of \( x^3 - x^2 \) and \( x^3 \) for \( 0 \leq x \leq 200 \). It is hard to distinguish between the two graphs confirming the conclusion reached in part b).

39. \( f(x) \cdot x^2 \cdot x \)

a) For values very close to 0, \( x \) is larger than \( x^2 \). Since for values of \( x \) less than 1 \( x^2 \) < \( x \).
b) For values of $x$ very close to $0$ $f(x)$ looks like $x$ since the $x^2$ can be "ignored".

c) Below is a graph of $y^2$ for $x$ and $y$ for $-0.01 \leq x \leq 0.01$. It is very hard to distinguish between the two graphs confirming our conclusion from part b).

41. $f(x) = x^3 - x$

\[

t(x) = 0 \\
x^3 - x = 0 \\
x(x^2 - 1) = 0 \\
x(x - 1)(x + 1) = 0 \\
x = 0 \\
x = 1 \\
x = -1
\]

43. $x = -1.831$, $x = 0.856$, and $x = 3.188$

45. $x = -10.153$, $x = -1.871$, $x = -0.821$, $x = -0.303$, $x = 0.098$, $x = 0.535$, $x = 1.219$, and $x = 3.297$

47. $y = -0.279x + 4.936$

49. $y = 0.942x^2 - 2.651x - 27.943$

51. $y = 0.237x^4 - 0.885x^3 - 29.224x^2 + 165.166x - 210.135$

Exercise Set 1.3

1. $y = |x|$ and $y = |x + 3|$
11. \( y = \sqrt{x} \)

13. \( y = \frac{\sqrt{x} - 1}{x - 1} \). It is important to note here that \( x = 1 \) is not in the domain of the plotted function.

15. \( y = \frac{x^2 - 1}{x - 1} \). It is important to note here that \( x = 1 \) is not in the domain of the plotted function.

33. \( x^\frac{2}{3} = \frac{1}{x} \cdot \frac{1}{\sqrt[3]{x}} \)

35. \( b^{\frac{3}{4}} = \frac{1}{\sqrt[4]{b^3}} \)

37. \( x^{\frac{4}{3}} = \frac{1}{\sqrt[3]{x^4}} \)

39. \( (x - 3)^{\frac{1}{2}} - \frac{1}{(x - 3)^{\frac{1}{2}}} = \frac{1}{\sqrt{x - 3}} \)

41. \( \frac{1}{\sqrt{x}} - \frac{1}{\sqrt[y]{y}} \)

43. \( 9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27 \)

45. \( 64^{\frac{3}{4}} = (\sqrt[4]{64})^3 = (4)^3 = 64 \)

47. \( 16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (2)^3 = 8 \)

49. The domain consists of all \( x \)-values such that the denominator does not equal 0, that is \( x - 5 \neq 0 \), which leads to \( x \neq 5 \). Therefore, the domain is \( \{x | x \neq 5\} \)

51. Solving for the values of \( x \) in the denominator that make it 0.

\[
\begin{align*}
x^2 - 5x + 6 &= 0 \\
(x - 3)(x - 2) &= 0
\end{align*}
\]

So \( x = 3 \) and \( x = 2 \)

Which means that the domain is the set of all \( x \)-values such that \( x \neq 3 \) or \( x \neq 2 \)

53. The domain of a square root function is restricted by the value where the radicand is positive. Thus, the domain of \( f(x) = \sqrt{5x^2 + 4} \) can be found by finding the solution to the inequality \( 5x^2 + 4 > 0 \).

\[
\begin{align*}
5x^2 + 4 &\geq 0 \\
5x^2 &\geq -4 \\
x^2 &\geq -\frac{4}{5}
\end{align*}
\]

55. To complete the table we will plug the given \( x \)-values into the equation

\[
T(20) = (20)^{\frac{1}{3}} - 50.623 \approx 54
\]

\[
T(30) = (30)^{\frac{1}{3}} - 50.623 = 80.168 \approx 86
\]

\[
T(40) = (40)^{\frac{1}{3}} - 50.623 = 125.516 \approx 126
\]

\[
T(50) = (50)^{\frac{1}{3}} - 50.623 = 168.132 \approx 168
\]

\[
T(100) = (100)^{\frac{1}{3}} - 50.623 = 416.269 \approx 417
\]

\[
T(150) = (150)^{\frac{1}{3}} - 50.623 = 709.154 \approx 709
\]

Therefore the table is given by

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>0</td>
<td>20</td>
<td>51</td>
<td>86</td>
<td>126</td>
<td>168</td>
<td>197</td>
<td>228</td>
<td>261</td>
<td>295</td>
<td>331</td>
<td>709</td>
</tr>
</tbody>
</table>

Now the graph
57. a) \( f(180) = 0.144(180)^{1/2} = 0.144(13.41640786) \approx 1.932 \text{ m}^2 \)

b) \( f(170) = 0.144(170)^{1/2} = 0.144(13.03840481) \approx 1.878 \text{ m}^2 \)

c) The graph

\[
\begin{align*}
x^2 + 7x + 9 &= 0 \\
x &= \frac{-7 \pm \sqrt{49 - 4(1)(9)}}{2} \\
x &= \frac{-7 \pm \sqrt{13}}{2} \\
\text{and} \\
x &= \frac{-7 + \sqrt{13}}{2}
\end{align*}
\]

63. \( P = 10000^{5/4} + 14000 \)

a) \( t = 37, P = 10000(37)^{5/4} + 14000 \approx 105254.0514 \)

\( t = 40, P = 10000(40)^{5/4} + 14000 \approx 114594.6744 \)

\( t = 50, P = 10000(50)^{5/4} + 14000 = 146357.3974 \)

b) Below is the graph of \( P \) for \( 0 \leq t \leq 50 \).

59. Let \( V \) be the velocity of the blood, and let \( A \) be the cross sectional area of the blood vessel. Then

\[ V = \frac{k}{A} \]

Using \( V = 30 \) when \( A = 3 \) we can find \( k \).

\( 30 = \frac{k}{3} \)

\( (30)(3) = k \)

\( 90 = k \)

Now we can write the proportional equation

\[ V = \frac{90}{A} \]

we need to find \( A \) when \( V = 0.026 \)

\[ 0.026 = \frac{90}{A} \]

\[ 0.026A = 90 \]

\[ A = \frac{90}{0.026} \]

\[ = 3401.538 \text{ m}^2 \]

61.

\[
\begin{align*}
x^2 + 7x + 9 &= 0 \\
x &= \frac{-7 \pm \sqrt{49 - 4(1)(9)}}{2} \\
x &= \frac{-7 \pm \sqrt{13}}{2} \\
\text{and} \\
x &= \frac{-7 + \sqrt{13}}{2}
\end{align*}
\]

65. A rational function is a function given by the quotient of two polynomial functions while a polynomial function is a function that has the form \( a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \). Since every polynomial function can be written as a quotient of two other polynomial function then every polynomial function is a rational function.

67. \( x = 2.6158 \) and \( x = -2.6158 \)

69. The function has no zeros

Exercise Set 1.4

1. \( (12r^2)(\frac{r}{144\pi}) = \frac{2}\pi \text{ rad} \)
3. \((240^\circ)(\frac{\pi}{180^\circ}) = \frac{4\pi}{3}\text{ rad}\)

5. \((540^\circ)(\frac{\pi}{180^\circ}) = 3\pi\text{ rad}\)

7. \((\frac{3\pi}{4})(\frac{180^\circ}{\pi}) = 135^\circ\)

9. \((\frac{3\pi}{2})(\frac{180^\circ}{\pi}) = 270^\circ\)

11. \((\frac{-\pi}{6})(\frac{180^\circ}{\pi}) = -60^\circ\)

13. We need to solve \(\theta_1 + \theta_2 = 360(k)\) for \(k\). If the solution is an integer then the angles are coterminal otherwise they are not coterminal.

\[
\begin{align*}
305 &= 15 + 360(k) \\
380 &= 360(k) \\
360 &= -k \\
1.05 &= k
\end{align*}
\]

Since \(k\) is not an integer, we conclude that \(15^\circ\) and \(395^\circ\) are not coterminal.

15. We need to solve \(\theta_1 + \theta_2 = 360(k)\) for \(k\). If the solution is an integer then the angles are coterminal otherwise they are not coterminal.

\[
\begin{align*}
107 &= -107 + 360(k) \\
214 &= 360(k) \\
214 &= -k \\
0.591 &= k
\end{align*}
\]

Since \(k\) is not an integer, we conclude that \(15^\circ\) and \(395^\circ\) are not coterminal.

17. We need to solve \(\theta_1 + \theta_2 = 2\pi(k)\) for \(k\). If the solution is an integer then the angles are coterminal otherwise they are not coterminal.

\[
\begin{align*}
\frac{\pi}{2} &= \frac{3\pi}{2} + 2\pi(k) \\
-\pi &= 2\pi(k) \\
-\pi &= k \\
2\pi &= k \\
\frac{1}{2} &= k
\end{align*}
\]

Since \(k\) is not an integer, we conclude that \(\frac{\pi}{2}\) and \(\frac{3\pi}{2}\) are not coterminal.

19. We need to solve \(\theta_1 + \theta_2 = 2\pi(k)\) for \(k\). If the solution is an integer then the angles are coterminal otherwise they are not coterminal.

\[
\begin{align*}
\frac{7\pi}{6} &= \frac{-5\pi}{6} + 2\pi(k) \\
2\pi &= 2\pi(k) \\
\frac{2\pi}{2} &= k \\
1 &= k
\end{align*}
\]
Since \( k \) is an integer, we conclude that \( \frac{7\pi}{6} \) and \( \frac{2\pi}{3} \) are coterminal.

21. \( \sin 34^\circ = 0.5592 \)
22. \( \cos 12^\circ = 0.9781 \)
25. \( \tan 5^\circ = 0.0875 \)
27. \( \cot 34^\circ = \frac{1}{\tan 34^\circ} = 1.4826 \)
29. \( \sec 23^\circ = \frac{1}{\cos 23^\circ} = 1.0861 \)
31. \( \sin \left( \frac{\pi}{3} \right) = 0.8660 \)
33. \( \tan \left( \frac{\pi}{4} \right) = 0.4816 \)
35. \( \sec \left( \frac{\pi}{4} \right) = \frac{1}{\cos \left( \frac{\pi}{4} \right)} = 2.4142 \)
37. \( \sin(2.3) = 0.7457 \)
39. \( t = \sin^{-1}(0.45) = 26.5717^\circ \)
41. \( t = \cos^{-1}(0.34) = 70.123^\circ \)
43. \( t = \tan^{-1}(2.34) = 66.8605^\circ \)
45. \( t = \sin^{-1}(0.59) = 0.6311 \)
47. \( t = \cos^{-1}(0.60) = 0.9273 \)
49. \( t - \tan^{-1}(0.11) = 0.1096 \)

51.
\[
\begin{align*}
\sin 57^\circ &= \frac{x}{40} \\
x &= 40 \sin 57^\circ \\
x &= 33.3468
\end{align*}
\]

53.
\[
\begin{align*}
\cos 50^\circ &= \frac{15}{x} \\
x &= 15 \cos 50^\circ \\
x &= 23.3359
\end{align*}
\]

55.
\[
\begin{align*}
\cos t &= \frac{10}{60} \\
t &= \cos^{-1} \left( \frac{10}{60} \right) \\
t &= 48.1897^\circ
\end{align*}
\]

57.
\[
\begin{align*}
\tan t &= \frac{18}{9.3} \\
t &= \tan^{-1} \left( \frac{18}{9.3} \right) \\
t &= 62.6761^\circ
\end{align*}
\]

59. We can rewrite \( 75^\circ - 30^\circ + 45^\circ \) then use a sum identity
\[
\begin{align*}
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos 75^\circ &= \cos(30^\circ + 45^\circ) \\
\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ \\
&= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
&= \frac{-1 + \sqrt{3}}{2\sqrt{2}}
\end{align*}
\]

61. Five miles is the same as \( 5 \cdot 5280 \) ft = 26,400 ft. The difference in elevation, \( y \), is
\[
\begin{align*}
\sin 4^\circ &= \frac{y}{26400} \\
y &= 26400 \sin 4^\circ \\
y &= 184.15 \text{ ft}
\end{align*}
\]

63. a) \( \cos 40^\circ = \frac{x}{150} \)
\[
\begin{align*}
x &= 150 \cos 40^\circ \\
x &= 114.907
\end{align*}
\]

b) \( \sin 40^\circ = \frac{y}{150} \)
\[
\begin{align*}
y &= 150 \sin 40^\circ \\
y &= 96.4181
\end{align*}
\]

c) \( z^2 = (x + 180)^2 + y^2 \)
\[
\begin{align*}
&= (114.907 + 180)^2 + (96.4181)^2 \\
&= 316.268
\end{align*}
\]

65.
\[
\begin{align*}
\nu &= \frac{77000 \cdot 100 \cdot \sec 60^\circ}{4000000} \\
&= \frac{7700000}{400000 \cos 60^\circ} \\
&= 45.494 \text{ cm/sec}
\end{align*}
\]

67. a) When we consider the two triangles we have a new triangle that has three equal angles which is the definition of an equilateral triangle.

b) The short leg of each triangle is given by \( 2 \sin(30^\circ) = 2 \left( \frac{1}{2} \right) \cdot 1 \)

c) The long leg \((L)\) is given by
\[
L^2 = L^2 + 1^2
\]
\[
L = \sqrt{3} \]
d) By considering all possible ratios between the long, 
short and hypotenuse of small triangles, we obtain the 
trigonometric functions of \( \frac{\pi}{4} \) - 30° and \( \frac{\pi}{3} \) - 60°.

69. a) The tangent of an angle is equal to the ratio of 
the opposite side to the adjacent side (of a right triangle), 
and for the small triangle that ratio is \( \frac{x}{y} \).

b) For the large right triangle, the opposite side is 10 
and the adjacent side is \( \frac{10}{\sqrt{3}} \). Thus the tangent 
is \( \frac{20}{\sqrt{3}} \).

c) Because the trigonometric functions depend on the 
ratios of the sides and not the size of triangle. Note 
that the answer in part b) is equivalent to that in part  
a) even though the triangle (in part b) was larger that 
that used in part a).

71. 
\[
\frac{\sin t}{\cos t} = \frac{y}{x} \quad \frac{y}{x}
\]
\[
\sin t = \frac{x}{r} \quad \cos t = \frac{y}{r}
\]
Thus 
\[
\sin t = \tan t
\]
and
\[
\frac{\cos t}{\sin t} = \frac{x}{y} \quad \cot t
\]

Thus 
\[
\frac{\cos t}{\sin t} = \cot t
\]

73. a) \( \sin(t) = \frac{y}{r} \) \( \frac{y}{r} \) \( \frac{y}{r} \), and \( \cos(t) = \frac{r}{r} \) \( \frac{r}{r} \) \( \frac{r}{r} \).

b) Consider the triangle made by the sides \( x \), \( y \), and 
\( y \). The angle \( \alpha \) has a value of \( 90 - \beta \) (completes a 
straight angle). The sum of angles in any triangle is 
180°. Therefore
\[
s + 90 \quad (90 - \beta) = 180
\]
\[
s + 180 - r = 180
\]
\[
s = r
\]
\[
s = r
\]

c) \( \cos(s) = \frac{r}{r} \), which means \( y = \cos(s)c \).

But from part a) \( y = \cos(t) \). Therefore \( y = \cos(s)\cos(t) \).

d) \( \sin(r) = \frac{y}{r} \), which means \( z = \sin(r)a \).

Using results from part a) and part b) we get \( \sin(r) = \sin(s) \) and \( u = \sin(t) \). Therefore \( z = \sin(s)\sin(t) \).

c) \( \cos(s + t) = \frac{\cos(t) - \sin(s)\sin(t)}{1} \). Replacing ur results for \( y \)
and \( z \) we get \( \cos(s + t) = \cos(s)\cos(t) - \sin(s)\sin(t) \).

75. Use \( \cos^2t + \sin^2t = 1 \) as follows
\[
\frac{\cos^2t + \sin^2t}{\cos^2t + \sin^2t} = \frac{1}{\cos^2t + \sin^2t}
\]
\[
1 + \tan^2t = \sec^2t
\]

77. Let \( 2\theta - \theta = \theta \)
\[
\sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)
\]
\[
\sin(2\theta) = \sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)
\]
\[
= 2\sin(\theta)\cos(\theta)
\]

79. Using the result from Exercise 78 part (c)
\[
\cos(2\theta) = 1 - 2\sin^2(\theta)
\]
\[
\cos(2\theta) - 1 = -2\sin^2(\theta)
\]
\[
\cos(2\theta) - 1 = \sin^2(\theta)
\]
\[
= \frac{1 - \cos(2\theta)}{2}
\]

78. a) \( V(0) = -\sin^3(0)\sin(0)\sin(0)\sin(0) = 0 \)

\( V(1) = \sin^3(\frac{\pi}{2})\sin^3(\frac{\pi}{2})\sin^3(\frac{\pi}{2})\sin^3(\frac{\pi}{2}) = 1 \)

b) When \( h = 0 \) the volume of the tree is zero since there 
is no height and therefore the proportion of volume 
under that height is zero. While at the top of the 
tree, \( h = 1 \), the proportion of volume under the tree 
is 1 since the entire tree volume falls below its height.

83. 
\[
V(\frac{1}{2}) = \sin^{-5,621}(\frac{\pi}{4})\sin(7,918)(\frac{\pi}{4})^2
\]
\[
\times\sin^{-19,411}(\frac{\pi}{4})\sin(7,918)(\frac{\pi}{2})
\]
\[
= 0.8219
\]

Exercise Set 1.5

1. \( 5\pi/1 \)
3. \(-\pi\)

5. \(13\pi/6\)

7. \(\cos(9\pi/2) = 0\)

9. \(\sin(-5\pi/6) = -\frac{1}{2}\)

11. \(\cos(5\pi) = -1\)

13. \(\tan(-4\pi/3) = -\sqrt{3}\)

15. \(\cos 125^\circ = -0.5736\)

17. \(\tan(-220^\circ) = -0.8391\)

19. \(\sec 286^\circ = -\frac{1}{\cos 286^\circ} = 3.62037\)

21. \(\sin(1.2\pi) = -0.58778\)

23. \(\cos(-1.9\lambda) = -0.932736\)

25. \(t = \sin^{-1}(1/2) = \frac{\pi}{6} + 2\pi \text{ and } \frac{5\pi}{6} + 2\pi\)

27. \(2t = \sin^{-1}(0) \cdot n\pi \text{ so } t = \frac{n\pi}{2}\)

29.

\[
\begin{align*}
\cos(3t + \frac{\pi}{4}) &= -\frac{1}{2} \\
3t + \frac{\pi}{4} &= \cos^{-1}(-\frac{1}{2}) \\
3t &= -\frac{\pi}{3} + 2n\pi \\
3t &= \frac{5\pi}{4} + 2n\pi \\
3t &= \frac{5\pi}{2} + 2n\pi \\
3t &= \frac{\pi}{3} + 2n\pi \\
\end{align*}
\]

31.

\[
\begin{align*}
3t &= \frac{13\pi}{12} + 2n\pi \\
t &= \frac{13}{3} \pi + \frac{2}{3} n\pi
\end{align*}
\]

33.

\[
\begin{align*}
2\sin^2 t - 5\sin t - 3 &= 0 \\
(2\sin t + 1)(\sin t - 3) &= 0 \\
The only solution comes from \\
(2\sin t + 1) &= 0 \\
\sin t &= -\frac{1}{2} \\
t &= \sin^{-1}(-\frac{1}{2}) \\
t &= \frac{7\pi}{6} + 2n\pi \\
\text{and} \\
t &= \frac{11\pi}{6} + 2n\pi
\end{align*}
\]

35.

\[
\begin{align*}
\cos^2 x + 5\cos x &= -6 \\
\cos^2 x + 5\cos x + 6 &= 0 \\
\cos x &= \frac{-5 \pm \sqrt{25 - 4(1)(6)}}{2} \\
&= \frac{-5 \pm 1}{2} \\
&= \frac{-5 - 1}{2} = -3 \\
&= \frac{-5 + 1}{2} = -2
\end{align*}
\]

Since both values are larger than one, then the equation has no solutions.

37. \(y = 2\sin 2t + 4\)

amplitude = 2, period = \(\frac{2\pi}{2} = \pi\), mid-line \(y = 4\)

maximum = \(4 + 2 = 6\), minimum = \(4 - 2 = 2\)

39. \(y = 5\cos(t/2) + 1\)

amplitude = 5, period = \(\frac{2\pi}{\frac{1}{2}} = 4\pi\), mid-line \(y = 1\)

maximum = \(1 + 5 = 6\), minimum = \(1 - 5 = -4\)

41. \(y = \frac{1}{2}\sin(3t) - 3\)

amplitude = \(\frac{1}{2}\), period = \(\frac{2\pi}{3}\), mid-line \(y = -3\)

maximum = \(-3 + \frac{1}{2} = -\frac{5}{2}\), minimum = \(-3 - \frac{1}{2} = -\frac{7}{2}\)

43. \(y = 4\sin(\pi t) + 2\)

amplitude = 4, period = \(\frac{2\pi}{\pi} = 2\), mid-line \(y = 2\)

maximum = \(2 + 4 = 6\), minimum = \(2 - 4 = -2\)
45. The maximum is 10 and the minimum is -3 so the amplitude is \( \frac{10 - (-3)}{2} = \frac{13}{2} \). The mid-line is \( y = 10 - 3 = 7 \), and the period is \( 2\pi \) (the distance from one peak to the next one) which means that \( b = \frac{2\pi}{2\pi} = 1 \). From the information above, and the graph, we conclude that the function is 
\[
y = \frac{13}{2} \sin t + 3 \]

47. The maximum is 1 and the minimum is -3 so the amplitude is \( \frac{1 - (-3)}{2} = 2 \). The mid-line is \( y = 1 - 2 = -1 \), and the period is \( 4\pi \) which means that \( b = \frac{2\pi}{4\pi} = \frac{1}{2} \). From the information above, and the graph, we conclude that the function is 
\[
y = 2\cos(\frac{1}{2}t) - 1 \]

49. 
\[
R = 0.339 + 0.088 \cos 40^\circ \cos 30^\circ \\
-0.196 \sin 40^\circ \sin 30^\circ - 0.182 \cos 9^\circ \cos 30^\circ \\
0.571945 \text{ mega joules/m}^2
\]

51. 
\[
R = 0.339 + 0.088 \cos 30^\circ \cos 55^\circ \\
-0.196 \sin 50^\circ \sin 55^\circ - 0.182 \cos 45^\circ \cos 55^\circ \\
0.234721 \text{ mega joules/m}^2
\]

53. Period is 5 so \( b = \frac{2\pi}{5}, k = 2500, a = 250 \). Therefore, the function is
\[
V(t) = 250 \cos \frac{2\pi t}{5} + 2500
\]

55. Since our lungs increase and decrease as we breathe then there is a maximum and minimum volume for the air capacity in our lungs. We have a regular period of time at which we breathe (inhal and exhale). These foreors are reasons why the cosine model is appropriate for describing lung capacity.

57. The frequency is the reciprocal of the period. Therefore, \( f = \frac{1}{5} \) \( 80 \pi \) 149 Hz.

59. The amplitude is given as 5.3, \( b = f = 2\pi \) where \( f \) is the frequency, \( b = 0.172 \cdot 2\pi = 1.08071 \), \( k = 143 \). Therefore, the function is
\[a(t) = 5.3 \cos(1.08071t) + 143 \]

61. \( x = \cos(140^\circ), y = \sin(140^\circ) \), \(-0.76601, 0.64279)\)

63. \( x = \cos(\frac{3\pi}{2}), y = \sin(\frac{3\pi}{2}) \), \(0.80902, -0.58779)\)

65. Rewrite \( 105^\circ \) as \( 45^\circ + 60^\circ \) and use a sum identity.
\[
\sin 105^\circ = \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ
\]
\[
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2} + \frac{\sqrt{6}}{2}
\]

67. a) From the graph we can see that the point with angle \( t \) has an opposite \( x \) and \( y \) coordinate than the point with angle \( t + \pi \). Since the \( x \) coordinate corresponds to the \( \cos \) of the angle which the point makes and the \( y \) coordinate corresponds to the \( \sin \) of the angle which the point makes it follows that \( \sin(t + \pi) = -\sin(t) \) and \( \cos(t + \pi) = -\cos(t) \).

b) 
\[
\sin(t + \pi) = \sin t \cos \pi + \cos t \sin \pi
= \sin t \cdot 1 + \cos t \cdot 0
= -\sin t
\]

and
\[
\cos(t + \pi) = \cos t \cos \pi - \sin t \sin \pi
= \cos t \cdot 1 - \sin t \cdot 0
= -\cos t
\]

c) 
\[
\tan(t + \pi) = \frac{\sin(t + \pi)}{\cos(t + \pi)}
= \frac{-\sin t}{-\cos t}
= \frac{\sin t}{\cos t} \tan t
\]

69. a) Since the radius of a unit circle is 1, the circumference of the unit circle is \( 2\pi \). Therefore any point \( t + 2\pi \) will have exactly the same terminal side as the point \( t \) that is to say that the points \( t \) and \( t + 2\pi \) are colinear on the unit circle. Therefore, \( \sin t = \sin(t + 2\pi) \) for all numbers \( t \).

b) \[g(t + 2\pi/b) = a\sin(b(t + 2\pi/b)) + k\]

from part a) \[a\sin(bt) + k\]

by definition \[g(t + 2\pi/b) = g(t)\]

c) Since the function evaluated at \( t + 2\pi/b \) has the same value as the function evaluated at \( t \) and \( 2\pi/b \neq 0 \) then \( t + 2\pi/b \) is evaluated after \( t \). Since we have a periodic function in \( g(t) \) it follows that the period of the function is implied to be \( 2\pi/b \).

71. Since the base, b, is small. \( f \) is large, and \( d \) is large, then the basilar membrane is affected mostly by high frequency sounds.

73. \[f = \frac{880(2^{9/12})}{2\pi} = 261.626\]

75. \[\frac{880(2^{9/12})}{2\pi} = 1760\]
\[\frac{9/12 - 1}{2} = 1700\]
\[\frac{9/12 - 1}{2} = 880\]
\[\frac{9/12 - 1}{2} = 2\]
Comparing exponents we can conclude that

\[
\frac{n}{12} - 1 = 1
\]
\[
\frac{n}{12} = 2
\]
\[
n = 24
\]

There are 24 notes above A above middle C.

77.

\[
\frac{880(2^{n/12})\pi}{2\pi} = 2200
\]
\[
2^{n/12-1} = \frac{2200}{880}
\]
\[
2^{n/12-1} = 2.5
\]
\[
\left(\frac{n}{12} - 1\right)\ln(2) = \ln(2.5)
\]
\[
\frac{n}{12} - 1 = \frac{\ln(2.5)}{\ln(2)}
\]
\[
\frac{n}{12} = \frac{\ln(2.5)}{\ln(2)} + 1
\]
\[
n = 12 \left( \frac{\ln(2.5)}{\ln(2)} + 1 \right)
\]
\[
n = 27.8631
\]

There are 28 notes above A above middle C.

79.  
   a) Left to the student
   b) \( y = \frac{1}{2} \cos(2t) + \frac{1}{2} \)
   c) We use the double angle identity obtained in Exercise 79 of Section 1.4 and solve for \( \cos^2(t) \) to obtain the model in part b).

81.  
   a) Left to the student
   b) Left to the student
   c) The horizontal shift moves every point of the original graph \( \frac{1}{4} \) units to the right.

83. Left to the student

85. Left to the student
Chapter 2
Differentiation

Exercise Set 2.1

1. The function is not continuous at \( x = 1 \) since the limit from the left of \( x \to 1 \) is not equal to the limit from the right of \( x \to 1 \) and therefore the limit of the function at \( x = 1 \) does not exist.

3. The function is continuous at every point in the given plot. Note that the graph can be traced without a jump from one point to another.

5. a) As we approach the \( x \)-value of 1 from the right we notice that the \( y \)-value is approaching a value of 1. Thus, \( \lim_{x \to 1^+} f(x) = 1 \). As we approach the \( x \)-value of 1 from the left we notice that the \( y \)-value is approaching a value of 2. Thus, \( \lim_{x \to 1^-} f(x) = 2 \). Since \( \lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \) then \( f(x) \) does not exist.

b) Reading the value from the graph \( f(1) = -1 \).

c) Since the \( \lim_{x \to 1} f(x) \) does not exist, then \( f(x) \) is not continuous at \( x = 1 \).

d) As we approach the \( x \)-value of -2 from the right we notice that the \( y \)-value is approaching a value of 3. Thus, \( \lim_{x \to -2^+} f(x) = 3 \). As we approach the \( x \)-value of -2 from the left we notice that the \( y \)-value is approaching a value of 3. Thus, \( \lim_{x \to -2^-} f(x) = 3 \). Since \( \lim_{x \to -2} f(x) = \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) \) then \( f(x) \) is continuous at \( x = -2 \).

e) Reading the value from the graph \( f(-2) = 3 \).

f) Since \( \lim_{x \to -2} f(x) = 3 \) and \( h(-2) = 0 \), then \( h(x) \) is not continuous at \( x = -2 \).

9. a) As we approach the \( x \)-value of 1 from the right we find that the \( y \)-value is approaching 3. Thus, \( \lim_{x \to 1^+} f(x) = 3 \)

b) As we approach the \( x \)-value of 1 from the left, we find that the \( y \)-value is approaching 3. Thus, \( \lim_{x \to 1^-} f(x) = 3 \)

c) Since \( \lim_{x \to 1} f(x) = 3 \) and \( \lim_{x \to 1^-} f(x) = 3 \) then \( \lim_{x \to 1^+} f(x) = 3 \)

d) From the given conditions \( f(1) = 2 \)

e) \( f(x) \) is not continuous at \( x = 1 \) since \( \lim_{x \to 1^-} f(x) \neq f(1) \)

f) \( f(x) \) is continuous at \( x = 2 \) since \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 2 \)

11. a) True. The values of \( g \) as we approach \( x = 0 \) from the right is the same as the value of the function at \( x = 0 \), which is 0.

b) True. The values of \( g \) as we approach \( x = 0 \) from the left is the same as the value of the function at \( x = 0 \), which is 0.

c) True. Since \( \lim_{x \to 0^+} f(x) = 0 \) and \( \lim_{x \to 0^-} f(x) = 0 \)

d) False. Since \( \lim_{x \to 0^+} f(x) = 0 \) and \( \lim_{x \to 0^-} f(x) = 1 \)

e) True. Since \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0 \)

f) False. Since \( \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) \)

g) True. Since \( \lim_{x \to 0} f(x) = 0 = f(0) \)

h) False. Since \( \lim_{x \to 0} f(x) \) does not exist.

13. a) False. As we approach the \( x \)-value of -2 from the right, we find that the \( y \)-value is approaching 2.

b) True. As we approach the \( x \)-value of -2 from the left, we find that the \( y \)-value is approaching 0.

c) False. Since \( \lim_{x \to -2^+} f(x) = 1 \) and \( \lim_{x \to -2^-} f(x) = 0 \)

d) False. Since \( \lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x) \)

e) False. Since \( \lim_{x \to -2^-} f(x) \) does not exist.

f) True. Since \( \lim_{x \to -2} f(x) = \lim_{x \to -2^-} f(x) = 0 \)

g) True. The graph indicates a point \( (0, 2) \) at \( (0, 2) \)

h) False. Since \( \lim_{x \to -2} f(x) \neq \lim_{x \to -2^-} f(x) \)

i) False. Since \( \lim_{x \to 0^+} f(x) \neq f(0) \)

j) True. Since \( \lim_{x \to -2} f(x) = \lim_{x \to -2^-} f(x) = f(-1) \)
19. a) True. As we approach the \( x \) value of 0 from the right we find that the \( y \) value is approaching 0, which is the value of the function at \( x = 0 \).

b) False. As we approach the \( x \) value of 0 from the left, we find that the \( y \) value is approaching 2 instead of 0.

c) False. Since \( \lim_{x \to 0^+} f(x) = 0 \) and \( \lim_{x \to 0^-} f(x) = 2 \)

\( \lim_{x \to 0} f(x) \) does not exist.

d) True. Since \( \lim_{x \to 2} f(x) = 4 = \lim_{x \to 2^+} f(x) \)

c) False. Since \( \lim_{x \to 2} f(x) \) does not exist.

e) True. Since \( \lim_{x \to 2} f(x) = 4 \) and \( \lim_{x \to 2^+} f(x) = 4 \)

g) False. Since \( \lim_{x \to 2} f(x) \) does not exist.

h) True. Since \( \lim_{x \to 2} f(x) = 4 \) and \( \lim_{x \to 2} f(x) = 4 \)

21. The function \( p \) is not continuous at \( x = 1 \) since the \( \lim_{x \to 1^{-}} p(x) \) does not exist. \( p \) is continuous at \( x = 1.5 \)

\( \lim_{x \to 1.5} p(x) = 0.6 \neq p(1.5) \). \( p \) is not continuous at \( x = 1 \) since the \( \lim_{x \to 1^{-}} p(x) \) does not exist. \( p \) is continuous at \( x = 2.01 \) since \( \lim_{x \to 2.01} p(x) = 0.8 = p(2.01) \).

22. \( \lim_{x \to 1} p(x) = 0.4 \), \( \lim_{x \to 1} p(x) = 0.6 \), therefore \( \lim_{x \to 1} p(x) \) does not exist.

23. \( \lim_{x \to 2} p(x) = 1 \) since \( \lim_{x \to 2.01} p(x) = 1 \), \( \lim_{x \to 2.01} p(x) = 1 \)

24. If we continue the pattern used for the taxi fare function, we see that for \( x = 2.3 \), which falls in the range of 2.2 and 2.4 miles, the fare will be $2.60, for \( x = 3 \), which falls in the range of 2.4 and 2.6 miles, the fare is $5.90.

For \( x = 2.6 \) and \( x = 3 \), we need to be careful since they act as a boundary of two possible fares. Therefore \( C \) is continuous at \( x = 2.3 \) since \( \lim_{x \to 2.3} C(x) = 5.60 = C(2.3) \). \( C \) is continuous at \( x = 2.5 \) since \( \lim_{x \to 2.5} C(x) = 5.90 = C(2.5) \).

\( C \) is not continuous at \( x = 2.6 \) since \( \lim_{x \to 2.6} C(x) = 6.20 \), \( \lim_{x \to 2.6} C(x) \) does not exist. \( C \) is not continuous at \( x = 2.5 \) since \( \lim_{x \to 2.5} C(x) \) does not exist. \( C \) is not continuous at \( x = 3 \) since \( \lim_{x \to 3} C(x) \) does not exist.

27. \( \lim_{x \to 0^+} C(x) = 2.30 \), \( \lim_{x \to 0^+} C(x) = 2.60 \).

29. \( \lim_{x \to 1^+} C(x) = 2.90 \), \( \lim_{x \to 1^+} C(x) = 2.90 \).

31. The population function, \( p(t) \), is discontinuous at \( t = 0.1 \), \( t = 0.3 \), \( t = 0.4 \), \( t = 0.5 \), \( t = 0.6 \), and at \( t = 0.8 \) since at these points the population function has a "jump" which means that the \( \lim_{t \to t^*} p(t) \) does not exist.

33. \( \lim_{t \to 1.5} p(t) = 12 \)

35. The population function, \( p(t) \), is discontinuous at \( t^* = 0.1 \), \( t^* = 0.3 \), \( t^* = 0.4 \), \( t^* = 0.5 \), \( t^* = 0.6 \), and at \( t^* = 0.8 \) since at these points the population function has a "jump" which means that the \( \lim_{t \to t^*} p(t) \) does not exist.

37. \( \lim_{t \to 0} p(t) = -35 \)

39. From the graph, the "I've got it" experience seems to occur after spending 20 hours on the task.

41. \( \lim_{t \to 20^+} N(t) = 100 \), \( \lim_{t \to 20^-} N(t) = 30 \), therefore \( \lim_{t \to 20} N(t) \) does not exist.

43. \( N(t) \) is discontinuous at \( t = 20 \) since \( \lim_{t \to 20^+} N(t) \) does not exist. \( N(t) \) is continuous at \( t = 30 \) since \( \lim_{t \to 30} N(t) = 100 = N(30) \).

45. A function may not be continuous if the function is not defined at one of the points in the domain, it may also not be continuous if the limit at a point does not exist, it also may not be continuous if the limit at a point is different than the value of the function at that point.

NOTE: See the graphs on page 77.

47. \( f(x) \) is continuous by C1 and C2, \( \lim_{x \to 1} f(x) = 0 \)

49. \( g(x) \) is continuous by C1, \( \lim_{x \to 1} g(x) = 1 \)

51. \( \cot x \) is continuous by C5, \( \lim_{x \to \frac{\pi}{2}} \cot x = \frac{1}{\sqrt{3}} \)

53. \( \csc x \) is continuous by C5, \( \lim_{x \to \frac{\pi}{2}} \csc x = \frac{1}{\sqrt{2}} \)

55. \( f(x) \) is continuous by C5, \( \lim_{x \to \frac{\pi}{2}} f(x) = \frac{\sqrt{3}}{\sqrt{2}} \)

57. \( g(x) \) is continuous by C5, \( \deflimit_{x \to \frac{\pi}{2}} g(x) = -\frac{1}{2} \)

59. Limit approaches 0

61. Limit approaches 1

63. Limit does not exist

Exercise Set 2.3

1. \( x^2 - 3 \) is a continuous function (it is a polynomial). Therefore, we can use direct substitution.

\[
\lim_{x \to 1} (x^2 - 3) = (1)^2 - 3 = 1 - 3 = -2
\]

3. The function \( f(x) = \frac{x}{x} \) is not continuous at \( x = 0 \) since the denominator equals zero. There are no algebraic simplifications that can be done to the function. To find the limit, we can either plug points that are approaching 0 from the right and the left and determine the limit from each side, or we can use the graph of the function to determine the limit (if it exists). Looking at the graph, we see that as \( x \)}
approaches 0 from the left, the $g$ values are becoming more and more negative, and as $x$ approaches 0 from the right, the $g$ values are becoming more and more positive. Therefore, since 
\[
\lim_{x \to 0^-} g(x) \neq \lim_{x \to 0^+} g(x),
\]
then \( \lim_{x \to 0} g(x) \) does not exist.

5. \( 2x + 5 \) is a continuous function (it is a polynomial). Therefore, we can use direct substitution.
\[
\lim_{x \to 3} (2x + 5) = 2(3) + 5 = 6 + 5 = 11
\]

7. The function \( \frac{x^2 - 25}{x + 5} \) is discontinuous at \( x = -5 \), but it can be simplified algebraically.
\[
\frac{x^2 - 25}{x + 5} = \frac{(x - 5)(x + 5)}{x + 5} = x - 5
\]
Therefore, \( \lim_{x \to -5} \frac{x^2 - 25}{x + 5} = \lim_{x \to -5} (x - 5) = -5 - 5 = -10 \)

9. Since \( \frac{1}{x} \) is continuous at \( x = -2 \) we can use direct substitution.
\[
\lim_{x \to -2} \frac{1}{x} = \frac{1}{-2} = -\frac{1}{2}
\]

11. The function \( \frac{\sqrt{x^2 + 9} - 3}{x - 2} \) is discontinuous at \( x = 2 \), but it can be simplified algebraically. The limit is then computed as follows:
\[
\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \frac{(x - 2)(x + 3)}{x - 2} = \frac{(x + 3)}{1} = x + 3
\]

13. Since \( \sqrt{x^2 - 17} \) is continuous at \( x = 5 \) we can use direct substitution.
\[
\lim_{x \to 5} \sqrt{x^2 - 17} = \sqrt{5^2 - 17} = \sqrt{25 - 17} = \sqrt{8} = 2
\]

15. \( \frac{\pi}{4} \sin \left( \frac{\pi}{4} \right) \) is \( \frac{\pi}{4} \sin \left( \frac{\pi}{4} \right) = \frac{\pi}{4} + \frac{1}{\sqrt{2}} \)

17. \( \lim_{x \to 0} \frac{1 + \sin x}{x} = \frac{1 + 0}{0} = 1 \)

19. Using the graph of \( \frac{1}{x^2} \) we find that the limit as \( x \) approaches 2 does not exist since the limit from the left of \( x - 2 \) does not equal the limit from the right of \( x - 2 \)

21. Since \( \frac{x^2 - 4x + 3}{x - 2} \) is continuous at \( x = 2 \) we can use direct substitution.
\[
\lim_{x \to 2} \frac{x^2 - 4x + 3}{x - 2} = \frac{3(2)^2 - 4(2) + 3}{2 - 2} = \frac{12 - 8 + 3}{0} = \frac{5}{0}
\]

23. The function \( \frac{x^2 + x - 6}{x^2 - 4} \) is discontinuous at \( x = 2 \). But we can simplify it algebraically first then find the limit as follows.
\[
\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{x + 3}{x + 2} = \frac{5}{4}
\]

25. Since we have a limit in terms of \( h \), we can treat \( x \) as a constant. To evaluate the limit we can use direct substitution (we have a polynomial in \( h \), which is continuous for all values of \( h \)).
\[
\lim_{h \to 0} \frac{6c^2 + 6ch + 2h^2}{6c^2 + 6h(0) + 2(0)^2} = \frac{6c^2 + 6c(0) + 2(0)^2}{6c^2 + 6h(0) + 2(0)^2}
\]

27. Since we have a limit in terms of \( h \), we can treat \( x \) as a constant. Since \( \frac{2x - h}{x^2(x + h)^2} \) is continuous at \( h = 0 \) we can use direct substitution.
\[
\lim_{h \to 0} \frac{-2x - h}{x^2(x + h)^2} = \frac{-2x - 0}{x^2(x + 0)^2} = \frac{-2x}{x^2} = -\frac{2}{x}\]
\[
\begin{align*}
29. \quad \lim_{x \to 0} \frac{\tan x}{x} &= \lim_{x \to 0} \frac{\sin x}{x} \cos x = 1 \cdot 1 = 1 \quad \text{Recall that } \lim_{x \to 0} \frac{\sin x}{x} = 1. \\
31. \quad \lim_{h \to 0} \frac{\sin x \cdot \sin h}{h} &= \sin x \lim_{h \to 0} \frac{\sin h}{h} = \sin x \cdot 1 = \sin x
\end{align*}
\]

33. \[
\lim_{x \to 0} \frac{x^2 + 3x}{x - 2x^3} = \lim_{x \to 0} \frac{x(x + 3)}{x(1 - 2x^3)} = \lim_{x \to 0} \frac{x + 3}{1 - 2x^3} = \frac{9 + 3}{1} = \frac{3}{1} = 3
\]

35. \[
\lim_{x \to 0} \frac{x \sqrt{x}}{x + 3x^2} = \lim_{x \to 0} \frac{x\sqrt{x}}{x(1 + x)} = \lim_{x \to 0} \frac{\sqrt{x}}{1 + x} = \frac{0}{1} = 0
\]

37. \[
\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 1)} = \lim_{x \to 2} \frac{1}{x + 1} = \lim_{x \to 2} \frac{1}{2 + 1} = \frac{1}{3} = 3
\]

39. \[
\lim_{x \to 3} \frac{x^2 - 9}{2x - 6} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{2(x - 3)} = \lim_{x \to 3} \frac{x + 3}{2} = \frac{6 + 3}{2} = \frac{3}{2} = 3
\]

41. \[
\frac{a^2 - 4}{\sqrt{a^2 + 5} - 3} = \frac{\sqrt{a^2 + 5} + 3}{\sqrt{a^2 + 5} + 3} \cdot \frac{a^2 - 4}{\sqrt{a^2 + 5} + 3}
\]

Thus, \[
\lim_{a \to -2} \frac{\sqrt{a^2 + 5} + 3}{3} = 6
\]

43. \[
\frac{\sqrt{3} - x - \sqrt{3}}{x} = \frac{x}{x} \cdot \frac{\sqrt{3} - x + \sqrt{3}}{\sqrt{3} - x + \sqrt{3}} = \frac{3 - x - 3}{\sqrt{3} - x + \sqrt{3}} = \frac{-1}{\sqrt{3} - x + \sqrt{3}}
\]

Thus, \[
\lim_{x \to 0} \frac{-1}{\sqrt{3} - x + \sqrt{3}} = \frac{-1}{2\sqrt{3}}
\]

45. Limit approaches \(\frac{4}{5}\)

47. \[
\frac{2 - \sqrt{x}}{4 - x} = \frac{2 - \sqrt{x}}{4 - x} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}}
\]

Thus, \[
\lim_{x \to 3} \frac{1}{2 + \sqrt{x}} = \frac{1}{4}
\]

Exercise Set 2.3

1. a) First we obtain the expression for \(f(x + h)\) with \(f(x) = 7x^2\)

\[
f(x + h) = 7(x + h)^2 = 7(x^2 + 2hx + h^2) = 7x^2 + 14xh + 7h^2
\]

Then

\[
\frac{f(x + h) - f(x)}{h} = \frac{(7x^2 + 14xh + 7h^2) - 7x^2}{h} = \frac{14xh + 7h^2}{h} = h(14x + 7h)
\]

b) For \(x = 4\) and \(h = 2\),

\[
14x + 7h = 14(4) + 7(2) = 56 + 14 = 70
\]

For \(x = 4\) and \(h = 1\),

\[
14x + 7h = 14(4) + 7(1) = 56 + 7 = 63
\]

For \(x = 4\) and \(h = 0.1\),

\[
14x + 7h = 14(4) + 7(0.1) = 56 + 0.7 = 56.7
\]

For \(x = 4\) and \(h = 0.01\),

\[
14x + 7h - 14(4) + 7(0.01) = 56 + 0.07 = 56.07
\]
3. a) First we obtain the expression for \(f(x + h)\) with
\[
f(x + h) = -7x^3
\]
\[
\frac{f(x + h) - f(x)}{h} = \frac{-7x^3 - 14xh - 7h^2}{h} - \frac{-7x^2 - 14xh - 7h^2}{h}
\]
Then
\[
\frac{f(x + h) - f(x)}{h} = \frac{-14xh - 7h^2}{h}
\]
\[
= \frac{-14x - 7h}{h}
\]
\[
= -14x - 7h
\]

b) For \(x = 4\) and \(h = 2\),
\[-14x - 7h = -14(4) - 7(2) = -56 - 14 = -70\]
For \(x = 4\) and \(h = 1\),
\[-14x - 7h = -14(4) - 7(1) = -56 - 7 = -63\]
For \(x = 4\) and \(h = 0.1\),
\[-14x - 7h = -14(4) - 7(0.1) = -56 - 0.7 = -56.7\]
For \(x = 1\) and \(h = 0.01\),
\[-14x - 7h = -14(1) - 7(0.01) = -56 - 0.07 = -56.07\]

5. a) First we obtain the expression for \(f(x + h)\) with \(f(x) = x^3\)
\[
f(x + h) = (x + h)^3
\]
\[
= x^3 + 3x^2h + 3xh^2 + h^3
\]
Then
\[
\frac{f(x + h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}
\]
\[
= \frac{21x^2h + 21xh^2 + 7h^3}{h}
\]
\[
= \frac{21x^2 + 21xh + 7h^2}{h}
\]
\[
= 21x^2 + 21xh + 7h^2
\]

b) For \(x = 4\) and \(h = 2\),
\[21x^2 + 21xh + 7h^2 = 21(4)^2 + 21(4)(2) + 7(2)^2 = 336 + 168 + 28 = 532\]
For \(x = 4\) and \(h = 1\),
\[21x^2 + 21xh + 7h^2 = 21(4)^2 + 21(4)(1) + 7(1)^2 = 336 + 84 + 7 = 427\]

7. a) First we obtain the expression for \(f(x + h)\) with \(f(x) = \frac{5}{x}\)
\[
f(x + h) = \frac{5}{x + h}
\]
Then
\[
\frac{f(x + h) - f(x)}{h} = \frac{\frac{5}{x + h} - \frac{5}{x}}{h}
\]
\[
= \frac{\frac{5}{x + h} \cdot x(x + h) - \frac{5}{x} \cdot x(x + h)}{h(x(x + h))}
\]
\[
= \frac{5x - 5x}{hx(x + h)}
\]
\[
= \frac{-5x}{hx(x + h)}
\]
\[
= \frac{-5}{x(x + h)}
\]

b) For \(x = 4\) and \(h = 2\),
\[\frac{-5}{x(x + h)} = \frac{-5}{4(4 + 2)} = \frac{-5}{26} \approx -0.208\]
For \(x = 4\) and \(h = 1\),
\[\frac{-5}{x(x + h)} = \frac{-5}{4(4 + 1)} = \frac{-5}{20} \approx -0.25\]
For \(x = 4\) and \(h = 0.1\),
\[\frac{-5}{x(x + h)} = \frac{-5}{4(4 + 0.1)} = \frac{-5}{15.4} \approx -0.325\]
For \(x = 4\) and \(h = 0.01\),
\[\frac{-5}{x(x + h)} = \frac{-5}{4(4 + 0.01)} = \frac{-5}{15.04} \approx -0.324\]

9. a) First we obtain the expression for \(f(x + h)\) with \(f(x) = -2x + 5\)
\[
f(x + h) = -2x + 5
\]
\[
\frac{f(x + h) - f(x)}{h} = \frac{(-2x - 2h + 5) - (-2x + 5)}{h}
\]
\[
= \frac{-2h}{h}
\]
\[
= -2
\]
b) Since the difference quotient is a constant, then the value of the difference quotient will be \(-2 \) for all the values of \( x \) and \( h \).

11. a) First we obtain the expression for \( f(x + h) \) with \( f(x) = x^2 - x \)

\[
f(x + h) = (x + h)^2 - (x + h) = x^2 + 2xh + h^2 - x - h
\]

Then

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - x - h) - (x^2 - x)}{h} = \frac{2xh + h^2 - h}{h} \cdot \frac{\lambda(2x + h - 1)}{h} = 2x + h - 1
\]

b) For \( x = 4 \) and \( h = 2 \),

\[
2x + h - 1 = 2(4) + 2 - 1 = 8 + 2 - 1 = 9
\]

For \( x = 4 \) and \( h = 1 \),

\[
2x + h - 1 = 2(4) + 2 - 1 = 8 + 1 - 1 = 8
\]

For \( x = 4 \) and \( h = 0.1 \),

\[
2x + h - 1 = 2(4) + 0.1 - 1 = 8 + 0.1 - 1 = 7.1
\]

For \( x = 4 \) and \( h = 0.01 \),

\[
2x + h - 1 = 2(4) + 0.01 - 1 = 8 + 0.01 - 1 = 7.01
\]

13. a) For the average growth rate during the first year we use the points \((0, 7.9)\) and \((12, 22.4)\)

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{22.4 - 7.9}{12 - 0} = \frac{14.5}{12} \approx 1.20834 \text{ pounds per month}
\]

b) For the average growth rate during the second year we use the points \((12, 22.4)\) and \((24, 27.8)\)

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{27.8 - 22.4}{24 - 12} = \frac{5.4}{12} = 0.45 \text{ pounds per month}
\]

c) For the average growth rate during the third year we use the points \((24, 27.8)\) and \((36, 31.5)\)

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{31.5 - 27.8}{36 - 24} = \frac{3.7}{12} \approx 0.30834 \text{ pounds per month}
\]

d) For the average growth rate during his first three years we use the points \((0, 7.9)\) and \((36, 31.5)\)

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{31.5 - 7.9}{36 - 0} = \frac{23.6}{36} \approx 1.967 \text{ pounds per month}
\]

e) The graph indicates that the highest growth rate out of the first three years of a boy’s life happens at birth (that is were the graph is the steepest).

15. a) For the average growth rate between ages 12 and 18 months

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{25.9 - 22.4}{18 - 12} = \frac{3.5}{6} = 0.583 \text{ pounds per month}
\]

b) For the average growth rate between ages 12 and 14 months (we use the point at 15 months)

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{21.5 - 22.4}{15 - 12} = \frac{0.9}{3} = 0.7 \text{ pounds per month}
\]

c) For the average growth rate between ages 12 and 13 months (we can approximate the value of \( y \) when \( x = 13 \) by reading it from the graph)

\[
\frac{y_2 - y_1}{x_2 - x_1} \approx \frac{23.21 - 22.4}{15 - 12} \approx \frac{0.83}{1} \approx 0.83 \text{ pounds per month}
\]

d) The average growth of a typical boy when he is 12 months old is about 0.9 pounds per month

17. a) Average rate of change from \( t = 0 \) to \( t = 8 \)

\[
\frac{N_2 - N_1}{t_2 - t_1} = \frac{10 - 0}{8 - 0} = \frac{10}{8} = 1.25 \text{ words per minute}
\]

Average rate of change from \( t = 8 \) to \( t = 16 \)

\[
\frac{N_2 - N_1}{t_2 - t_1} = \frac{20 - 10}{16 - 8} = \frac{10}{8} = 1.25 \text{ words per minute}
\]

Average rate of change from \( t = 16 \) to \( t = 24 \)

\[
\frac{N_2 - N_1}{t_2 - t_1} = \frac{25 - 20}{24 - 16} = \frac{5}{8} = 0.625 \text{ words per minute}
\]
Average rate of change from \( t = 21 \) to \( t = 32 \)
\[
\frac{N_2 - N_1}{t_2 - t_1} = \frac{25 - 25}{32 - 24} = \frac{0}{8} = 0 \text{ words per minute}
\]

Average rate of change from \( t = 32 \) to \( t = 36 \)
\[
\frac{N_2 - N_1}{t_2 - t_1} = \frac{25 - 25}{36 - 32} = \frac{0}{4} = 0 \text{ words per minute}
\]

b) The rate of change becomes 0 after 24 minutes because the number of words memorized does not change and remains at 25 words, that means that there is no change in the number of words memorized after 24 minutes.

19. a) When \( t = 3 \), \( s = 16(3)^2 = 16(9) = 144 \) feet
b) When \( t = 5 \), \( s = 16(5)^2 = 16(25) = 400 \) feet
c) Average velocity = \( \frac{400 - 144}{5 - 3} = \frac{256}{2} = 128 \) feet per second

21. a) Population A: The average growth rate = \( \frac{500 - 0}{4 - 0} = \frac{500}{4} = 125 \text{ million per year} \)
Population B: The average growth rate = \( \frac{500 - 0}{4 - 0} = \frac{500}{4} = 125 \text{ million per year} \)
b) We would not detect the fact that the population grows at different rates. The calculation shows the populations growing at the same average growth rate, since for either population we used the same points to calculate the average growth rate (0,0), and (4, 500).

c) Population A:
Between \( t = 0 \) and \( t = 1 \), Average Growth Rate = \( \frac{250 - 0}{1 - 0} = 250 \text{ million per year} \)
Between \( t = 1 \) and \( t = 2 \), Average Growth Rate = \( \frac{125 - 250}{2 - 1} = -95 \text{ million per year} \)
Between \( t = 2 \) and \( t = 3 \), Average Growth Rate = \( \frac{50 - 125}{3 - 2} = -50 \text{ million per year} \)
Between \( t = 3 \) and \( t = 4 \), Average Growth Rate = \( \frac{0 - 50}{4 - 3} = -30 \text{ million per year} \)

Population B:
Between \( t = 0 \) and \( t = 1 \), Average Growth Rate = \( \frac{250 - 0}{1 - 0} = 250 \text{ million per year} \)
Between \( t = 1 \) and \( t = 2 \), Average Growth Rate = \( \frac{125 - 250}{2 - 1} = -95 \text{ million per year} \)
Between \( t = 2 \) and \( t = 3 \), Average Growth Rate = \( \frac{50 - 125}{3 - 2} = -50 \text{ million per year} \)
Between \( t = 3 \) and \( t = 4 \), Average Growth Rate = \( \frac{0 - 50}{4 - 3} = -30 \text{ million per year} \)

d) It is clear from part (c) that the first population has different growing rates depending on which interval of time we choose. Therefore, the statement “the population grew by 125 million each year” does not convey how population went through periods were the population increased and periods were the population decreased.

23. The rate of change in the period between 1800 and 1850 is similar to that of 1930 to 1950 in the sense that they both exhibit steady increases in the population. The drastic drop in the population shortly after 1850 is similar to the drop in population seen near 1975.

25.
\[
\frac{f(x + h) - f(x)}{h} = \frac{a(x + 1)^2 + bx + c}{h} - \frac{ax^2 + bx + c}{h} = \frac{a(x^2 + 1) - ax}{h} + \frac{x + c}{h}
\]

27.
\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} = \frac{h}{\sqrt{x + h} + \sqrt{x}}
\]

29.
\[
\frac{f(x + h) - f(x)}{h} = \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h} = \frac{x^2 - (x + h)^2}{h(x^2(x + h)^2)} = \frac{x^2 - (x + h)^2}{h^2(x^2 + 1)} = \frac{2x - h}{x^2(x + h)^2}
\]

31.
\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)(x + 1)}{(x + h)(x + 1) - x(x + h)} = \frac{x^2 + x + h + x - x^2 - xh}{h(1 + x)(1 + x + h)} = \frac{1}{(1 + x)(1 + x + h)}
\]
33. 
\[
\frac{f(x + h) - f(x)}{h} = \frac{\frac{1}{\sqrt{x + h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\sqrt{x} - \sqrt{x + h}}{h \sqrt{x} \sqrt{x + h}} = \frac{\sqrt{x} - \sqrt{x + h}}{h \sqrt{x} \sqrt{x + h}} \frac{\sqrt{x} + \sqrt{x + h}}{\sqrt{x} + \sqrt{x + h}} = \frac{x - (x + h)}{h \sqrt{x} \sqrt{x + h} (\sqrt{x} + \sqrt{x + h})} = -\frac{1}{\sqrt{x} \sqrt{x + h} (\sqrt{x} + \sqrt{x + h})}
\]

Exercise Set 2.4
1. a-b) \( f(x) = 5x^2 \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{5(x + h)^2 - 5x^2}{h} = \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h} = \lim_{h \to 0} \frac{h(10x + 5h)}{h} = \lim_{h \to 0} 10x + 5h = 10x
\]

\[d) f'(-2) = 10(-2) = -20, \\
f'(0) = 10(0) = 0, \\
f'(1) = 10(1) = 10. \text{ These slopes are in agreement with the slopes of the tangent lines drawn in part (b).}
\]

3. a-b) \( f(x) = -5x^2 \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{-5(x + h)^2 - (-5x^2)}{h} = \lim_{h \to 0} \frac{-5x^2 - 10xh - 5h^2 - (-5x^2)}{h} = \lim_{h \to 0} \frac{h(-10x - 5h)}{h} = \lim_{h \to 0} -10x - 5h = -10x
\]

\[d) f'(-2) = -10(-2) = 20, \\
f'(0) = -10(0) = 0, \\
f'(1) = -10(1) = -10. \text{ These slopes are in agreement with the slopes of the tangent lines drawn in part (b).}
\]

5. a-b) \( f(x) = x^3 \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2
\]
7. a-b) \( f(x) = 2x + 3 \)

c)
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h) + 3 - (2x + 3)}{h} = \lim_{h \to 0} \frac{2h}{h} = 2
\]

\( f'(1) = 2 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

d) \( f'(-2) = -2 \)
\( f'(0) = 2 \)
\( f'(1) = 2 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

9. a-b) \( f(x) = 4x \)

c)
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4(x+h) - 4x}{h} = \lim_{h \to 0} \frac{4h}{h} = 4
\]

\( f'(1) = 4 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

d) \( f'(-2) = -3 \)
\( f'(0) = 4 \)
\( f'(1) = 4 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

11. a-b) \( f(x) = x^2 + x \)

c)
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} = \lim_{h \to 0} \frac{2xh + h^2 + x + h + 1}{h} = \lim_{h \to 0} \frac{2xh + h^2 + x + h + 1}{2x + 1}
\]

\( f'(0) = 0 \)
\( f'(1) = 3 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

13. a-b) \( f(x) = 2x^2 + 3x - 2 \)

c)
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 + 3(x+h) + 2}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 + 3h}{h} = \lim_{h \to 0} \frac{4x + 2h + 3}{1}
\]

\( f'(0) = 3x \)
\( f'(1) = 3 \). These slopes are in agreement with the slopes of the tangent lines drawn in part (b).
15. a-b) \( f(x) = \frac{1}{x} \)

c) \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{1}{x + h} - \frac{1}{x} \]
\[ = \lim_{h \to 0} \frac{x - (x + h)}{hx(x + h)} \]
\[ = \lim_{h \to 0} \frac{-1}{x(x + h)} \]
\[ = \frac{-1}{x^2} \]

d) \( f'(-2) = -\frac{1}{2}^2 = \frac{1}{4} \)
\( f'(0) = \) does not exist.
\( f'(1) = \) does not exist.
These slopes are in agreement with the slopes of the tangent lines drawn in part (b).

17. \( f(x) = \max \)

19. \( f(x) = x^2 \). From Example 3, \( f'(x) = 2x \). For the point \((3,9)\) we have \( f'(3) = 2(3) = 6 = m \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 9 &= 6(x - 3) \\
y &= 6x - 18 + 9 \\
y &= 6x - 9
\end{align*}
\]
For the point \((-1,1)\) we have \( f'(-1) = 2(-1) = -2 \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 1 &= -2(x - (-1)) \\
y - 1 &= -2x - 2 \\
y &= -2x - 1
\end{align*}
\]
For the point \((10,100)\) we have \( f'(10) = 2(10) = 20 \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 100 &= 20(x - 10) \\
y &= 20x - 200 + 100 \\
y &= 20x - 100
\end{align*}
\]
21. From Exercise 14, \( f'(x) = -\frac{x}{x^2} \). For the point \((1,5)\) we have \( f'(1) = -\frac{1}{5} = -5 = m \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 5 &= -5(x - 1) \\
y &= -5x + 5 + 5 \\
y &= -5x + 10
\end{align*}
\]
For the point \((-1,5)\) we have \( f'(-1) = -\frac{1}{1^2} = -1 \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 5 &= -1(x - 1) \\
y &= -5x - 5 + 5 \\
y &= -5x - 10
\end{align*}
\]
For the point \((100,0.05)\), which can be rewritten as \((100,\frac{1}{20})\), we have \( f'(100) = -\frac{1}{2000} = -\frac{5}{100} \). So the equation of the tangent line is
\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - \frac{1}{20} &= -\frac{1}{2000}(x - 100) \\
y &= -\frac{x}{2000} + \frac{1}{20} + \frac{1}{20} \\
y &= -\frac{x}{2000} + \frac{3}{20} \\
y &= -\frac{x}{2000} + \frac{1}{10}
\end{align*}
\]
23. First let us find the expression for \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - (x + h)^2 - (4 - x^2)}{h} = \lim_{h \to 0} \frac{4 - x^2 - 2ch - h^2 - 4 + x^2}{h} = \lim_{h \to 0} \frac{-2ch - h^2}{h} = \lim_{h \to 0} \frac{-2ch}{h} = \lim_{h \to 0} (-2x - h) = -2x
\]

For the point \((-1, 3)\) we have \( f'(-1) = -2(-1) = 2 = m \). So the equation of the tangent line is

\[
y - y_1 = m(x - x_1) = 2(x + 1) = 2x + 2
\]

For the point \((0, 4)\) we have \( f'(0) = -2(0) = 0 \). So the equation of the tangent line is

\[
y - y_1 = m(x - x_1) = 0(x - 0) = 0
\]

For the point \((5, -21)\) we have \( f'(5) = -2(5) = -10 \). So the equation of the tangent line is

\[
y - y_1 = m(x - x_1) = -10(x - 5) = -10x + 50
\]

25. The function is not differentiable at \( x_4 \) since it is discontinuous, \( x_5 \) since it has a corner, \( x_6 \) since it has a corner, and \( x_7 \) since it has a vertical tangent.

27. The function is not differentiable at integer values of \( x \) since the function is not continuous at integer values of \( x \).

29. The function is differentiable for all values in the domain.

31. The function is differentiable for all values in the domain.

33. As the points \( Q \) get closer to \( P \) the secant lines are getting closer to the tangent line at point \( P \).

35.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(h + x + 1)} - \frac{1}{2x}}{h} = \lim_{h \to 0} \frac{\frac{x^2 - x - 2xh - h^2}{h(x + 2x + h + 1)}}{h} = \lim_{h \to 0} \frac{-h(2x - h)}{h(x + 2x + h + 1)} = \lim_{h \to 0} \frac{-2x}{x^2(2x + 2x + h + 1)} = \frac{-2}{x^3}
\]

37.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{e^{xh}}{x + h} - \frac{e^x}{x}}{h} = \lim_{h \to 0} \frac{(x + h)(1 + x) - x(1 + x + h)}{h(1 + x)(1 + x + h)} = \lim_{h \to 0} \frac{x + x^2 + h + hh - x-x^2 - xh}{h(1 + x)(1 + x + h)} = \lim_{h \to 0} \frac{h}{h(1 + x)(1 + x + h)} = \lim_{h \to 0} \frac{1}{(1 + x)(1 + x + h)} = \frac{1}{(1 + x)^2}
\]

39.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x + h} - \sqrt{x}} = \lim_{h \to 0} \frac{1}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{\sqrt{x + h} + \sqrt{x}}{h(x + x + h)} = \lim_{h \to 0} \frac{-1}{\sqrt{x + h} + \sqrt{x}(\sqrt{x + h})} = \frac{-1}{2\sqrt{x}}
\]

41. The function \( f(x) \) will be not differentiable at \( x = -3 \).

43. \( f'(-2) = 12 \), \( f'(0) = 0 \), \( f'(4) = 48 \)

45. \( f'(-1) = -2 \), \( f'(2) = \frac{-2}{3} \), \( f'(0) = \frac{-2}{10} \)

47. \( f'(-2) = -6 \), \( f'(1) = 0 \), \( f'(4) = 6 \)
Exercise Set 2.5

1. \( \frac{dy}{dx} = 7x^2 - 1 = 7x^6 \)

2. \( \frac{dy}{dx} = 3 - 2x^2 - 1 = 6x \)

3. \( \frac{dy}{dx} = 4 \cdot 3x^{3-1} = 12x^2 \)

4. \( \frac{dy}{dx} = 3 \cdot \frac{3}{2}x^{3/1} - 1 = 2x^{1/1} \)

5. Rewrite as \( y = x^{1/4} \). \( \frac{dy}{dx} = \frac{1}{4}x^{-3/4} - \frac{3}{2}x^{-1/4} \)

11. \( \frac{dy}{dx} = 4 \cos x \)

12. \( \frac{dy}{dx} = \cos x - 1x^{-2} - \cos x - \frac{x}{2} \)

13. \( \frac{dy}{dx} = 2(2x + 1)^{2-1}(2) = 4(2x + 1) \)

14. \( f'(x) = 0.25(3.2x^2 - 1) - 0.8x^2 \)

15. \( f'(x) = 2 \times 2 \sin x \times 12 \sin x \)

16. \( f'(x) = -\sqrt{x} \sin x \)

17. Rewrite as \( f(x) - 5x^{-1} + \frac{1}{x} \), \( f'(x) = -5x^{-2} + \frac{1}{x} \)

18. \( f'(x) = -2x^{-1/2} - x^{-3/1} - \frac{1}{2}x^{-5/4} - \frac{1}{2}x^{-3/2} \)

19. \( f(x) = x^{1/4} - x^{3/5} \)

20. \( f'(x) = -\frac{5}{2}x^{-3/4} - \frac{3}{2}x^{-2/4} \)

21. \( f(x) = 4x - 3 - 2x^{-1} \)

22. \( f'(x) = 4x + 2x^{-1} \)

23. \( p(x) = 3x^{3/4} - 2x^{-2} \)

24. \( p'(x) = -24x^{-5} + 6x^{-1} \)

25. \( s(x) = 3\sqrt{2} \cos x - 2\sqrt{2} \sin x \)

26. \( s'(x) = -3\sqrt{2} \sin x - 2\sqrt{2} \cos x \)

27. \( q'(x) = 8 \left( \frac{\sqrt{x}}{x} \right) \cos x + \sqrt{2} \left( \frac{\sqrt{x}}{x} \right) \sin x \)

28. \( U(x) = \sin x \times 2x^{1/2} - 3x^{-1/2} \)

29. \( U'(x) = \cos x - x^{-1/2} - \frac{1}{2}x^{-1/2} \)

30. \( f'(x) = -3 \sin x \)

31. The slope at (0, 4) is \( f'(0) = 0 \).

32. So the equation of the tangent line is

\[
\begin{align*}
\frac{y - y_1}{y - y_1} &= m(x - x_1) \\
y - y_1 &= 0(x - 0) \\
y - y_1 &= 0 \\
y &= 0
\end{align*}
\]

33. \( f''(x) = \frac{2}{3}x^{1/3} + \frac{3}{4}x^{-2/3} \)

The slope at (8, 20) is \( f''(8) = \frac{2}{3}(8)^{1/3} + \frac{3}{4}(8)^{-2/3} = \frac{12}{9} \)

So the equation of the line is

\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 20 &= \frac{17}{9}(x - 8) \\
y &= \frac{17x}{9} - \frac{68}{3} + 20 \\
y &= \frac{17x}{9} - \frac{68}{3} + 20
\end{align*}
\]

34. Rewrite \( y = \left( \frac{x^{2/3}}{x^{1/3}} \right)^{1/3} = x^{-2/3} \)

35. \( \frac{dy}{dx} = \frac{5}{12}x^{-2/3} \sin x \)

36. Use the trigonometric identity \( 1 + \tan^2 x = \sec^2 x \) to rewrite \( y = 3\sec^2 x \cos x = 3 \sin x \)

37. \( \frac{dy}{dx} = -3 \sin x \)

38. Use the same identity for cosine to rewrite \( y = \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \)

39. \( \frac{dy}{dx} = -\frac{3}{2} \cos x + \frac{1}{2} \sin x \)

40. The slope at (1, 0) is \( \frac{dy}{dx} \big|_{x=1} = 0 \)

41. The equation of the line is

\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 0 &= 0(x - 1) \\
y &= 0
\end{align*}
\]

42. Let \( f(x) = \cos x \) then

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \cos(x + h) - \cos(x) \\
&= \lim_{h \to 0} \cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x) \\
&= \cos(x) \cdot \lim_{h \to 0} \cos(h) - \sin(x) \cdot \lim_{h \to 0} \sin(h) \\
&= -\sin(x)
\end{align*}
\]

43. Left to the student

44. The tangent line is horizontal at \( x = \pm 0.6922 \)

45. The tangent line is horizontal at \( x = -0.3456 \) and at \( x = 1.9289 \)
Exercise Set 2.6

1. a) \( r(t) = \frac{a(t)}{a} = 3t^2 + 1 \)
b) \( a(t) \) is not given.
c) \( \frac{dr}{dt} = 6t \quad \text{feet per second} \)
   \( a(t) = 6(1) = 6 \quad \text{feet per squared seconds} \)

2. a) \( v(t) = \frac{dr}{dt} = -20 \quad \text{feet per second} \)
b) \( a(t) \) is not given.
c) \( \frac{dv}{dt} = -20 \quad \text{feet per squared seconds} \)

3. a) \( r(t) = \frac{oa(t)}{oa} = 5 + 2\cos t \)
b) \( a(t) = \frac{oa(t)}{oa} = -2\sin t \)
c) Find \( r(\frac{\pi}{3}) \) and \( a(\frac{\pi}{3}) \)
   \[ r\left(\frac{\pi}{3}\right) = 5 + 2\cos\left(\frac{\pi}{3}\right) = 5 + 2 \cdot \frac{1}{2} = 6 \]
   \[ a\left(\frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{3}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3} \approx -1.732 \text{ m/sec} \]
d) Find \( t \) when \( r(t) = 3 \)
   \[ 3 = 5 + 2\cos t \]
   \[ -2 = 2\cos t \]
   \[ \cos t = -1 \]
   \[ t = \cos^{-1}(-1) = \pi + 2n\pi \]

4. a) \( \frac{dN}{dt} = -20 + 300 \)
b) Since \( a \) is counted in thousands we need to find \( N(10) \)
   \[ N(10) = -(10)^2 + 300(10) + 6 \]
   \[ = -100 + 3000 + 6 \]
   \[ = 2906 \]
   There will be 2906 units sold after spending $10000 on advertising.

c) At \( a = 10, \frac{dN}{da} = -2(10) + 300 = -20 + 300 = 280 \) units per thousand dollars spent on advertising.

d) The rate of change of the number of units sold depends on the amount spent on advertising according to the equation \( \frac{dN}{da} = -20 + 300 \) which means that for every 1 thousand dollars spent on advertising, there will be a change in the units sold of \(-20 \times 300\) units. If $10000 is spent on advertising, then there will be a 280 unit increase in the number of units sold.

9. a) \( \frac{dF}{dt} = 1.61 - 0.09968 \cdot 0.0018t^2 \)
b) \( a(0) = 7.60 + 1.61(0) - 0.0184(0)^2 + 0.0006(0)^3 = 7.60 \) pounds

c) \( \frac{dF}{dt}|_{t=0} = 1.61 - 0.09968(0) + 0.0184(0)^2 = 1.61 \) pounds per month

d) \( a(12) = 7.60 + 1.61(12) - 0.0184(12)^2 + 0.0006(12)^3 = 20.987 \) pounds

e) \( \frac{dF}{dt}|_{t=12} = 1.61 - 0.09968(12) + 0.0184(12)^2 = 0.708 \) pounds per month

f) Average rate of change: \( \frac{v(12) - v(9)}{12 - 9} = \frac{20.87 - 7.61}{12 - 9} = 6.116 \) pounds per month

g) \[
\frac{1.61 - 0.09968t + 0.0018t^2}{2(0.09968)} = \frac{0.0018t^2 - 0.0006t + 0.018t}{0.0018} = t
\]

11. a) \( \frac{dC}{dt} = 2\pi \)
b) For every increase of one centimeter of the radius the Healing wound circumference increases by 2\(\pi\) centimeters.

13. a) \( \frac{dT}{dt} = -0.2t + 1.2 \)
b) At \( t = 1.5 \)

\[ T(1.5) = -0.1(1.5)^2 + 1.2(1.5) + 98.6 \]
\[ = -0.1(2.25) + 1.2(1.5) + 98.6 \]
\[ = -0.225 + 1.8 + 98.6 \]
\[ = 99.175 \text{ degrees} \]

d) \( \frac{dT}{dt}|_{t=1.5} = -0.2(1.5) + 1.2 = 0.9 \) degrees per day

15. a) \( \frac{dC}{dt} = 1.311^{0.31} \)
b) For every increase of \( W \) in body weight the territorial area of an animal increases by an amount equal to \( 1.311^{0.31} \)

17. First rewrite \( R(Q) \) as follows \( R(Q) = \frac{5}{2}Q^2 - \frac{1}{3}Q^3 \)
a) \( \frac{dR}{dQ} = \frac{5}{2}Q - \frac{1}{3}Q^2 \) \( Q - Q^2 \)

b) For every increase \( Q \) in the dosage there will be a change in the reaction of the body that dosage change equal to the amount \( kQ - Q^2 \)

19. a) \( \frac{dA}{dt} = 0.08 \)
b) The rate of change for the median age of women at first marriage is constant at 0.08. That is, each year the median age of women at first marriage is increasing by 0.08 years

21. 'The average rate of change of a function is the value of the difference quotient evaluated over a period of time, while the instantaneous rate of change is the value of the slope of the tangent line at that particular instant.'
Exercise Set 2.7

1. Method One: \( x^3 \cdot x^8 = x^{11} \), \( \frac{dy}{dx} = 11x^{10} \)
   Method Two (product rule):
   \[
   \frac{dy}{dx} = x^3 \cdot 8x^7 + 3x^2 \cdot x^8 = 8x^{10} + 3x^{10} = 11x^{10}
   \]

3. Method One: \( x^{5/2} = x^{5/2} \), \( \frac{dy}{dx} = \frac{3}{2}x^{3/2} \)
   Method Two (product rule):
   \[
   \frac{dy}{dx} = x^{5/2} + (1)x^{3/2} = \frac{3}{2}x^{3/2} + \frac{3}{2}x^{3/2} = \frac{3}{2}x^{3/2}
   \]

5. Method One: \( \frac{d^2y}{dx^2} = x^3 \), \( \frac{dy}{dx} = 3x^2 \)
   Method Two (quotient rule):
   \[
   \frac{dy}{dx} = \frac{x^3(8x^7) - 5x^4(x^8)}{(x^8)^2} = \frac{8x^{12} - 5x^{12}}{x^{16}} = \frac{3x^{12}}{x^{16}} = \frac{3x^4}{x^4} = 3x^2
   \]

7. Method One: \( y = x^2 - 25 \), \( \frac{dy}{dx} = 2x \)
   Method Two (product rule):
   \[
   \frac{dy}{dx} = (x + 5)(1) + (1)(x - 5) = x + 5 + x - 5 = 2x
   \]

9. \( y = (8x^5 - 3x^2 + 20)(8x^4 - 3x^{1/2}) \)
   \[
   \frac{dy}{dx} = (8x^5 - 3x^2 + 20)(32x^3 - \frac{3}{2}x^{-1/2}) + (40x^4 - 6x)(8x^4 - 3x^{1/2}) = (8x^5 - 3x^2 + 20)(32x^3 - \frac{3}{2}x^{1/2}) + (40x^4 - 6x)(8x^4 - 3x^{1/2})
   \]

11. \( f(x) = (x^{1/2} - x^{1/3})/2x + 3 \)
   \[
   f'(x) = \left( \frac{1}{2}x^{1/2} - \frac{1}{3}x^{-2/3} \right)(2x + 3) + \left( \sqrt{x} - \sqrt[3]{x} \right)(2)
   = 2(\sqrt{x} - \sqrt[3]{x}) + (2x + 3) \left( \frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x}} \right)
   \]

13. \( f(x) = x^{1/2} \tan x \)
   \[
   f'(x) = x^{1/2} \sec^2 x \cdot \tan x \cdot \left( \frac{1}{2}x^{-1/2} \right) = \sqrt{x} \sec^2 x \cdot \left( \frac{1}{2\sqrt{x}} \right) \tan x
   \]

15. \((2x + 3)^2 = (2x + 3)(2x + 3) = 4x^2 + 12x + 9 \)
   \[
   f'(x) = 8x + 12
   \]

17. \( g(x) = (0.02x^2 + 1.3x - 11.7)(4.1x + 11.3) \)
   \[
   g'(x) = (0.02x^2 + 1.3x - 11.7)(4.1) + (0.04x + 1.3)(4.1x + 11.3)
   \]

19. \( g(x) = \sec x \, \csc x \)
   \[
   g'(x) = \sec x (-\csc x \cot x) + \csc x (\sec x \tan x) = \frac{-1}{\sin^2 x} - \frac{1}{\cos^2 x} = \sec^2 x - \csc^2 x
   \]
21. \( q(x) = \frac{\sin x}{1 + \cos x} \)

\[
q'(x) = \frac{(1 + \cos x)(\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}
\]

\[
= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}
\]

\[
= \frac{\cos x + 1}{1 + \cos x}
\]

\[
= \frac{1}{1 + \cos x}
\]

23. \( s(t) = \tan^2 t \)

\[
s'(t) = 2\tan t \cdot \sec^2 t
\]

25. \( f(x) = (e^x + 2e^{-x})(x^2 - 3) \)

\[
f'(x) = (e^x + 2e^{-x})(2x) - (1 - 2x^2)(x^2 - 3)
\]

\[
= 2e^x + 4e^{-x} - 2x - 6e^{-2x} - 3x^2 + 6
\]

\[
= 3x^2 - 6e^{-2x} + \frac{6}{x^2}
\]

27. You could use the quotient rule, but a better technique to use would be to rewrite the function as follows: \( q(x) = \frac{6}{x^2} - \frac{6}{x} \) then

\[
q'(x) = \frac{6}{x^2} + \frac{6}{x^2}
\]

29. \( y = e^x + \frac{1}{2x - 1} \)

\[
dy = \left(2x - 1\right) e^x - \frac{12}{\left(2x - 1\right)^2}
\]

31. \( w = \frac{3x - 1}{\sqrt{2x + 1}} \)

\[
dw = \left(2x + 1\right)^{3/2} - \frac{3}{2} \left(2x + 1\right)^{1/2}
\]

33. \( f(x) = \frac{x}{x + 1} \cdot \frac{x^2}{2x + 1} \)

\[
f'(x) = \frac{(x + 1)(2x) - (x^2 + 3x - 1)(2)}{(x + 1)^2}
\]

\[
= 2x^2 + 6x - 2x - 3 - 2x^2 - 6x + 8
\]

\[
= \frac{6x - 6}{(x + 1)^2}
\]

35. \( y = \frac{\sin t}{1 + \cos t} \)

Which can be rewritten as \( y = \frac{\sin t}{1 + \cos t} \) multiplying every term by \( \cos t \) to clear the fractions gives

\( y = \frac{\cos t}{1 + \cos t} \)

which has a derivative of \( \frac{dy}{dx} = \frac{1}{1 + \cos t} \) (see problem 21 for details)

37. \( w = \frac{t \tan^2 x + \sin x}{\sqrt{t}} \)

\[
\frac{dw}{dt} = \frac{\sqrt{t}(\sec^2 x + \tan x \cos x + \sin x) - (\tan x \sec^2 x + \sin x)}{\sqrt{t}^3}
\]

39. \( y = \frac{1 + t^{1/2}}{1 - t^{1/2}} \)

\[
\frac{dy}{dt} = \frac{(1 - t^{1/2})(1/2) - (1 + t^{1/2})(-1/2)}{(1 - t^{1/2})^2}
\]

\[
= \frac{1}{2}\frac{(1 - t^{1/2})}{(1 - t^{1/2})^2}
\]

\[
= \frac{1}{2}\frac{t^{1/2}}{(1 - t^{1/2})^2}
\]

41. \( f(t) = \tan t \tan t \)

\[
f'(t) = (\tan t)(\sec^2 t) + (\tan t)(\sec^2 t) \tan t
\]

\[
= \frac{1}{\cos^2 t} + \frac{\tan t}{\cos^2 t} \tan t
\]

43. - 83. Left to the student.

85. a) \( f(x) = \frac{x}{x + 1} \)

\[
f'(x) = \frac{(x + 1)(1) - x(1)}{(x + 1)^2}
\]

\[
= \frac{1}{(x + 1)^2}
\]

b) \( g(x) = \frac{1}{x + 1} \)

\[
g'(x) = \frac{(x + 1)(0) - (-1)(1)}{(x + 1)^2}
\]

\[
= \frac{1}{(x + 1)^2}
\]

87. \( f(x) = \sin^2 x + \cos^2 x \)

a) \( f'(x) = 2\sin x \cos x + 2\cos x (-\sin x) \)

\[
= 2\sin x \cos x - 2\sin x \cos x
\]

\[
= 0
\]
b) Since \( \sin^2 x + \cos^2 x = 1 \) (fundamental trigonometric identity) we would expect the derivative to be zero since we are taking a derivative of a constant, which is always zero.

\[
89. \quad y = \frac{x}{x^2 + 1}
\]

\[
\frac{dy}{dx} = \frac{(x^2 + 1)(1) - 2x(x)}{(x^2 + 1)^2} = 0 - \frac{2x}{(x^2 + 1)^2} = \frac{-2x}{(x^2 + 1)^2}
\]

For the point \((0, 2)\), \( \frac{dy}{dx}_{x=0} = m = \frac{-2(0)}{(0^2 + 1)^2} = 0 \). The tangent line is

\[
y - y_1 = m(x - x_1)
\]

\[
y - 2 = 0(x - 0)
\]

\[
y - 2 = 0
\]

\[
y = 2
\]

For the point \((-2, -1)\), \( \frac{dy}{dx}_{x=-2} = m = \frac{-2(-2)}{((-2)^2 + 1)^2} = \frac{4}{5} = 0.8 \). The tangent line is

\[
y - y_1 = m(x - x_1)
\]

\[
y - (-1) = 0.8(x - (-2))
\]

\[
y + 1 = 0.8x + 1.6
\]

\[
y - 0.8x - 0.6 = 0
\]

\[
y = 0.8x + 0.6
\]

93. \( y = x \sin x \)

\[
\frac{dy}{dx} = \cos x + x \sin x
\]

When \( x = \frac{\pi}{4} \)

\[
\frac{dy}{dx}_{x=\frac{\pi}{4}} = m = \frac{\pi}{4} \cos \left(\frac{\pi}{4}\right) + \sin \left(\frac{\pi}{4}\right)
\]

\[
= \frac{\pi}{4} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}
\]

\[
= \frac{\sqrt{2} \pi}{8} + \frac{\sqrt{2}}{2}
\]

\[
= \frac{\sqrt{2}(\pi + 4)}{8}
\]

The tangent line is

\[
y - y_1 = m(x - x_1)
\]

\[
y - \frac{\sqrt{2}\pi}{8} = \frac{\sqrt{2}(\pi + 4)}{8}(x - \frac{\pi}{4})
\]

\[
y = \frac{\sqrt{2}(\pi + 4)}{8}x - \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}(\pi + 4)}{8}\frac{\pi}{4}
\]

\[
y = \frac{\sqrt{2}(\pi + 4)}{8}x - \frac{\pi^2}{16\sqrt{2}} + \frac{\sqrt{2}\pi}{8}
\]

\[
y = \frac{\sqrt{2}(\pi + 4)}{8}x - \frac{\pi^2}{16\sqrt{2}}
\]

95. a) \( T(t) = \frac{d}{dt} \) \( t^4 + 98.6 \)

\[
\frac{dT}{dt} = \frac{(2t^2 + 4t)(4t^3) - 98.6(3t^2)}{(2t^2 + 1)^2} = 0
\]

\[
= \frac{4t^2 + 4 - 98.6}{(2t^2 + 1)^2}
\]

\[
= \frac{-4t^2 + 4}{(2t^2 + 1)^2}
\]

b) When \( t = 2 \) hours

\[
T' = \frac{4(2)^2}{2^2 + 1} + 98.6
\]

\[
= \frac{16}{5} + 98.6
\]

\[
= 100.2 \text{ degrees}
\]
c) When \( t = 2 \) hours
\[
\frac{dI}{dt} = -4(2)^2 + 4
\]
\[
= -16 + 4
\]
\[
= -12
\]
\[
= -2.4 \text{ degrees per hour}
\]

97. a) Since \( \frac{d}{dt} \tan t \) then \( s(t) = 100 \tan t \)
b) \( \frac{ds(t)}{dt} = 100 \sec^2 t \)
c) \[
\begin{align*}
100 \sec^2 t &= 200 \\
\sec^2 t &= 2 \\
\frac{1}{\cos^2 t} &= 2 \\
\cos^2 t &= \frac{1}{2} \\
\cos t &= \pm \frac{1}{\sqrt{2}} \\
&= \pm \frac{\sqrt{2}}{4} \pi
\end{align*}
\]

99. \( g(x) = \frac{x^2 + 1 + \tan x}{(x^2 - 1)^2} \)
\[
g'(x) = \frac{(2x^2 + 1) \sec^2 x (x^2 - 1) + 2x \tan x}{(x^2 - 1)^2} - \frac{2x \tan x}{(x^2 - 1)^2} - \frac{(x^4 - 1) \sec^2 x}{(x^2 - 1)^2} + \frac{2 \tan x}{(x^2 - 1)^2} - \frac{(x^4 - 1) \sec^2 x}{(x^2 - 1)^2} + \frac{2 \tan x}{(x^2 - 1)^2}
\]

103. \( g'(x) = \frac{(x + \cos x) - x \sec x + \cos x}{(x + \cos x)^2}
\]
\[
= \frac{(x + \cos x) - x \sec x + \cos x}{(x + \cos x)^2}
\]

105. Let \( y = (x - 1)(x - 2)(x - 3) \)

a) \[
\frac{dy}{dx} = (x - 1)(x - 2)(x - 3) + (x - 2)(x - 1)(x - 3) + (x - 1)(x - 2)(x - 3)
\]
\[
= 3x^2 - 12x + 12
\]

b) \( y = (2x + 1)(3x - 5)(-x + 3) \)
\[
\frac{dy}{dx} = (2x + 1)(3x - 5)(-1) + (2x - 3)(-x + 3)(1)
\]
\[
= 12x^2 - 5x - 18x + 15
\]

101. \( s(t) = \frac{\tan t}{\sec t} \)
\[
s'(t) = \frac{\sec^2 t - \tan t \sec t}{\sec^2 t - \tan^2 t}
\]
\[
= \frac{\sec^2 t - \tan t \sec t}{\sec^2 t - \tan^2 t}
\]
\[
= \frac{\sec^2 t \sec t + \tan t \sec t - \sec t \tan t}{\sec^2 t + \tan^2 t}
\]
\[
= \frac{\sec^2 t \sec t + \tan t \sec t - \sec t \tan t}{\sec^2 t + \tan^2 t}
\]
\[
= \frac{\sec^2 t \sec t + \tan t \sec t - \sec t \tan t}{\sec^2 t + \tan^2 t}
\]
107. No horizontal tangent lines

109. \((0.2, 0.75)\) and \((-0.2, -0.75)\)

111. \((1, 2)\) and \((-1, -2)\)

Exercise Set 2.8

1. \(y = (2x + 1)^2\)
   Method One (chain rule):
   \[
   \frac{dy}{dx} = 2(2x + 1)(2) = (2x + 1)
   \]
   Method Two (product rule): \(y = (2x + 1)(2x + 1)\)
   \[
   \frac{dy}{dx} = (2x + 1)(2) + (2x + 1)(2) = 4x + 2 + 4x + 2 = 8x + 4
   \]

Method Three (expand first):
\[
\frac{dy}{dx} = \frac{4x^2 + 4x + 1}{2(2x + 1)} = 2x + 1
\]

3. \(y = (1 - x)^5\)
   \[
   \frac{dy}{dx} = 55(1 - x)^5(-1) = -55(1 - x)^5
   \]

5. \(\sec^2 x\)
   \[
   \frac{dy}{dx} = 2 \sec x \tan x = 2 \tan x \sec x
   \]

7. \(y = \sqrt{1 - 3x} - (1 - 3x)^{1/2}\)
   \[
   \frac{dy}{dx} = \frac{1}{2} (1 - 3x)^{-1/2}(-3) = -3(1 - 3x)^{-1/2}
   \]

9. \(y = \frac{3}{\sqrt{2x^2 + 1}} = 2(3x^2 + 1)^{-1}\)
   \[
   \frac{dy}{dx} = \frac{2(1)(6x)}{(3x^2 + 1)^{3/2}}
   \]

11. \(s(t) = t(2t + 3)^{1/2}\)
   \[
   s'(t) = \frac{t}{\sqrt{2t + 3}}(1 + (2t + 3)^{1/2}(2)) = \frac{t}{\sqrt{2t + 3}}(1 + (2t + 3)^{1/2})(1)
   \]

13. \(s(t) = \sin \left(\frac{2}{5} t + \frac{\pi}{3}\right)\)
   \[
   s'(t) = \frac{\pi}{5} \cos \left(\frac{2}{5} t + \frac{\pi}{3}\right)
   \]

15. \(g(x) = (1 + x^3)^2 - (1 + x^4)^4\)
   \[
   g'(x) = 3(1 + x^3)(3x^2) - 4(1 + x^4)^3(3x^2) = 9x^2(1 + x^3)^2(1 + x^3)^3
   \]

17. \(y = \sqrt{1 - \sec x} - (1 - \sec x)^{1/2}\)
   \[
   \frac{dy}{dx} = \frac{1}{2} (1 - \sec x)^{-1/2}(-\sec x \tan x) = \frac{1}{2} \sec x \tan x
   \]

\[
= \frac{1}{2} \sqrt{1 - \sec x}
\]
19. \( g(x) = (2x - 1)^{1/2} + (1 - x)^2 \)
\[
g'(x) = \frac{1}{3} (2x - 1)^{-1/2} (2) + 2(1 - x)(-1) = \frac{2}{3 \sqrt{2x - 1}^2} - 2(1 - x) \]

21. \( y = x^3 \sin x + 5x \cos x + 4x e^{-x} \)
\[
dy = x^3 \cos x + 3x^2 \sin x + 5x (-\sin x) + 5 \cos x - 4 \sec x \tan x \]
\[
\hat{y} = 3x^2 \cos x + 3x^2 \sin x - 5x \sin x + 5 \cos x + 4 \sec x \tan x \]

23. \( y = (x^2 + x^{-1})^{1/2}(2x^2 + 3x + 5) \)
\[
dy = [x^2 + x^{-1}]^{1/2} (4x + 3) + \frac{1}{2} [x^2 + x^{-1}]^{-1/2} (2x^2 + 3x^2) (2x^2 + 3x + 5) - (4x + 3) \sqrt{x^2 + x^{-1}} \cdot \frac{(2x^2 + 3x^2)(2x^2 + 3x + 5)}{2 \sqrt{x^2 + x^{-1}}} \]

25. \( f(t) = \cos \sqrt{t} \)
\[
f'(t) = -\sin \sqrt{t} \left( \frac{1}{2} t^{-1/2} \right) = -\sin \sqrt{t} \cdot \frac{3}{2t^{1/2}} \]

27. \( f(x) = (3x + 2)(2x + 5)^{1/3} \)
\[
f'(x) = (3x + 2) \left( \frac{2}{3} (2x + 5)^{-1/3} (2) \right) + 3(2x + 5)^{1/3} \]
\[
\frac{3x + 2}{\sqrt{2x + 5}} \cdot \frac{3 \sqrt{2x + 5}}{5} \]

29. \( y = \cos(4x) \)
\[
dy = -\sin x \cos(4x) \]

31. \( y = \cos^3(4t) \)
\[
dy = \frac{1}{3} (\cos(4t))^{-1/2} (-\sin(4t)) (4) = -2 \sin(4t) \cos(4t) \]

33. \( f(x) = \frac{x^3 + 2x^2 + x + 2}{(2x + 1)^4} \)
\[
f'(x) = \frac{[x^3 + 2x^2 + x + 2][3x^2 + 2x + 3x - 1] - (3x^2 + 2x + 3x - 1)[(2x + 1)^1]}{(2x + 1)^4} \]
\[
\frac{3(x^3 + 2x^2 + x - 1)(2x + 1)^1}{(2x + 1)^4} \]
\[
\frac{16(x^3 + 2x^2 + 3x - 1)(2x + 1)^1}{(2x + 1)^4} \]

35. \( y = \frac{(2x + 1)(x^2 + 4)}{(x^2 + 1)^{1/2}} \)
\[
dy = \frac{1}{2} \left( 3x - 1 \right) [x(3x + 1)(3) - (3x - 1)(5)] \]
\[
= \frac{15x - 9 - 15x + 20}{2 \sqrt{3x^2 + 2x + 3x - 1}} \]
\[
- \frac{20}{2 \sqrt{3x^2 + 2x + 3x - 1}} \]

37. \( r(x) = x(0.01x^2 + 2.391x - 8.51)^5 \)
\[
r'(x) = -x [5(0.01x^2 + 2.391x - 8.51)^4(0.01x^2 + 2.391)] + (0.01x^2 + 2.391x - 8.51)^5 \]
\[
= 5x(0.01x^2 + 2.391)(0.01x^2 + 2.391x - 8.51)^4(1) \]
\[
= 5x(0.01x^2 + 2.391x - 8.51)^4(1) \]
\[
= 5x(0.01x^2 + 2.391x - 8.51)^4(1) \]
\[
= 5x(0.01x^2 + 2.391x - 8.51)^4(1) \]
\[
= 5x(0.01x^2 + 2.391x - 8.51)^4(1) \]
\[
= 5x(0.01x^2 + 2.391x - 8.51)^4(1) \]

39. \( y = \cos(5x - \cos 5x))^{1/5} \)
\[
dy = \frac{1}{5} \left( \cos(5x - \cos 5x)^{4/5} \right) \]
\[
= \sin(5x - \cos 5x) \]

41. \( y = \sin(\sin^3x^2) \)
\[
dy = \cos(\sin^3x^2) \cdot 4 \sin x \cos x \cdot \tan x \cdot 2x \]
\[
= 8x \sin x \cos x \cdot \tan x \cdot 2x \]

43. \( y = (2x^2 + 1)^{1/3} + 1 \)
\[
dy = \frac{1}{3} (2x^2 + 1)^{-2/3} \left( \frac{1}{3} (2x^2 + 1)^{-2/3} \cdot 2x \right) \]
\[
= \frac{x}{6 \sqrt{2x^2 + 1}} \cdot \frac{2x}{(2x^2 + 1)^{1/3}} \]

45. \( y = \sin^2x \)
\[
dy = \sin(2x) \cdot \cos(2x) \cdot 2 \cdot 2x \]
\[
= 4x \sin(2x) \cos(2x) \]

47. \( f(x) = \cos(4x) \sin(2x + 4)) \)
\[
f'(x) = 3 \sin^2(2x + 4) \cdot (4x \sin(2x + 4)) \cdot (2x + 4) \]
\[
= 3 \sin^2(2x + 4) \cdot (2x + 4) \cdot (2x + 4) \]
\[
= 3 \sin^2(2x + 4) \cdot (2x + 4) \cdot (2x + 4) \]
\[
= 3 \sin^2(2x + 4) \cdot (2x + 4) \cdot (2x + 4) \]
49. \( y = \sqrt{\sec^4 x + x} \)
\[
\frac{dy}{dx} = \frac{1}{2\sqrt{\sec^4 x + x}} \cdot 4\sec^3 x (\sec x \tan x + 1)
\]
\[
= \frac{2\sec x \tan x + 1}{\sqrt{\sec^4 x + x}}
\]

51. \( y = u^{1/2}, \quad u = x^2 - 1 \)
\[
\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2}\sqrt{u}^{-1/2}
\]
\[
= \frac{1}{2\sqrt{u}} \cdot 2x
\]
\[
\frac{dy}{dx} = \frac{du}{dx} = \frac{2x}{2\sqrt{u}}
\]
\[
= \frac{2x}{\sqrt{x^2 - 1}}
\]

53. \( y = u^{50}, \quad u = 4x^2 - 2x^2 \)
\[
\frac{dy}{du} = 50u^{49}
\]
\[
\frac{du}{dx} = 12x^2 - 4x
\]
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{50u^{49}(12x^2 - 4x)}{12x^2 - 4x}
\]
\[
= 50(4x^4 - 2x^2)^{49}(12x^2 - 4x)
\]

55. \( y = u(u + 1), \quad u = x^2 - 2x \)
\[
\frac{dy}{du} = u \cdot 1 + 1 \cdot (u + 1)
\]
\[
= u + u + 1
\]
\[
= 2u + 1
\]
\[
\frac{du}{dx} = 3x^2 - 2
\]
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u + 1)(3x^2 - 2)
\]
\[
= (2x^3 - 4x + 1)(3x^2 - 2)
\]

57. \( y = \sqrt{x^2 + 3x} = (x^2 + 3x)^{1/2} \)
\[
\frac{dy}{dx} = \frac{1}{2}(x^2 + 3x)^{-1/2}(2x + 3)
\]
\[
= \frac{2x + 3}{2\sqrt{x^2 + 3x}}
\]
When \( x = 1 \),
\[
\frac{dy}{dx} = \frac{2 \cdot 1 + 3}{2\sqrt{1^2 + 3 \cdot 1}}
\]
\[
= \frac{2 + 3}{2\sqrt{1}}
\]
\[
= \frac{5}{2\sqrt{1}}
\]
\[
= \frac{5}{2}
\]

Thus, at (1, 2), \( m = \frac{5}{2} \). We use point-slope equation.

61. \( f(x) = 3x^2 \cos x \)
\[
\frac{y - y_1}{x - x_1} = m(x - x_1)
\]
\[
y - 2 = \frac{5}{4}(x - 1)
\]
\[
y - 2 = \frac{5}{4}x - \frac{5}{4}
\]
\[
y = \frac{5}{4}x + \frac{3}{4}
\]

63. \( f(x) = \frac{x^2}{(1 + x)^5} \)

a)
\[
f'(x) = \frac{(1 + x)^5(2x) - x^2(5(1 + x)^4)}{(1 + x)^{10}}
\]
\[
= \frac{2x - 5x^2}{(1 + x)^5}
\]

b)
\[
f'(x) = x^2 \frac{(5(1 + x)^4) - 2x(1 + x)^5}{(1 + x)^{10}}
\]
\[
= \frac{-5x^2 + 2x}{(1 + x)^6}
\]
\[
= \frac{-5x^2 + 2x}{(1 + x)^6}
\]
c) The results in the previous parts are the same.

65. Using the Chain Rule:

Let \( y = f(u) \). Then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[3u^2 \cdot (8u^3)\]

\[3(2x^4 + 1)^2(8x^3)\]

Substituting \(2x^4 + 1\) for \(u\)

When \( x = -1 \):

\[
\frac{dy}{dx} = 3[2(-1)^4 + 1]^2[8(-1)^4] = 3(2 + 1)^2(-8) = 3 \cdot 3^2(-8) = -216
\]

Finding \( f'(u(x)) \):

\( f \circ g(x) = f(g(x)) \cdot f'(2x^4 + 1) \cdot (2x^4 + 1)^4 \)

Then \( (f \circ g)'(x) = 3(2x^4 + 1)^2(8u^3) \) and

\( (f \circ g)'(-1) = -216 \) as above.

67. Using the Chain Rule:

Let \( y = f(u) \cdot \sqrt{u} \cdot u^{1/3} \). Then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[-\frac{1}{3}u^{-2/3} \cdot (-6x) \]

\[-2x \cdot u^{-2/3} \]

\[-2x(1 - 3x^2)^{-2/3} \]

Substituting \(1 - 3x^2\) for \(u\)

When \( x = 2 \):

\[
\frac{dy}{dx} = 2 \cdot 2(1 - 3 \cdot 2^2)^{-2/3} \approx -4
\]

Finding \( f'(u(x)) \):

\( f \circ g(x) = f(1 - 3x^2) \cdot \sqrt{1 - 3x^2}, \) or

\( (1 - 3x^2)^{1/3} \)

Then \( (f \circ g)'(x) = \frac{1}{3}(1 - 3x^2)^{-2/3}(-6x) = -2x(1 - 3x^2)^{-2/3} \) and

\( (f \circ g)'(2) = -1(-11)^{-2/3} \approx -0.8087 \) as above.

69. \( A = 1000(1 + i)^2 \)

a) \[
\frac{dA}{dt} = 1000(3(1 + i)^2)
\]

\[-3000(1 + i)^2\]

b) \( \frac{dA}{dt} \) represents the rate at which the amount of investment is changing with respect to an annual interest rate \(i\).

71. \( D = 0.85A(c + 25), \) \( r = (10 - y) \frac{d^2}{72r}\)

a) To find \( D \) as a function of \(c\), we substitute \(5\) for \(A\) in the formula for \(D\):

\[
D = 0.85A(c + 25)
\]

\[
= 0.85(5)(c + 25)
\]

\[
= 4.25c + 106.25
\]

To find \(r\) as a function of \(w\), we substitute \(45\) for \(y\) and \(0.6\) for \(x\) in the formula for \(r\):

\[
r = \frac{140 - 45}{2(0.6)}
\]

\[
= \frac{95}{2(0.6)}
\]

\[
= 2.199w
\]

b) \( \frac{dD}{dc} = 4.25 \)

c) \( \frac{dr}{dw} = 2.199 \)

d) First we find \(D \circ c(w)\).

\(D \circ c(w) = D(c(w))\)

\[1.25(2.199w) + 106.25\]

\[= 3.4575w + 106.25\]

Then we have

\[
\frac{dD}{dw} = 3.4575 \approx 3.46\]

b) \( \frac{dD}{dw} \) represents the rate of change of the dosage with respect to the patient’s weight. For each additional kilogram of weight, the dosage is increased by about 3.35 mg.

73. a) January 2009 corresponds to \(t = 52\)

\(C'(52) = 0.71 + 0.02376(52 - 10814\pi \cos(2\pi t))\)

\(C''(52) = -1.4218 \text{ ppm} \text{ per} \text{ week} \)

b) July 2009 corresponds to \(t = 52.5\)

\(C'(52.5) = 0.71 + 0.02376(52.5 - 10814\pi \cos(105\pi))\)

\( = 5.3847 \text{ ppm} \text{ per} \text{ week} \)

75. \(y = (x^2 + 4)^8 + 3\sqrt{x}\)

\[
\frac{dy}{dx} = 4(x^2 + 4)^8 \cdot 8(x^2 + 4)^7 \cdot (2x) + \frac{3}{2\sqrt{x}}
\]

\[-4(x^2 + 4)^8 + 3\sqrt{x} \left( 16x(x^2 + 4)^7 + \frac{3}{2\sqrt{x}} \right) \]

77. Let \(y = \sin(\sin(x))\) then

\[
\frac{dy}{dx} = \cdot \cos(\sin(x)) \cdot \sin(x) \cdot \cos x
\]

\[-\cos x \cos(\sin(x)) \cos(\sin(x))\]
79. Let \( y = \tan(\cot(\sec 3x)) \) then
\[
\frac{dy}{dx} = \sec^2(\cot(\sec 3x)) \cdot -\tan(\sec 3x) \cdot 3 \sec 3x \tan 3x
= -3 \sec 3x \tan 3x \sec^2(\sec 3x) \sec^2(\cot(\sec 3x))
\]

81. \( y = (\sin \left( \frac{2\pi}{3} + 3 \right) )^{1/5} \) is a constant, which means \( \frac{dy}{dx} = 0 \).

83.
\[
sin(a + x) = \sin a \cos x + \cos a \sin x
\]
\[
\frac{d}{dx} (\sin(a + x)) = \frac{d}{dx} (\sin a \cos x + \cos a \sin x)
= -\sin a \sin x + \cos a \cos x
= \cos a \cos x - \sin a \sin x
= \cos(a + x)
\]

85. Let \( Q(x) = \frac{N(x)}{D(x)} \). Then we can write
\[
Q(x) = N(x) \cdot |D(x)|^{-1}
\]
using the property of negative exponents. Now we use the product differentiation rule
\[
Q'(x) = \frac{N(x) \cdot D'(x) + N'(x) \cdot D(x)}{|D(x)|^2}
\]
\[
= \frac{N(x) \cdot D'(x) - N'(x) \cdot D(x)}{|D(x)|^2}
\]
87. \((-2.145, -7.728) \) and \((2.145, 7.728)\)

89.
\[
f'(x) = \frac{x - 2x}{2\sqrt{4 - x^2}} + \sqrt{4 - x^2}
= \frac{\sqrt{4 - x^2} + 4 - x^2}{\sqrt{4 - x^2}}
= \frac{4 - 2x^2}{\sqrt{4 - x^2}}
\]

Exercise Set 2.9

1. \( y = 3x + 5 \)
\[
\frac{dy}{dx} = 3
\]
\[
\frac{d^2y}{dx^2} = 0
\]

3. \( y = -3(2x + 2)^{-1} \)
\[
\frac{dy}{dx} = 3(2x + 2)^{-2}(2) = 6(2x + 2)^{-2} = \frac{6}{(2x + 2)^2}
\]
\[
\frac{d^2y}{dx^2} = -12(2x + 2)^{-3}(2) = \frac{-24}{(2x + 2)^3}
\]

5. \( y = (2x + 1)^{1/3} \)
\[
\frac{dy}{dx} = \frac{1}{3}(2x + 1)^{-2/3}(2) = \frac{2}{3(2x + 1)^{2/3}}
\]
\[
\frac{d^2y}{dx^2} = \frac{2}{3}(-3)(2x + 1)^{-5/3}(2) = \frac{-8}{9(2x + 1)^{5/3}}
\]

7. \( f(x) = (4 - 3x)^{-1} \)
\[
f'(x) = -4(4 - 3x)^{-2}(-3) = 12(4 - 3x)^{-5}
\]
\[
f''(x) = -60(4 - 3x)^{-6}(-3) = \frac{180}{(4 - 3x)^6}
\]

9. \( y = \sqrt{x + 1} - (x + 1)^{1/2} \)
\[
\frac{dy}{dx} = \frac{1}{2}(x + 1)^{-1/2} \cdot 1
= \frac{1}{2(x + 1)^{-1/2}}
\]
\[
\frac{d^2y}{dx^2} = -\frac{1}{4}(x + 1)^{-3/2} \cdot 1
= -\frac{1}{4(x + 1)^{-3/2}}
\]
\[
= -\frac{1}{4\sqrt{x + 1}^{3/2}}
\]

11. \( f(x) = (2x + 9)^{11} \)
\[
f'(x) = 16(2x + 9)^{10}(2) = 32(2x + 9)^{10}
\]
\[
f''(x) = 480(2x + 9)^{14}(2) = 960(2x + 9)^{14}
\]

13. \( g(x) = \sec(3x + 1) \)
\[
g'(x) = 3\sec(3x + 1) \tan(3x + 1)(3)
= 3\sec(3x + 1) \tan(3x + 1)
\]
\[
g''(x) = 9\sec^3(3x + 1) + 9\sec(3x + 1) \tan(3x + 1)(3)
= 9\sec(3x + 1)[\sec^2(3x + 1) + \tan^2(3x + 1)]
15. \( f(x) = \sec(2x + 3) + 4x^2 + 3x - 7 \)

\[
f'(x) = 2 \sec(2x + 3) \tan(2x + 3)(3) + 8x + 3
\]

\[
f''(x) = 2 \sec(2x + 3) \sec^2(2x + 3)(2) + \tan(2x + 3) \sec(2x + 3)(3) + 8x \sin(2x + 3) + \sec(2x + 3) \tan(2x + 3)(3) + 8
\]

17. \( y = ax^2 + bx + c \)

\[
\frac{dy}{dx} = 2ax + b
\]

\[
\frac{d^2y}{dx^2} = 2a
\]

19. \( y = \sqrt{x^2 + 1} \cdot (x^2 + 1)^{3/2} \)

\[
\frac{dy}{dx} = \frac{3}{2}(x^2 + 1)^{1/2} \cdot (x^2 + 1)^{1/2}
\]

\[
\frac{d^2y}{dx^2} = \frac{3}{4} \left( \frac{x^2}{(x^2 + 1)^{5/4}} + \frac{3}{2(x^2 + 1)^{1/4}} \right)
\]

21. \( f(x) = \{4x + 3\} \cos x \)

\[
f'(x) = (4x + 3) (-\sin x) + 4 \cos x
\]

\[
\{4x + 3\} \sin x + 4 \cos x
\]

23. \( s(t) = \cos(\alpha t + b) \)

\[
s'(t) = -\sin(\alpha t + b) \cdot \alpha - \alpha \sin(\alpha t + b)
\]

\[
s''(t) = -\alpha \cos(\alpha t + b)
\]

25. \( \frac{14x^2 + 22x + 9}{9} + (3t^2 + 1)^{3/2} \)

\[
\frac{dy}{dx} = \frac{2}{7} \left( t^2 + 3 \right)^{-1/2} \cdot (2t) + \frac{1}{3} (3t^2 + 1)^{-2/3} (6t)
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{7} (t^2 + 3)^{-3/2} + 2(3t^2 + 1)^{-2/3}
\]

27. \( y = x^3 \)

\[
\frac{dy}{dx} = 4x^3
\]

\[
\frac{d^2y}{dx^2} = 12x^2
\]

\[
\frac{d^3y}{dx^3} = 24x
\]

29. \( y = x^6 - x^3 + 2x \)

\[
\frac{dy}{dx} = 6x^5 - 3x^2 + 2
\]

\[
\frac{d^2y}{dx^2} = 30x^4 - 6x
\]

\[
\frac{d^3y}{dx^3} = 120x^3 - 6
\]

\[
\frac{d^4y}{dx^4} = 360x^2
\]

\[
\frac{d^5y}{dx^5} = 720x
\]

31. \( (x^2 - 5)^{10} \)

\[
\frac{dy}{dx} = 40(x^2 - 5)^9 \cdot 2x
\]

\[
= 20x(x^2 - 5)^9
\]

\[
\frac{d^2y}{dx^2} = 20x \cdot 9(x^2 - 5)^8 \cdot 2x + 20(x^2 - 5)^9
\]

\[
= 20(x^2 - 5)^8 \cdot 18x^2 + 20(x^2 - 5)^9
\]

\[
= 20(x^2 - 5)^8(19x^2 - 5)
\]

33. \( \sec(2x + 3) \)

\[
\frac{dy}{dx} = \sec(2x + 3) \tan(2x + 3)(2)
\]

\[
= 2 \sec(2x + 3) \tan(2x + 3)
\]

\[
\frac{d^2y}{dx^2} = 4 \sec(2x + 3) \sec^2(2x + 3)(2) + 2 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \tan(2x + 3)(2)
\]

\[
= 4 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \tan(2x + 3)(2)
\]

\[
= 4 \sec^3(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \tan(2x + 3)(2)
\]

\[
= 8 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \sec(2x + 3)(2) + 2 \sec(2x + 3) \tan(2x + 3)(2)
\]
35. \( s(t) = 10 \cos(3t + 2) - 4 \sin(3t + 2) \)

\[
\begin{align*}
\alpha(t) &= 10[-\sin(3t + 2)(3)] - 4[\cos(3t + 2)(3)] \\
&= -30 \sin(3t + 2) - 12 \cos(3t + 2) \\
\gamma(t) &= -30[\cos(3t + 2)(3)] - 12[-\sin(3t + 2)(3)] \\
&= -90 \cos(3t + 2) - 36 \sin(3t + 2) \\
&= 9[10 \cos(3t + 2) - 4 \sin(3t + 2)] \\
&= 9s(t)
\end{align*}
\]

37. \( s(t) = t^3 + t^2 + 2t \)

\[
\begin{align*}
v(t) &= s'(t) = 3t^2 + 2t + 2 \\
a(t) &= s''(t) = 6t + 2
\end{align*}
\]

39. \( w(t) = 0.0000758s^3 - 0.05062^2 - 1.82t + 8.15 \)

The acceleration of a function that depends on time is the second derivative of the function with respect to time.

\[
\begin{align*}
w'(t) &= 0.0002574t^2 - 0.1192t + 1.82 \\
w''(t) &= 0.0004588t - 0.1192
\end{align*}
\]

41. \( P(t) = 100000(1 + 0.6t + t^2) \)

\[
\begin{align*}
P'(t) &= 100000(0.6 + 2t) \\
P''(t) &= 100000(2)
\end{align*}
\]

\[= 200000\]

43. \( y = \frac{x^2}{(x-1)^{3/2}} \)

\[
y' &= \frac{\sqrt{x-1}(1) - x \cdot \frac{4}{2\sqrt{x-1}}}{x-1} \\
&= \frac{2(x-1) - x}{2(x-1)^{3/2}} \\
&= \frac{x - 2}{2(x-1)^{3/2}} \\
y'' &= \frac{2(x-1)^{3/2}(1) - (x-2) \left[ \frac{3}{2} \cdot \frac{2}{(x-1)^{1/2}} \right]}{4(x-1)^2} \\
&= \frac{2(x-1)^{3/2} - 3(x-2)(x-1)^{3/2}}{4(x-1)^2} \\
&= \frac{(x-1)^{3/2}[2(x-1) - 3(x-2)]}{(x-1)^3} \\
&= 4 - x \\
&= (x-1)^{3/2} \\
y''' &= \frac{4(x-1)^{3/2}(1) - (4 - x) \left[ \frac{3}{2} \cdot \frac{3}{2}(x-1)^{1/2} \right]}{16(x-1)^5} \\
&= \frac{(x-1)^{3/2}[4(x-1) - 10(4-x)]}{16(x-1)^5} \\
&= \frac{4x - 40 + 10x}{16(x-1)^{5/2}} \\
&= \frac{3x - 18}{16(x-1)^{5/2}}
\]

45. \( f(x) = \frac{x}{x-1} \)

\[
f'(x) &= \frac{(x-1)(1) - x(1)}{(x-1)^2} \\
&= \frac{1}{(x-1)^2} \\
&= -(x-1)^{-2} \\
f''(x) &= -(-2(x-1)^{-3}) \\
&= \frac{2}{(x-1)^3}
\]

47. \( y = \sin x \)

a) \( \frac{dy}{dx} = \cos x \)

b) \( \frac{d^2y}{dx^2} = -\sin x \)

c) \( \frac{d^3y}{dx^3} = -\cos x \)

d) \( \frac{d^4y}{dx^4} = \sin x \)

e) \( \frac{d^5y}{dx^5} = \sin x \)

f) \( \frac{d^6y}{dx^6} = \sin x \)

g) \( \frac{d^7y}{dx^7} = \cos x \)

49. Functions that have the form \( f(x) = 3 \sin x + 4 \cos x \) where \( A \) and \( B \) are constants, will satisfy the condition of their second derivative being the negative of the original function.

51. \( f(x) = \frac{x+3}{x-2} \)

\[
f'(x) &= \frac{(x-2)(1) - (x+3)(1)}{(x-2)^2} \\
&= \frac{-5}{(x-2)^2} \\
&= \frac{10}{(x-2)^3} \\
f''(x) &= -30(x-2)^{-4} \\
f'''(x) &= \frac{120(x-2)^{-5}}{(x-2)^2} \\
f^{(4)}(x) &= -600(x-2)^{-6} \\
f^{(5)}(x) &= \frac{3600}{(x-2)^7}
\]
Chapter 3

Application of Differentiation

Exercise Set 3.1

1. \( f(x) = x^2 - 4x + 5 \). First, find the critical points (values of \( x \) at which the derivative is zero or undefined).

\[ f'(x) = 2x - 4 \]

\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):

\[
2x - 4 = 0 \\
2x = 4 \\
x = 2
\]

The only critical point is at \( x = 2 \). We use \( 2 \) to divide the real number line into two intervals, \( A: (-\infty, 2) \) and \( B: (2, \infty) \):

\[ x \]
\[ 2 \]
\[ A \]
\[ B \]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test 0, \( f'(0) = 2 \cdot 0 - 4 = -4 < 0 \)
B: Test 3, \( f'(3) = 2 \cdot 3 - 4 = 2 > 0 \)

We see that \( f(x) \) is decreasing on \( (-\infty, 2) \) and increasing on \( (2, \infty) \), and the change from decreasing to increasing indicates that a relative minimum occurs at \( x = 2 \). We substitute into the original equation to find \( f(2) \):

\[ f(2) = 2^2 - 4 \cdot 2 + 5 = 1 \]

Thus, there is a relative minimum at \( (2, 1) \). We use the information obtained to sketch the graph.

5. \( f(x) = 1 + 6x + 3x^2 \)

First, find the critical points.

\[ f'(x) = 6 + 6x \]

\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):

\[ 6 + 6x = 0 \]

The only critical point is at \( x = -1 \). We use \( -1 \) to divide the real number line into two intervals, \( A: \left( -\infty, -\frac{1}{2} \right) \) and \( B: \left( -\frac{1}{2}, \infty \right) \):

\[ x \]
\[ -1 \]
\[ \frac{1}{2} \]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test 0, \( f'(-1) = 6 - 1 \cdot 4 > 0 \)
B: Test 1, \( f'(1) = 6 - 2 \cdot 1 < 0 \)

We see that \( f(x) \) is increasing on \( \left( -\infty, -\frac{1}{2} \right) \) and decreasing on \( \left( -\frac{1}{2}, \infty \right) \), so there is a relative maximum at \( x = -\frac{1}{2} \). We find \( f\left(-\frac{1}{2}\right) \):

\[ f\left(-\frac{1}{2}\right) = 1 - 2 \cdot -\frac{1}{2} = 2 \]

Thus, there is a relative maximum at \( \left( -\frac{1}{2}, 2 \right) \). We use the information obtained to sketch the graph.
\[ 6x = -6 \]
\[ x = -1 \]

The only critical point is at \( x = -1 \). We use \(-1\) to divide the real number line into two intervals, \( A: (-\infty, -1) \) and \( B: (-1, \infty) \):

We use a test value in each interval to determine the sign of the derivative in each interval.

- For \( A: -2 \)
  \[ f'(2) = 2 + 6(-2) = -10 \]
  \[ f(0) = 0 + 6(-2) = -12 \]
  We see that \( f(x) \) is decreasing on \((-\infty, -1)\) and increasing on \((-1, \infty)\), so there is a relative minimum at \( x = -1 \). We find \( f(-1) \):
  \[ f(-1) = 1 + 6(-1) + 3(-1)^2 = -2 \]
  Thus, there is a relative minimum at \((-1, -2)\). We use the information obtained to sketch the graph.

7. \( f(x) = x^3 - x^2 - x + 2 \)

First, find the critical points.

\[ f'(x) = 3x^2 - 2x - 1 \]

\[ f'(x) \text{ exists for all real numbers. We solve } f'(x) = 0: \]

\[ 3x^2 - 2x - 1 = 0 \]

\[ (3x + 1)(x - 1) = 0 \]

\[ 3x + 1 = 0 \]

\[ 3x = -1 \]

\[ x = -\frac{1}{3} \]

\[ y \]

\[ x - 1 = 0 \]

\[ x = 1 \]

The critical points are at \( x = -\frac{1}{3} \) and \( x = 1 \). We use them to divide the real number line into three intervals.

- \( A: (-\infty, -\frac{1}{3}) \), \( B: (-\frac{1}{3}, 1) \), and \( C: (1, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.

- For \( A: -2 \)
  \[ f'(-2) = 3(-2)^2 - 2(-2) - 1 = 13 > 0 \]
  \[ f(0) = 3(0)^2 - 2(0) - 1 = -1 < 0 \]
  \[ f(1) = 1^3 - 1^2 - 1 + 2 = 1 > 0 \]
  \[ f(2) = 2^3 - 2(2) - 1 + 2 = 1 > 0 \]

Then, we find \( f(1) \):

\[ f(1) = 1^3 - 1^2 - 1 + 2 = 1 \]

9. \( f(x) = x^3 - 3x + 6 \)

First, find the critical points.

\[ f'(x) = 3x^2 - 3 \]

\[ f'(x) \text{ exists for all real numbers. We solve } f'(x) = 0: \]

\[ 3x^2 - 3 = 0 \]

\[ x^2 - 1 = 0 \]

\[ (x + 1)(x - 1) = 0 \]

\[ x + 1 = 0 \]

\[ x = -1 \]

\[ x - 1 = 0 \]

\[ x = 1 \]
Or
\[ x - 1 = 0 \]
\[ x = 1 \]

The critical points are at \( x = -1 \) and \( x = 1 \). We use them to divide the real number line into three intervals: A: \( (-\infty, -1) \), B: \( (-1, 1) \), and C: \( (1, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.
A: Test \(-2\). \( f'( -2) = 3(-2)^2 - 3 \cdot 12 - 3 = 9 > 0 \)
B: Test 0. \( f'(0) = 3 \cdot 0^2 - 3 \cdot 0 - 3 = -3 < 0 \)
C: Test \(2\). \( f'(2) = 3 \cdot 2^2 - 3 \cdot 12 - 3 = 9 > 0 \)

We see that \( f(x) \) is increasing on \( (-\infty, -1) \), decreasing on \( (-1, 1) \), and increasing again on \( (1, \infty) \), so there is a relative maximum at \( x = -1 \) and a relative minimum at \( x = 1 \). We find \( f(-1) \):
\[ f(-1) = (-1)^3 - 3(-1) + 6 = -1 + 3 + 6 = 8 \]
Then we find \( f(1) \):
\[ f(1) = 1^3 - 3 \cdot 1 + 6 = 1 - 3 + 6 = 4 \]

There is a relative maximum at \( (-1, 8) \), and there is a relative minimum at \( (1, 4) \). We use the information obtained to sketch the graph.

11. \( f(x) = 2x^2 \)
First, find the critical points.
\[ f'(x) = 4x \]
\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):
\[ 4x = 0 \]
\[ x = 0 \]

The only critical point is at \( x = 0 \). We use 0 to divide the real number line into two intervals, A: \( (-\infty, 0) \) and B: \( (0, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.
A: Test 0. \( f'(0) = 4 \cdot 0 = 0 \)
B: Test -1, \( f'(-1) = 4(-1) = -4 < 0 \)

We see that \( f(x) \) is increasing on \( (-\infty, -1) \) and decreasing on \( (-1, \infty) \), so there is no change from decreasing to increasing or from increasing to decreasing. Therefore, the function does not have a relative extremum.

13. \( f(x) = 0.02x^2 + 1.3x + 2.31 \) First, find the critical points.
\[ f'(x) = 0.04x + 1.3 \]
\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):
\[ 0.04x + 1.3 = 0 \]
\[ 0.04x = -1.3 \]
\[ x = -32.5 \]

The only critical point is at \( x = -32.5 \). We use -32.5 to divide the real number line into two intervals, A: \( (-\infty, -32.5) \) and B: \( (-32.5, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.
A: Test 0. \( f'(0) = 0.04 \cdot 0 + 1.3 = 1.3 > 0 \)
B: Test -100, \( f'(-100) = 0.04 \cdot (-100) + 1.3 = -27 < 0 \)

We see that \( f(x) \) is decreasing on \( (-\infty, -32.5) \) and increasing on \( (-32.5, \infty) \), and the change from decreasing to increasing indicates that a relative minimum occurs at \( x = -32.5 \). We substitute into the original equation to find \( f(-32.5) \):
\[ f(-32.5) = 0.02(-32.5)^2 + 1.3(-32.5) + 2.31 \]
\[ = 21.125 - 42.25 + 2.31 \]
\[ = -18.815 \]

Thus, there is a relative minimum at \( (-32.5, -18.815) \). We use the information obtained to sketch the graph.
15. \( f(x) = x^4 - 2x^3 \)

\[ f'(x) = 4x^3 - 6x^2 \]

\( f'(x) \) exists for all real numbers. Solve \( f'(x) = 0 \).

\[
\begin{align*}
4x^3 - 6x^2 &= 0 \\
2x^3 - 3x^2 &= 0 \\
x^2(2x - 3) &= 0 \\
&\quad x = 0 \\
&\quad x = \frac{3}{2}
\end{align*}
\]

The critical points are \( x = 0 \) and \( x = \frac{3}{2} \). Use them to divide the real number line into three intervals, A: \( (-\infty, 0) \), B: \( (0, \frac{3}{2}) \), and C: \( \left( \frac{3}{2}, \infty \right) \).

- A: Test -1, \( f'(-1) = 4(-1)^3 - 6(-1)^2 = -10 < 0 \)
- B: Test 1, \( f'(1) = 4 \cdot 1^3 - 6 \cdot 1^2 = -2 < 0 \)
- C: Test 2, \( f'(2) = 4 \cdot 2^3 - 6 \cdot 2^2 = 8 > 0 \)

Since \( f(x) \) is decreasing on both \( (-\infty, 0) \) and \( (0, \frac{3}{2}) \) and is increasing on \( \left( \frac{3}{2}, \infty \right) \), there is no relative extremum at \( x = 0 \) but there is a relative minimum at \( x = \frac{3}{2} \).

\[
f\left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^4 - 2\left( \frac{3}{2} \right)^3 = \frac{81}{16} - \frac{27}{4} = \frac{-27}{16}
\]

There is a relative minimum at \( \left( \frac{3}{2}, -\frac{27}{16} \right) \). We sketch the graph.

17. \( f(x) = x\sqrt{8 - x^2} \)

The domain of this function is between \( [-\sqrt{8}, \sqrt{8}] \)

First, find the critical points.

\[
f'(x) = x \left( \frac{1}{2} (8 - x^2)^{-1/2} (-2x) \right) + (8 - x^2)^{1/2} (1)
\]

\[
= \frac{-x^2}{(8 - x^2)^{1/2}} + (8 - x^2)^{1/2}
\]

\( f'(x) \) does not exist for \( x = \pm \sqrt{8} \). We solve \( f'(x) = 0 \):

\[
\frac{-x^2}{(8 - x^2)^{1/2}} + (8 - x^2)^{1/2} = 0
\]

\[
x = \pm \sqrt{8}
\]

\[ 8 - x^2 = -x^2 \]

\[ 8 = 2x^2 \]

\[ 4 = x^2 \]

\[ \pm 2 = x \]

The critical points are at \( x = \pm \sqrt{8}, x = -2 \) and \( x = 2 \). We use them to divide the real number line into three intervals, A: \( (-\sqrt{8}, -2) \), B: \( (-2, 2) \), C: \( (2, \sqrt{8}) \).

We use a test value in each interval to determine the sign of the derivative in each interval.

- A: Test -2.5, \( f'(-2.5) = \frac{-(2.5)^2}{(8 - (2.5)^2)^{1/2}} + (8 - (2.5)^2)^{1/2} > 0 \)
- B: Test 0, \( f'(0) = \frac{-0^2}{(8 - 0^2)^{1/2}} + (8 - 0^2)^{1/2} = 0 \)
- C: Test 2.5, \( f'(2.5) = \frac{(2.5)^2}{(8 - (2.5)^2)^{1/2}} + (8 - (2.5)^2)^{1/2} < 0 \)

We see that \( f(x) \) is decreasing on \( (-\sqrt{8}, -2) \), and on \( (2, \sqrt{8}) \), and increasing on \( (-2, 2) \), so there is a relative minimum at \( x = -2 \) and a relative maximum at \( x = 2 \).

We find \( f(-2) \):

\[
f(-2) = (-2)\sqrt{8 - (-2)^2} = -4
\]

Then we find \( f(2) \):

\[
f(2) = (2)\sqrt{8 - (2)^2} = 4
\]

There is a relative minimum at \( (-2, -4) \), and there is a relative maximum at \( (2, 4) \). We use the information obtained to sketch the graph.
19. \( f(x) = 1 - x^{2/3} \)

First, find the critical points.

\[ f'(x) = -\frac{2}{3}x^{-1/3} = -\frac{2}{3\sqrt[3]{x}} \]

\( f'(x) \) does not exist for \( x = 0 \). The equation \( f'(x) = 0 \) has no solution, so the only critical point is at \( x = 0 \). We use it to divide the real number line into two intervals: A: \((-\infty, 0)\) and B: \((0, \infty)\).

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \(-1\), \( f'(-1) = \frac{2}{3\sqrt[3]{-1}} = -\frac{2}{3(-1)} = \frac{2}{3} > 0 \)

B: Test \(1\), \( f'(1) = -\frac{2}{3\sqrt[3]{1}} = -\frac{2}{3} < 0 \)

We see that \( f(x) \) is increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\), so there is a relative maximum at \( x = 0 \).

We find \( f(0) \):

\[ f(0) = 1 - 0^{2/3} = 1 - 0 = 1 \]

There is a relative maximum at \((0, 1)\). We use the information obtained to sketch the graph.

21. \( f(x) = \frac{-8}{x^2 + 1} = -8(x^2 + 1)^{-1} \)

First, find the critical points.

\[ f'(x) = -8(-1)(x^2 + 1)^{-2}(2x) = \frac{16x}{(x^2 + 1)^2} \]

\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):

\[ \frac{16x}{(x^2 + 1)^2} = 0 \]

\[ 16x = 0 \]

\[ x = 0 \]

The only critical point is at \( x = 0 \). We use it to divide the real number line into two intervals, A: \((-\infty, 0)\) and B: \((0, \infty)\).

23. \( f(x) = \frac{4x}{x^2 + 1} \)

First, find the critical points.

\[ f'(x) = \frac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2} = \frac{4 - 4x^2}{(x^2 + 1)^2} \]

\( f'(x) \) exists for all real numbers. We solve \( f'(x) = 0 \):

\[ \frac{4 - 4x^2}{(x^2 + 1)^2} = 0 \]

\[ 4 - 4x^2 = 0 \]

\[ x^2 = 1 \]

\[ x = \pm 1 \]

Or

\[ 1 + x = 0 \]

\[ x = -1 \]
The critical points are at \( x = -1 \) and \( x = 1 \). We use them to divide the real number line into three intervals, \( A: (-\infty, -1) \), \( B: (-1, 1) \), and \( C: (1, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.

**A:** Test \(-2\), \( f'(-2) = \frac{4 - 4\cdot(-2)^2}{[(-2)^2 + 1]^2} = -\frac{12}{25} < 0 \)

**B:** Test \(0\), \( f'(0) = \frac{4 - 4\cdot 0^2}{(0^2 + 1)^2} = \frac{4}{1} = 4 > 0 \)

**C:** Test \(2\), \( f'(2) = \frac{4 - 4\cdot 2^2}{(2^2 + 1)^2} = -\frac{12}{25} < 0 \)

We see that \( f(x) \) is decreasing on \(( -\infty, -1) \), increasing on \((-1, 1)\), and decreasing again on \((1, \infty)\), so there is a relative minimum at \( x = -1 \) and a relative maximum at \( x = 1 \).

We find \( f(-1) \):

\[
f(-1) = \frac{4(-1)}{(-1)^2 + 1} = \frac{-4}{2} = -2
\]

Then we find \( f(1) \):

\[
f(1) = \frac{4 - 4\cdot 1}{1^2 + 1} = \frac{4}{2} = 2
\]

There is a relative minimum at \((-1, -2)\), and there is a relative maximum at \((1, 2)\). We use the information obtained to sketch the graph.

**27.** \( f(x) = \sqrt{x^2 + 2x + 5} = \sqrt{(x^2 + 2x + 5)^{1/2}} \)

First, find the critical points.

\[
f'(x) = \frac{1}{2}(x^2 + 2x + 5)^{-1/2}(2x + 2) = \frac{x + 1}{\sqrt{x^2 + 2x + 5}}
\]

\(f'(x)\) exists for all \( x \) values. We solve \( f'(x) = 0 \),

\[
f'(c) = 0
\]

\[
\frac{x + 1}{\sqrt{x^2 + 2x + 5}} = 0
\]

\[
x + 1 = 0
\]

\[
x = -1
\]

So the only critical point is at \( x = -1 \). We use it to divide the real number line into two intervals, \( A: (-\infty, -1) \), and \( B: (-1, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.

**A:** Test \(-2\), \( f'(-2) = \frac{(-2) + 1}{\sqrt{(-2)^2 + 2(-2) + 5}} = -\frac{1}{\sqrt{5}} < 0 \)

**B:** Test \(0\), \( f'(0) = \frac{0 + 1}{\sqrt{(0)^2 + 2(0) + 5}} = \frac{1}{\sqrt{5}} > 0 \)

We see that \( f(x) \) is decreasing on \(( -\infty, -1) \) and increasing on \((-1, \infty)\), so the function has a relative minimum at \( x = -1 \).

We find \( f(-1) \):

\[
f(-1) = \sqrt{(-1)^2 + 2(-1) + 5} = \sqrt{1 - 2 + 5} = \sqrt{4} = 2
\]

There is a relative minimum at \((-1, 2)\). We use the information obtained to sketch the graph.
29. \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \cdot (x^2 + 1)^{-1/2} \)

First, find the critical points.

\[
f'(x) = -\frac{1}{2} (x^2 + 1)^{-3/2} (2x - \frac{2x}{\sqrt{(x^2 + 1)^3}})
\]

\( f'(x) \) exists for all \( x \) values. We solve \( f'(x) = 0 \).

\[
f'(x) = 0
\]

\[
\frac{2x}{\sqrt{(x^2 + 1)^3}} = 0
\]

\[
x = 0
\]

So the only critical point is at \( x = 0 \). We use it to divide the real number line into two intervals, \( A: (-\infty, 0) \) and \( B: (0, \infty) \).

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test -1, \( f'(-1) = \frac{2(-1)}{\sqrt{((-1)^2 + 1)^3}} = \frac{-2}{\sqrt{8}} < 0 \)

B: Test 1, \( f'(1) = \frac{2(1)}{\sqrt{(1^2 + 1)^3}} = \frac{2}{\sqrt{8}} > 0 \)

We see that \( f(x) \) is decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\), so the function has a relative minimum at \( x = 0 \).

We find \( f(0) \):

\[
f(0) = \sqrt{(0)^2 + 1} = \sqrt{1} = 1
\]

There is a relative minimum at \((0, 1)\). We use the information obtained to sketch the graph.

31. \( f(x) = \sin x \)

First, find the critical points.

\( f'(x) = \cos x \)

\( f'(x) \) exists for all \( x \) values. We solve \( f'(x) = 0 \).

\[
f'(x) = 0
\]

\[
\cos x = 0
\]

\[
x = \frac{\pi}{2}
\]

So the only critical points are at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \) and there might be extrema points at the end points \( x = 0 \) and \( x = 2\pi \). We use them to divide the real number line into three intervals, \( A: [0, \frac{\pi}{2}] \), \( B: (\frac{\pi}{2}, \frac{3\pi}{2}) \), and \( C: (\frac{3\pi}{2}, 2\pi] \).

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test 0, \( f'(0) = \cos(0) = 1 > 0 \)

B: Test \( \pi \), \( f'(\pi) = \cos(\pi) = -1 < 0 \)

C: Test \( 2\pi \), \( f'(2\pi) = \cos(2\pi) = 1 > 0 \)

We see that \( f(x) \) is decreasing on \((-\infty, \frac{\pi}{2})\) and increasing on \((\frac{\pi}{2}, \frac{3\pi}{2})\), we also see that \( f(x) \) is increasing on \((\frac{3\pi}{2}, \infty)\), so the function has a relative maximum at \( x = \frac{\pi}{2} \) and a relative minimum at \( x = \frac{3\pi}{2} \).

We find \( f(\frac{\pi}{2}) \):

\[
f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1
\]

We find \( f(\frac{3\pi}{2}) \):

\[
f(\frac{3\pi}{2}) = -\sin(\frac{3\pi}{2}) = -1
\]

There is a relative maximum at \((\frac{\pi}{2}, 1)\) and there is a relative minimum at \((\frac{3\pi}{2}, -1)\). We use the information obtained to sketch the graph.
33. \( f(x) = \sin x - \cos x \) First, find the critical points.

\( f'(x) = \cos x + \sin x \)

\( f'(x) \) exists for all \( x \) values. We solve \( f'(x) = 0 \).

\[
\begin{align*}
f'(x) &= 0 \\
\cos x + \sin x &= 0 \\
\cos x &= -\sin x \\
x &= \frac{3\pi}{4}
\end{align*}
\]

So the only critical points are at \( x = \frac{3\pi}{4} \) and \( x = \frac{7\pi}{4} \). We use them to divide the real number line into three intervals:

A: \([0, \frac{3\pi}{4})\), B: \((\frac{3\pi}{4}, \frac{7\pi}{4})\), C: \((\frac{7\pi}{4}, 2\pi]\).

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \(0\), \( f'(0) = \cos(0) + \sin(0) = 1 > 0 \)

B: Test \(\frac{\pi}{4}\), \( f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} > 0 \)

C: Test \(2\pi\), \( f'(2\pi) = \cos(2\pi) + \sin(2\pi) = 1 > 0 \)

We see that \( f(x) \) is increasing on \([0, \frac{\pi}{4})\) and on \((\frac{\pi}{4}, 2\pi]\) and decreasing on \((\frac{3\pi}{4}, \frac{5\pi}{4})\) so the function has a relative maximum at \( x = \frac{3\pi}{4} \) and a relative minimum at \( x = \frac{7\pi}{4} \).

We find \( f(\frac{3\pi}{4}) \):

\[
\begin{align*}
f\left(\frac{3\pi}{4}\right) &= \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) \\
&= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) \\
&= \frac{2}{\sqrt{2}} \approx 1.414
\end{align*}
\]

We find \( f\left(\frac{7\pi}{4}\right) \):

\[
\begin{align*}
f\left(\frac{7\pi}{4}\right) &= \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) \\
&= -\frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right) \\
&= -\frac{2}{\sqrt{2}} \approx -1.414
\end{align*}
\]

There is a relative maximum at \((\frac{3\pi}{4}, \frac{2}{\sqrt{2}})\) and there is a relative minimum at \((\frac{7\pi}{4}, \frac{2}{\sqrt{2}})\). We use the information obtained to sketch the graph.

35. \( f(x) = \cos 2x \) First, find the critical points.

\( f'(x) = -2\sin 2x \)

\( f'(x) \) exists for all \( x \) values. We solve \( f'(x) = 0 \).

\[
\begin{align*}
f'(x) &= 0 \\
-2\sin 2x &= 0 \\
2x &= 0 \\
x &= 0
\end{align*}
\]

and

\[
\begin{align*}
2x &= \pi \\
x &= \frac{\pi}{2}
\end{align*}
\]

and

\[
\begin{align*}
2x &= 3\pi \\
x &= \frac{3\pi}{2}
\end{align*}
\]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \(\pi\), \( f'(\pi) = -2\sin(2\pi) = 0 \)

B: Test \(\frac{\pi}{4}\), \( f'(\frac{\pi}{4}) = -2\sin\left(\frac{3\pi}{4}\right) = -2 > 0 \)

C: Test \(\frac{3\pi}{4}\), \( f'(\frac{3\pi}{4}) = -2\sin\left(\frac{5\pi}{4}\right) = -2 > 0 \)

D: Test \(\frac{7\pi}{4}\), \( f'(\frac{7\pi}{4}) = -2\sin\left(\frac{7\pi}{2}\right) = 2 > 0 \)

We see that \( f(x) \) is decreasing on \([0, \frac{\pi}{4})\) and \((\frac{3\pi}{4}, \frac{\pi}{2})\) and the function is increasing on \((\frac{\pi}{2}, \pi)\) and
Exercise Set 3.1

(3π, 2π) so the function has a relative maximum at x = 0
(end point), x = π and x = 2π (end point) and a relative
minimum at x = π/2 and x = 3π/2.

We find f'(0):

\[ f(0) = \cos(2 \cdot 0) \]
\[ = 1 \]

We find f'(π):

\[ f(\pi) = \cos(2 \cdot \pi) \]
\[ = 1 \]

We find f'(2π):

\[ f(2\pi) = \cos(2 \cdot 2\pi) \]
\[ = 1 \]

We find f'(\frac{\pi}{2}):

\[ f(\frac{\pi}{2}) = \cos(2 \cdot \frac{\pi}{2}) \]
\[ = \cos(\pi) \]
\[ = -1 \]

We find f'(\frac{3\pi}{2}):

\[ f(\frac{3\pi}{2}) = \cos(2 \cdot \frac{3\pi}{2}) \]
\[ = \cos(3\pi) \]
\[ = -1 \]

There is a relative maximum at (0, 1), (π, 1) and (2π, 1)
and there is a relative minimum at (\frac{\pi}{2}, -1) and (\frac{3\pi}{2}, -1).

We use the information obtained to sketch the graph.

\[ f(x) = x + \cos 2x \]

First, find the critical points.

\[ f'(x) = 1 - 2 \sin 2x \]

\[ f'(x) \] exists for all \( x \) values. We solve \( f'(x) = 0 \).

\[ f'(x) = 0 \]
\[ 1 - 2 \sin 2x = 0 \]
\[ \sin 2x = 1 \]
\[ x = \frac{\pi}{12} \]

and

\[ x = \frac{5\pi}{12} \]

So the only critical points are at x = \frac{\pi}{12}, x = \frac{5\pi}{12}, x = \frac{13\pi}{12},
and x = \frac{17\pi}{12} and there might be extreme points at the end
points x = 0 and x = 2π. We use them to divide the real
number line into five intervals: A: \[ [0, \frac{\pi}{12}] \]
B: \[ (\frac{\pi}{12}, \frac{\pi}{4}] \]
C: \[ (\frac{\pi}{4}, \frac{13\pi}{12}] \]
D: \[ (\frac{13\pi}{12}, \frac{17\pi}{12}] \]
and E: \[ (\frac{17\pi}{12}, 2\pi] \]

We use a test value in each interval to determine the sign
of the derivative in each interval.

A: Test 0. \( f''(0) = 1 - 2 \sin(2 \cdot 0) = 1 > 0 \)

B: Test \frac{\pi}{12}. \( f''(\frac{\pi}{12}) = 1 - 2 \sin(2 \cdot \frac{\pi}{4}) = 1 > 0 \)

C: Test \pi. \( f''(\pi) = 1 - 2 \sin(2 \cdot \pi) = 1 > 0 \)

D: Test \frac{13\pi}{12}. \( f''(\frac{13\pi}{12}) = 1 - 2 \sin(2 \cdot \frac{13\pi}{12}) = -1 < 0 \)

E: Test 2π. \( f''(2\pi) = 1 - 2 \sin(2 \cdot 2\pi) = 1 > 0 \)

We see that \( f(x) \) is increasing on \[ [0, \frac{\pi}{12}] \], \[ (\frac{\pi}{12}, \frac{13\pi}{12}] \], and \[ (\frac{13\pi}{12}, 2\pi] \] and decreasing on \[ (\frac{\pi}{2}, \frac{\pi}{12}] \], \[ (\frac{\pi}{4}, \frac{13\pi}{12}] \] and \[ (\frac{13\pi}{12}, \frac{17\pi}{12}] \) so the function has a relative maximum at \( x = \frac{\pi}{12} \) and \( x = \frac{13\pi}{12} \)
and a relative minimum at \( x = \frac{5\pi}{12} \), \( x = \frac{\pi}{2} \), and \( x = 2\pi \).

We find \( f(\frac{\pi}{12}) \):

\[ f(\frac{\pi}{12}) = \frac{\pi}{12} + \cos(2 \cdot \frac{\pi}{12}) \approx 1.128 \]

We find \( f(\frac{5\pi}{12}) \):

\[ f(\frac{5\pi}{12}) = \frac{5\pi}{12} + \cos(2 \cdot \frac{5\pi}{12}) \approx 0.443 \]

We find \( f(\frac{13\pi}{12}) \):

\[ f(\frac{13\pi}{12}) = \frac{13\pi}{12} + \cos(2 \cdot \frac{13\pi}{12}) \approx -1.269 \]

We find \( f(\frac{17\pi}{12}) \):

\[ f(\frac{17\pi}{12}) = \frac{17\pi}{12} + \cos(2 \cdot \frac{17\pi}{12}) \approx 3.585 \]

We find \( f(2\pi) \):

\[ f(2\pi) = 2\pi + \cos(2 \cdot 2\pi) \approx 7.383 \]

There is a relative maximum at \( (\frac{\pi}{12}, 1.128) \) and \( (\frac{13\pi}{12}, -1.269) \)
and a relative minimum at \( (\frac{5\pi}{12}, 0.443) \), \( (\frac{17\pi}{12}, 3.585) \), and
39. \( f(x) = \frac{x}{4} + \cos \frac{2x}{3} \). Find the critical values \( f'(x) = \frac{1}{3} - \frac{2}{3} \sin \frac{2x}{3} \). We solve \( f'(x) = 0 \)

\[
\frac{1}{3} - \frac{2}{3} \sin \frac{2x}{3} = 0
\]

\[
\frac{2x}{3} = \frac{\sin^{-1}(\frac{1}{2})}{2}
\]

\[
\frac{2x}{3} = \frac{\pi}{6}
\]

\[
x = \frac{\pi}{4}
\]

and

\[
\frac{2x}{3} = \frac{5\pi}{6}
\]

\[
x = \frac{5\pi}{4}
\]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \( \frac{\pi}{4} \), \( f'(\frac{\pi}{6}) = \frac{1}{3} - \frac{2}{3} \sin \frac{\pi}{9} = -0.105 > 0 \)

B: Test \( \pi \), \( f'(\pi) = \frac{1}{3} - \frac{2}{3} \sin \frac{2\pi}{3} = -0.323 < 0 \)

C: Test \( \frac{5\pi}{4} \), \( f'(\frac{5\pi}{6}) = \frac{1}{3} - \frac{2}{3} \sin \frac{\pi}{2} = 0.333 > 0 \)

We see that \( f(x) \) is increasing on \( (0, \frac{\pi}{4}) \) and \( (\frac{\pi}{4}, \frac{5\pi}{4}) \) and decreasing on \( (\frac{\pi}{4}, \frac{5\pi}{4}) \) so the function has a relative maximum at \( x = \frac{\pi}{4} \) and a relative minimum at \( x = \frac{5\pi}{4} \). We find \( f(\frac{\pi}{4}) \)

\[
f(\frac{\pi}{4}) = \frac{\pi}{12} + \cos(\frac{\pi}{6}) = 1.128
\]

We find \( f(\frac{5\pi}{4}) \)

\[
f(\frac{5\pi}{4}) = \frac{5\pi}{12} + \cos(\frac{5\pi}{6}) = 0.443
\]

There is a relative maximum at \( (\frac{\pi}{4}, 1.128) \) and a relative minimum at \( (\frac{5\pi}{4}, 0.443) \). We sketch the graph.

41. \( f(x) = \frac{\sin x}{2 - \sin x} \)

Find the critical values

\[
f'(x) = \frac{(2 - \sin x)(-\cos x) - \sin x(-\cos x)(-\sin x)}{(2 - \sin x)^2}
\]

\[
= \frac{-2 \sin x + \sin^2 x + \cos^2 x}{(2 - \sin x)^2}
\]

\[
= \frac{1 - 2 \sin x}{(2 - \sin x)^2}
\]

We solve \( f'(x) = 0 \)

\[
\frac{1 - 2 \sin x}{(2 - \sin x)^2} = 0
\]

\[
1 - 2 \sin x = 0
\]

\[
\sin x = \frac{1}{2}
\]

\[
x = \frac{\pi}{6}
\]

and

\[
x = \frac{5\pi}{6}
\]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \( \frac{\pi}{6} \), \( f'(\frac{\pi}{12}) = 0.196 > 0 \)

B: Test \( \pi \), \( f'(\pi) = -0.248 < 0 \)

C: Test \( \frac{5\pi}{12} \), \( f'(\frac{11\pi}{12}) = 0.159 > 0 \)

We see that \( f(x) \) is increasing on \( (0, \frac{\pi}{6}) \) and \( (\frac{5\pi}{6}, 2\pi) \) and decreasing on \( (\frac{\pi}{6}, \frac{5\pi}{6}) \) so the function has a relative maximum at \( x = \frac{\pi}{6} \) and a relative minimum at \( x = \frac{5\pi}{6} \).

We find \( f(\frac{\pi}{6}) \)

\[
f(\frac{\pi}{6}) = \frac{\cos(\frac{\pi}{6})}{2 - \sin(\frac{\pi}{6})}
\]
We find \( f'\left(\frac{\pi}{6}\right) \)
\[
\begin{align*}
  f'\left(\frac{\pi}{6}\right) &= \frac{\cos\left(\frac{3\pi}{6}\right)}{2 - \sin\left(\frac{3\pi}{6}\right)} \\
  &= \frac{-\sqrt{3}}{2 - \frac{1}{2}} \\
  &= \frac{-\sqrt{3}}{3}
\end{align*}
\]

There is a relative maximum at \( \left(\frac{\pi}{6}, -\sqrt{3}\right) \) and a relative minimum at \( \left(\frac{5\pi}{6}, \frac{\sqrt{3}}{2}\right) \). We sketch the graph.

43. \( f(x) = \sin x - \sin^2 x \)

Find the critical values
\[
f'(x) = \cos x - 2 \sin x \cos x \\
  \cos x(1 - 2 \sin x)
\]

We solve \( f'(x) = 0 \)
\[
\begin{align*}
  \cos x(1 - 2 \sin x) &= 0 \\
  \cos x &= 0 \\
  x &= \frac{\pi}{2} \\
  \text{and} \\
  1 - 2 \sin x &= 0 \\
  \sin x &= \frac{1}{2} \\
  x &= \frac{\pi}{6}, \frac{5\pi}{6} \\
  \text{and} \\
  x &= \frac{\pi}{2}
\end{align*}
\]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \( \frac{\pi}{6} \) \( f'\left(\frac{\pi}{6}\right) = \frac{1}{18} > 0 \)

B: Test \( \frac{\pi}{4} \) \( f'\left(\frac{\pi}{4}\right) = -0.293 < 0 \)

C: Test \( \frac{3\pi}{4} \) \( f'\left(\frac{3\pi}{4}\right) = 0.366 > 0 \)

D: Test \( \pi \) \( f'\left(\pi\right) = -1 < 0 \)

E: Test \( \frac{7\pi}{4} \) \( f'\left(\frac{7\pi}{4}\right) = 1.707 < 0 \)

We see that \( f(x) \) is increasing on \( \left[0, \frac{\pi}{6}\right) \), \( \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \), and \( \left(\frac{5\pi}{6}, \frac{7\pi}{4}\right] \) and decreasing on \( \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \) and \( \left(\frac{3\pi}{4}, \frac{5\pi}{6}\right] \), so the function has a relative maximum at \( x = \frac{\pi}{6} \) and \( x = \frac{5\pi}{6} \) and a relative minimum at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{4} \).

We find \( f\left(\frac{\pi}{6}\right) \)
\[
\begin{align*}
  f\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) \\
  &= -\frac{1}{4}
\end{align*}
\]

We find \( f\left(\frac{3\pi}{6}\right) \)
\[
\begin{align*}
  f\left(\frac{3\pi}{6}\right) &= \sin\left(\frac{3\pi}{6}\right) - \sin^2\left(\frac{3\pi}{6}\right) \\
  &= -\frac{1}{4}
\end{align*}
\]

We find \( f\left(\frac{\pi}{2}\right) \)
\[
\begin{align*}
  f\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right) \\
  &= 0
\end{align*}
\]

We find \( f\left(\frac{3\pi}{4}\right) \)
\[
\begin{align*}
  f\left(\frac{3\pi}{4}\right) &= \sin\left(\frac{3\pi}{4}\right) - \sin^2\left(\frac{3\pi}{4}\right) \\
  &= -\frac{1}{4}
\end{align*}
\]

There is a relative maximum at \( \left(\frac{\pi}{6}, \frac{1}{4}\right) \) and \( \left(\frac{3\pi}{6}, \frac{1}{4}\right) \) and a relative minimum at \( \left(\frac{\pi}{2}, 0\right) \) and \( \left(\frac{3\pi}{4}, -2\right) \). We sketch the graph.
45. \( f'(x) = 9 \sin x - 4 \sin^3 x \)

Find the critical values:

\[
f'(x) = 9 \cos x - 12 \sin^2 x \cos x
= 3 \cos x (3 - 4 \sin^2 x)
\]

We solve \( f'(x) = 0 \):

\[
3 \cos x (3 - 4 \sin^2 x) = 0
\]

\[\cos x = 0\]
\[x = \frac{\pi}{2}\]

and

\[x = \frac{3\pi}{2}\]

\[3 - 4 \sin^2 x = 0\]
\[\sin^2 x = \frac{3}{4}\]
\[\sin x = \pm \frac{\sqrt{3}}{2}\]

\[x = \frac{\pi}{2}\]

and

\[x = \frac{3\pi}{2}\]

and

\[x = \frac{5\pi}{2}\]

We use a test value in each interval to determine the sign of the derivative in each interval.

A: Test \( \frac{\pi}{6} \), \( f'(\frac{\pi}{6}) = 2.121 > 0 \)

B: Test \( \frac{\pi}{4} \), \( f'(\frac{\pi}{4}) = -0.568 < 0 \)

C: Test \( \frac{\pi}{12} \), \( f'(\frac{\pi}{12}) = 0.568 > 0 \)

D: Test \( \pi \), \( f'(\pi) = -9 < 0 \)

E: Test \( \frac{7\pi}{12} \), \( f'(\frac{7\pi}{12}) = 0.568 > 0 \)

F: Test \( \frac{5\pi}{6} \), \( f'(\frac{5\pi}{6}) = -0.568 < 0 \)

G: Test \( \frac{11\pi}{12} \), \( f'(\frac{11\pi}{12}) = -5.760 > 0 \)

We see that \( f(x) \) is increasing on \([0, \frac{\pi}{6}]\), \([\frac{\pi}{6}, \frac{\pi}{4}]\), \([\frac{\pi}{4}, \frac{\pi}{12}]\) and \([\frac{\pi}{12}, 2\pi]\) and decreasing on \([\frac{\pi}{12}, \frac{\pi}{6}]\), \([\frac{\pi}{6}, \frac{\pi}{4}]\) and \([\frac{\pi}{4}, \frac{\pi}{12}]\) so the function has a relative maximum at \( x = \frac{\pi}{12} \), \( x = \frac{2\pi}{3} \) and \( x = \frac{5\pi}{6} \) and a relative minimum at \( x = \frac{\pi}{6} \) and \( x = \frac{2\pi}{3} \).

We find \( f'(\frac{\pi}{3}) \):

\[
f'(\frac{\pi}{3}) = 9 \sin\left(\frac{\pi}{3}\right) - 4 \sin^3\left(\frac{\pi}{3}\right)
= \frac{9\sqrt{3}}{2} - \frac{12\sqrt{3}}{8}
= \frac{3\sqrt{3}}{8}
\]

We find \( f(\frac{2\pi}{3}) \):

\[
f(\frac{2\pi}{3}) = 9 \sin\left(\frac{2\pi}{3}\right) - 4 \sin^3\left(\frac{2\pi}{3}\right)
= \frac{9\sqrt{3}}{2} - \frac{12\sqrt{3}}{8}
= \frac{3\sqrt{3}}{8}
\]

We find \( f(\frac{3\pi}{2}) \):

\[
f(\frac{3\pi}{2}) = 9 \sin\left(\frac{3\pi}{2}\right) - 4 \sin^3\left(\frac{3\pi}{2}\right)
= -9 - 4(-1)
= 5
\]

We find \( f(\frac{5\pi}{6}) \):

\[
f(\frac{5\pi}{6}) = 9 \sin\left(\frac{5\pi}{6}\right) - 4 \sin^3\left(\frac{5\pi}{6}\right)
= \frac{9\sqrt{3}}{2} + \frac{12\sqrt{3}}{8}
= -3\sqrt{3}
\]

There is a relative maximum at \( (\frac{\pi}{3}, 3\sqrt{3}) \), \( (\frac{2\pi}{3}, 3\sqrt{3}) \) and \( (\frac{\pi}{2}, -5) \) and a relative minimum at \( (\frac{\pi}{6}, 3), (\frac{5\pi}{6}, -3\sqrt{3}) \) and \( (\frac{3\pi}{2}, -3\sqrt{3}) \). We sketch the graph.

47. - 91. Left to the student.
93. \( h(d) = -0.002d^2 + 0.8d + 6.6 \)
\[ h'(d) = -0.004d + 0.8 \]
Solve \( h'(d) = 0 \).
\[-0.004d + 0.8 = 0 \]
\[-0.004d = -0.8 \]
\[ d = 20 \]
A: Test 100, \( f'(50) = 0.1 > 0 \)
B: Test 300, \( f'(50) = -0.4 < 0 \)
The function is increasing on to the left of \( d = 200 \) and decreasing to the right of \( d = 200 \) therefore there is a relative maximum at \( d = 200 \). We find \( h(200) \)
\[ h(200) = -0.002(200)^2 + 0.8(200) + 6.6 \]
\[ = -80 + 160 + 6.6 \]
\[ = 86.6 \]
There is a relative maximum at \((200, 86.6)\)

95. \( f(t) = -0.1t^2 + 1.2t + 98.6, 0 \leq t \leq 12 \)
\[ f'(t) = -0.2t + 1.2 \]
f"(t) exists for all real numbers. Solve \( f''(t) = 0 \).
\[-0.2t + 1.2 = 0 \]
\[-0.2t = -1.2 \]
\[ t = 6 \]
The only critical point is at \( t = 6 \). We use it to divide the interval \([0, 12]\) (the domain of \( f(t) \)) into two intervals. A: \([0, 6]\) and B: \([6, 12]\).
A: Test 0, \( f'(0) = -0.2(0) + 1.2 = 1.2 > 0 \)
B: Test 7, \( f'(7) = -0.2(7) + 1.2 = -0.2 < 0 \)
Since \( f(t) \) is increasing on \([0, 6]\) and decreasing on \([6, 12]\), there is a relative maximum at \( x = 6 \).
\[ f(6) = -0.1(6)^2 + 1.2(6) + 98.6 = 102.2 \]
There is a relative maximum at \((6, 102.2)\). We sketch the graph.

97. Let us consider the intervals \([a, b], [c, d]\) and consider the sign of the slope of the tangent line. Next consider the intervals \([b, c]\) and \([d, e]\) and again consider the sign of the slope of the tangent line. The sign of the slope of the tangent line at a point, which is the value of the derivative of the function at that point, informs us whether the function is increasing or decreasing.

99. The critical values occur when the slope of the tangent line at the point is either zero or undefined (which usually indicates a maximum or a minimum).
(2, 1) a relative minimum
(4, 7) a relative maximum

101. Relative minimum at \((-5, 425)\) and \((4, -304)\) and a relative maximum at \((-2, 560)\).
3. \( f(x) = 2x^3 - 3x^2 - 36x + 28 \)
   a) Find \( f'(x) \) and \( f''(x) \).
   \[
   f'(x) = 6x^2 - 6x - 36
   \]
   \[
   f''(x) = 12x - 6
   \]
   b) Find the critical points of \( f \).
   Since \( f'(x) \) exists for all values of \( x \), the only critical points are where 
   \[
   6x^2 - 6x - 36 = 0
   \]
   \[
   6(x + 2)(x - 3) = 0
   \]
   \[
   x + 2 = 0 \quad \text{or} \quad x - 3 = 0
   \]
   \[
   x = -2 \quad \text{or} \quad x = 3 \quad \text{Critical points}
   \]
   Then
   \[
   f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 28
   \]
   \[
   = -16 - 12 + 72 + 28
   \]
   \[
   = 72.
   \]
   and
   \[
   f(3) = 2 \cdot 3^3 - 3 \cdot 3^2 - 36 \cdot 3 + 28
   \]
   \[
   = 54 - 27 - 108 + 28
   \]
   \[
   = -53.
   \]
   These give the points \((-2, 72)\) and \((3, -53)\) on the graph.

   c) Use the Second-Derivative Test:
   \[
   f''(-2) = 12(-2) - 6 = -30 < 0, \text{ so } (-2, 72) \text{ is a relative maximum.}
   \]
   \[
   f''(3) = 12 \cdot 3 - 6 = 30 > 0, \text{ so } (3, -53) \text{ is a relative minimum.}
   \]
   Then if we use the points \(-2\) and \(3\) to divide the real number line into three intervals, \((-\infty, -2), \ (-2, 3), \text{ and } (3, \infty), \) we know that \( f \) is increasing on \((-\infty, -2), \) decreasing on \((-2, 3), \) and increasing again on \((3, \infty).\)

   d) Find the possible inflection points.
   \[
   f''(x) \text{ exists for all values of } x, \text{ so we solve } f''(x) = 0.
   \]
   \[
   12x - 6 = 0
   \]
   \[
   x = \frac{1}{2} \quad \text{Possible inflection point}
   \]
   Then
   \[
   f \left( \frac{1}{2} \right) = 2 \left( \frac{1}{2} \right)^3 - 3 \left( \frac{1}{2} \right)^2 - 36 \left( \frac{1}{2} \right) + 28
   \]
   \[
   = \frac{1}{4} - \frac{3}{4} - 18 + 28
   \]
   \[
   = \frac{19}{2}.
   \]
   This gives the point \(\left( \frac{1}{2}, \frac{19}{2} \right)\) on the graph.

   v) To determine the concavity we use the possible inflection point \(\frac{1}{2}\), to divide the real number line into two intervals, \(A: \left( -\infty, \frac{1}{2} \right) \) and \(B: \left( \frac{1}{2}, \infty \right) \). Test a point in each interval.
   A: Test 0, \( f''(0) = 12 \cdot 0 - 6 = -6 < 0 \)
   B: Test 1, \( f''(1) = 12 \cdot 1 - 6 = 6 > 0 \)
   Then \( f \) is concave down on \(\left( -\infty, \frac{1}{2} \right)\) and concave up on \(\left( \frac{1}{2}, \infty \right). \) \( \frac{1}{2}, \frac{19}{2} \) is an inflection point.

   f) Sketch the graph using the preceding information.

5. \( f(x) = \frac{8}{3}x^3 - 2x + \frac{1}{3} \)
   a) Find \( f'(x) \) and \( f''(x) \).
   \[
   f'(x) = 8x^2 - 2
   \]
   \[
   f''(x) = 16x
   \]
b) Find the critical points of $f$.

Now $f'(x) = 8x^2 - 2$ exists for all values of $x$, so the only critical points of $f$ are where $8x^2 - 2 = 0$.

\[8x^2 - 2 = 0\]
\[x^2 = \frac{1}{4}\]
\[x = \pm \frac{1}{2}\]

Critical points

Then $f'\left(-\frac{1}{2}\right) = \frac{8}{3}\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + \frac{1}{3} = \frac{1}{3}$
and $f'\left(\frac{1}{2}\right) = \frac{8}{3}\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right) + \frac{1}{3} = \frac{1}{3}$

These give the points $\left(-\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$ on the graph.

c) To determine the concavity, we use the possible inflection point, 0, to divide the real number line into two intervals: A: $(-\infty, 0)$ and B: $(0, \infty)$. Test a point in each interval.

A: Test -1, $f''(-1) = 16(-1) = -16 < 0$
B: Test 1, $f''(1) = 16(1) = 16 > 0$

Then $f'(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$, so $\left(0, \frac{1}{3}\right)$ is an inflection point.

f) Sketch the graph using the preceding information.

7. $f(x) = 3x^4 - 10x^3 + 10x^2$

a) $f'(x) = 12x^3 - 30x^2 + 20x$
\[f''(x) = 36x^2 - 72x = 12x(3x - 6) = 12x(3x - 6) = 0\]

b) Since $f'(x)$ exists for all values of $x$, the only critical points are where $f'(x) = 0$.

$12x^3 - 30x^2 + 20x = 0$
\[12x(x^2 - 4x + 3) = 0\]
\[12x(x - 1)(x - 3) = 0\]

$x = 0$ or $x = 1$ or $x = 3$

Then $f(0) = 3(0)^4 - 10(0)^3 + 18(0)^2 = 0$,
\[f(1) = 3(1)^4 - 10(1)^3 + 18(1)^2 = 15\]
and $f(3) = 3(3)^4 - 10(3)^3 + 18(3)^2 = 27$

These give the points $(0, 0)$, $(1, 5)$, and $(3, 27)$ on the graph.

c) Use the Second-Derivative Test:

\[f''(0) = 36(0)^2 - 72(0) = 36 > 0, \text{ so } (0, 0) \text{ is a relative minimum.}\]
\[f''(1) = 36(1)^2 - 72(1) = -24 < 0, \text{ so } (1, 5) \text{ is a relative maximum.}\]
\[f''(3) = 36(3)^2 - 72(3) + 36 = 72 > 0, \text{ so } (3, 27) \text{ is a relative minimum.}\]

Then if we use the points 0, 1, and 3 to divide the real number line into four intervals, $(-\infty, 0)$, $(0, 1)$, $(1, 3)$, and $(3, \infty)$, we know that $f$ is decreasing on $(-\infty, 0)$ and on $(1, 3)$ and is increasing on $(0, 1)$ and $(3, \infty)$. 

d) \( f''(x) \) exists for all values of \( x \), so the only possible inflection points are where \( f''(x) = 0 \):

\[
36x^2 - 96x + 36 = 0
\]

\[
12(3x^2 - 8x + 3) = 0
\]

\[
3x^2 - 8x + 3 = 0
\]

Using the quadratic formula, we find \( x = \frac{4 \pm \sqrt{16}}{3} \), so \( x \approx 0.45 \) or \( x \approx 2.22 \) are possible inflection points.

Then \( f(0.45) \approx 2.31 \) and \( f(2.22) \approx -13.48 \), so \((0.45, 2.31)\) and \((2.22, -13.48)\) are two more points on the graph.

e) To determine the concavity, we use the points \((0.45, 2.31)\) and \((2.22, -13.48)\). Test a point in each interval.

A: Test 0, \( f''(0) = 36 - 96 - 0 + 36 = 36 > 0 \)

B: Test 1, \( f''(1) = 36 - 96 - 96 + 1 + 36 = -24 < 0 \)

C: Test 3, \( f''(3) = 36 - 96 - 96 + 3 + 36 = 72 > 0 \)

Then \( f \) is concave up on \((-\infty, 0.45)\), concave down on \((0.45, 2.22)\), and concave up on \((2.22, \infty)\). The equation \( f''(x) = 0 \) has no solution, so the only possible inflection point is \(-1\). We have already found \(-1\) in step (b).

e) To determine the concavity, we use \(-1\) to divide the real number line into two intervals as in step (e).

A: Test \(-2\), \( f''(-2) = \frac{-2}{9\sqrt{(-2+1)^4}} = \frac{-2}{9} < 0 \)

B: Test 0, \( f''(0) = \frac{2}{9\sqrt{(0+1)^4}} = \frac{2}{9} > 0 \)

Then \( f \) is concave down on both intervals, so there is no inflection point.

f) Sketch the graph using the preceding information.

\[f(x) = x^4 - 6x^2\]

a) \( f'(x) = 4x^3 - 12x \)

\( f''(x) = 12x^2 - 12 \)

b) Since \( f'(x) \) exists for all values of \( x \), the only critical points are where \( 4x^3 - 12x = 0 \),

\( 4x(x^2 - 3) = 0 \)

\( x = 0 \) or \( x^2 - 3 = 0 \)

\( x = 0 \) or \( x = \pm \sqrt{3} \)

The critical points are \(-\sqrt{3}, 0, \) and \( \sqrt{3} \).

\( f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 \)

\( = -9 - 6 \cdot 3 = -9 - 18 = -27 \)

\( f(0) = 0^4 - 6 \cdot 0^2 = 0 - 0 = 0 \)

\( f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 \)

\( = 9 - 6 \cdot 3 = 9 - 18 = -9 \)

These give the points \((\sqrt{3}, -9), (0, 0), \) and \((-\sqrt{3}, -9)\) on the graph.
c) Use the Second-Derivative Test:
\[ f''(\sqrt{3}) = 12(\sqrt{3})^2 - 12 = 12 \cdot 3 = 36 > 0, \]
so \( f''(\sqrt{3}) > 0 \) is a relative minimum.

Since \( f''(\sqrt{3}) > 0 \), then \( f(x) \) is increasing on \((\sqrt{3}, \infty)\) and is decreasing on \((\sqrt{3}, 0)\).

Then if we use the values of \( -\sqrt{3}, 0, \sqrt{3} \) to divide the real number line into four intervals, \((-\infty, -\sqrt{3}), (-\sqrt{3}, 0), (0, \sqrt{3}), (\sqrt{3}, \infty)\), we know that \( f(x) \) is decreasing on \((-\infty, -\sqrt{3})\) and \((0, \sqrt{3})\) and is increasing on \(( -\sqrt{3}, 0)\) and \((\sqrt{3}, \infty)\).

d) Since \( f''(x) \) exists for all values of \( x \), the only possible inflection points are where \( 12x^2 - 12 = 0 \),
\[
12x^2 - 12 = 0
\]
\[
x^2 - 1 = 0
\]
\[
(x + 1)(x - 1) = 0
\]
\[
x = -1 \text{ or } x = 1
\]
The possible inflection points are \(-1, 1\).

The critical points are \(-1, 1\). Since \( f''(x) = 36(x^2 - 1) \) and \( f''(-1) = 36(-2) = -72 < 0 \),
\[
 f''(x) = 36(x^2 - 1) = 36(-2) = -72 < 0
\]

These give the points \((-1, -5)\) and \((1, 5)\) on the graph.

c) To determine the concavity we use the points \(-1\) and \(1\) to divide the real number line into three intervals, \(-\infty, -1\), \((-1, 1)\), and \((1, \infty)\). Test a point in each interval.

A: Test \(-2, f''(-2) = 12(-2)^2 - 12 = 36 > 0\)
B: Test 0, \( f''(0) = 12(0)^2 - 12 = -12 < 0\)
C: Test 2, \( f''(2) = 12(2)^2 - 12 = 24 > 0\)

We see that \( f(x) \) is concave up on the intervals \((-\infty, -1)\) and \((1, \infty)\) and concave down on the interval \((-1, 1)\), so \((-1, -5)\) and \((1, 5)\) are inflection points.

e) Sketch the graph using the preceding information. By solving, \( x^3 - 6x^2 = 0 \) we can find the \( x \)-intercepts.
They are helpful in graphing.
\[
x^3 - 6x^2 = 0
\]
\[
x^2(x - 6) = 0
\]
\[
x = 0 \text{ or } x = 6
\]
\[
x = 0 \text{ or } x = \pm \sqrt{6}
\]
The \( x \)-intercepts are \((0, 0)\), \((-\sqrt{6}, 0)\), and \((\sqrt{6}, 0)\).

13. \( f(x) = 3x^4 + 4x^2 \)

a) \( f'(x) = 12x^3 + 8x \)
\[
f''(x) = 36x^2 + 24x
\]

b) Since \( f'(x) \) exists for all values of \( x \), the only critical points of \( f \) are where \( 12x^3 + 8x = 0 \),
\[
12x(x^2 + 2) = 0
\]
\[
12x(x - 1)(x + 2) = 0
\]
\[
\text{or } x = 0 \text{ or } x = 1 \text{ or } x = -2
\]
The critical points are 0 and -1.

The critical points are 0 and -1.

The critical points are 0 and -1.

The critical points are 0 and -1.

The critical points are 0 and -1.

These give the points \((0, 0)\) and \((-1, -1)\) on the graph.

c) Use the Second-Derivative Test:
\[
f''(-1) = 36(-1)^2 + 24(-1) - 36 - 24 = 12 > 0, \text{ so } (-1, -1) \text{ is a relative minimum.}
\]
\[
f''(0) = 36(0)^2 + 24(0) - 36 - 24 = 0 \text{, so this test fails.}
\]

We will use the First-Derivative Test. Use 0 to divide the interval \((-1, \infty)\) into two intervals, \((-1, 0)\) and \((0, \infty)\). Test a point in each interval.

A: Test \(-\frac{1}{2}, f'(\frac{-1}{2}) = 12\left(-\frac{1}{12}\right)^2 + 12\left(\frac{1}{2}\right)^2 \geq 0
\]

B: Test 2, \( f'(-2) = 12(2)^2 + 12(2)^2 = 144 > 0
\)

Since \( f(x) \) is increasing on both intervals, \((0, \infty)\) and \((-1, 0)\), we know that \( f(x) \) is increasing on \((\infty, -1)\).

d) Now \( f''(x) \) exists for all values of \( x \), so the only possible inflection points are where \( 36x^2 + 24x = 0 \),
\[
36x^2 + 24x = 0
\]
\[
12x(3x + 2) = 0
\]
\[
x = 0 \text{ or } x = -\frac{2}{3}
\]
The possible inflection points are 0 and \(-\frac{2}{3}\).
\[ f(\theta) = 3 \cdot 0^4 + 4 \cdot 0^4 = 0 \quad \text{Already found in step (b)} \]

\[
f\left( -\frac{2}{3} \right) = 3\left( -\frac{2}{3} \right)^4 + 4\left( -\frac{2}{3} \right)^3
= \frac{3 \cdot 16}{81} + 4 \cdot \left( -\frac{8}{27} \right)
= \frac{16}{27} - \frac{32}{27}
= -\frac{16}{27}
\]

This gives one additional point \(-\frac{2}{3}, -\frac{16}{27}\) on the graph.

e) To determine the concavity we use \(-\frac{2}{3}\) and 0 to divide the real number line into three intervals, A: \((-\infty, -\frac{2}{3})\), B: \((-\frac{2}{3}, 0)\), and C: \((0, \infty)\). Test a point in each interval.

A: Test \(-1\), \(f''(-1) = 36(-1)^2 + 24(-1) = 12 > 0\)

B: Test \(-\frac{1}{2}\), \(f''\left(-\frac{1}{2}\right) = 36\left(-\frac{1}{2}\right)^2 + 24\left(-\frac{1}{2}\right) = -3 < 0\)

C: Test 1, \(f''(1) = 36 \cdot 1^2 + 24 \cdot 1 = 60 > 0\)

We see that \(f\) is concave up on the intervals \((-\infty, -\frac{2}{3})\) and \((0, \infty)\) and concave down on the interval \((-\frac{2}{3}, 0)\), so \(-\frac{2}{3}, -\frac{16}{27}\) and \((0, 0)\) are both inflection points.

f) Sketch the graph using the preceding information.

By solving \(3x^4 + 4x^3 = 0\) we can find \(x\)-intercepts. They are helpful in graphing.

\(3x^4 + 4x^3 = 0\)
\(x^3(3x + 4) = 0\)
\(x^3 = 0\) or \(3x + 4 = 0\)
\(x = 0\) or \(x = -\frac{4}{3}\)

The intercepts are \((0, 0)\) and \(-\frac{4}{3}, 0\).

15. \(f(x) = x^4 - 6x^2 - 135x\)

a) \(f'(x) = 3x^2 - 12x - 135\)
\(f''(x) = 6x - 12\)

b) Since \(f'(x)\) exists for all values of \(x\), the only critical points of \(f\) are where \(3x^2 - 12x - 135 = 0\),
\(3x^2 - 12x - 135 = 0\)
\(x^2 - 4x - 45 = 0\)
\((x - 9)(x + 5) = 0\)
\(x = 9\) or \(x = -5\)

The critical points are 9 and -5.

\(f(9) = 9^4 - 6 \cdot 9^2 - 135 \cdot 9\)
\(-729 - 486 - 1215 = -2430\)
\(f(-5) = (-5)^4 - 6(-5)^2 - 135(-5)\)
\(-625 - 30 - 135 = -860\)

These give the points \((9, -2430)\) and \((-5, -860)\) on the graph.

c) Use the Second-Derivative Test:
\(f''(-5) = 6(-5) - 12 = -30 - 12 = -42 < 0\), so \((-5, -860)\) is a relative maximum.
\(f''(9) = 6 \cdot 9 - 12 = 54 - 12 = 42 > 0\), so \((9, -2430)\) is a relative minimum.

Then if we use the points -5 and 0 to divide the real number line into three intervals, \((-\infty, -5)\), \((-5, 9)\), and \((9, \infty)\), we know that \(f\) is increasing on \((-\infty, -5)\) and \(f\) is decreasing on \((-5, 9)\).

d) Now \(f''(x)\) exists for all values of \(x\), so the only possible inflection points are where \(6x - 12 = 0\).
\(6x = 12\)
\(x = 2\) Possible inflection point

\(f(2) = 2^4 - 6 \cdot 2^2 - 135 \cdot 2\)
\(= 16 - 24 - 270\)
\(= -286\)

This gives another point \((2, -286)\) on the graph.

e) To determine the concavity we use 2 to divide the real number line into two intervals, A: \((-\infty, 2)\) and B: \((2, \infty)\). Test a point in each interval.

A: Test 0, \(f''(0) = 6 \cdot 0 - 12 = -12 < 0\)

B: Test 3, \(f''(3) = 6 \cdot 3 - 12 = 6 > 0\)

We see that \(f\) is concave down on \((-\infty, 2)\) and concave up on \((2, \infty)\), so \((2, -286)\) is an inflection point.

f) Sketch the graph using the preceding information.
Exercise Set 3.2

17. \( f(x) = \frac{-x}{\sqrt{x^2 + 1}} \)

a) \( f'(x) = \frac{(x^2 + 1)(1) - 2x \cdot x}{(x^2 + 1)^2} \) Quotient Rule

\[ f'(x) = \frac{2}{(x^2 + 1)^{3/2}} \]

b) Since \( f'(x) \) exists for all real numbers, the only critical points are where \( f'(x) = 0. \)

\[ 1 - x^2 \]
\[ (x^2 + 1)^{3/2} = 0 \]

\[ 1 - x^2 = 0 \quad x = 1 \quad or \quad 1 - x = 0 \]

\[ x = 1 \quad x = 1 \quad Critical \ points \]

Then \( f(-1) = \frac{-1}{(1^2 + 1)^{3/2}} = \frac{-1}{2} \) and \( f(1) = \frac{1}{(1^2 + 1)^{3/2}} \) are on the graph.

c) Use the Second-Derivative Test:

\[ f''(x) = \frac{2(x^2 + 1)^{3/2} - 2x(2x)(x^2 + 1)}{[(x^2 + 1)^{3/2}]^2} \]

\[ f''(1) = \frac{2 - 6}{[(1^2 + 1)^{3/2}]^2} = \frac{-4}{8} = -\frac{1}{2} < 0, \quad so \quad \left( 1, \frac{1}{2} \right) \quad is \quad a \quad relative \ minimum. \]

Then if we use \(-1 \) and \( 1 \) to divide the real number line into three intervals, \((-\infty, -1), (-1, 1), \) and \((1, \infty), \) we know that \( f \) is decreasing on \((-\infty, -1)\) and on \((1, \infty)\) and is increasing on \((-1, 1)\).

d) \( f''(x) \) exists for all real numbers, so the only possible inflection points are where \( f''(x) = 0. \)

\[ \frac{2x^3 - 6x}{(x^2 + 1)^{3/2}} = 0 \]

\[ 2x(x^2 - 3) = 0 \]

\[ 2x = 0 \quad or \quad x^2 - 3 = 0 \]

\[ x = 0 \quad or \quad x = \pm \sqrt{3} \]

Possible inflection points

\[ f(\sqrt{3}) = \frac{-\sqrt{3}}{(\sqrt{3})^2 + 1} = \frac{-\sqrt{3}}{4} \]

\[ f(0) = \frac{0}{1} = 0 \]

\[ f(\sqrt{3}) = \frac{\sqrt{3}}{(\sqrt{3})^2 + 1} = \frac{\sqrt{3}}{4} \]

These give the points \( (-\sqrt{3}, -\frac{\sqrt{3}}{4}), (0, 0), \) and \( (\sqrt{3}, \frac{\sqrt{3}}{4}) \) on the graph.

e) To determine the concavity, we use \(-\sqrt{3}, 0, \) and \( \sqrt{3} \) to divide the real number line into four intervals, A: \((-\infty, -\sqrt{3})\), B: \((-\sqrt{3}, 0)\), C: \((0, \sqrt{3})\), and D: \((\sqrt{3}, \infty)\). Test a point in each interval.

A: \( Test: -2, \quad f''(-2) = \frac{-3}{125} < 0 \)

B: \( Test: -1, \quad f''(-1) = \frac{-1}{2} > 0 \)

C: \( Test: 1, \quad f''(1) = \frac{-1}{2} < 0 \)

D: \( Test: 2, \quad f''(2) = \frac{-1}{125} > 0 \)

Then \( f \) is concave down on \((-\infty, -\sqrt{3})\) and on \((0, \sqrt{3})\) and is concave up on \((-\sqrt{3}, 0)\) and on \((\sqrt{3}, \infty)\), so \( (-\sqrt{3}, -\frac{\sqrt{3}}{4}), (0, 0), \) and \( (\sqrt{3}, \frac{\sqrt{3}}{4}) \) are all inflection points.

f) Sketch the graph using the preceding information.
19. \( f(x) = (x - 1)^3 \)

a) \( f'(x) = 3(x - 1)^2 \)  
\( f''(x) = 3 \cdot 2(x - 1)(1) = 6(x - 1) \)

b) Since \( f'(x) \) exists for all real numbers, the only critical points are where \( f'(x) = 0 \).

\[ 3(x - 1)^2 = 0 \]
\[ (x - 1)^2 = 0 \]
\[ x - 1 = 0 \]
\[ x = 1 \]

Critical point

Then \( f(1) = (1 - 1)^3 = 0 \), so \((1, 0)\) is on the graph.

c) The Second-Derivative Test fails since \( f''(1) = 0 \), so we use the First-Derivative Test. Use 1 to divide the real number line into two intervals, A: \((-\infty, 1)\) and B: \((1, \infty)\). Test a point in each interval:

A: Test 0, \( f'(0) = 3(0 - 1)^2 = 3 > 0 \)

B: Test 2, \( f'(2) = 3(2 - 1)^2 = 3 > 0 \)

Since \( f \) is increasing on both intervals, \((1, 0)\) is not a relative extremum.

d) \( f''(x) \) exists for all real numbers, so the only possible inflection points are where \( f''(x) = 0 \).

\[ 6(x - 1) = 0 \]
\[ x - 1 = 0 \]
\[ x = 1 \]

Possible inflection point

From step (b), we know that \((1, 0)\) is on the graph.

e) To determine the concavity, we use 1 to divide the real number line as in step (c).

A: Test 0, \( f''(0) = 6(0 - 1) = -6 < 0 \)

B: Test 2, \( f''(2) = 6(2 - 1) = 6 > 0 \)

Then \( f \) is concave down on \((-\infty, 1)\) and concave up on \((1, \infty)\), so \((1, 0)\) is an inflection point.

l) Sketch the graph using the preceding information.

21. \( f(x) = x^2(1 - x)^2 \)

a) \( f'(x) = 2x(1 - x)^2 + 4x^3(1 - x) \)
\[ = x^2 - 2x^3 + x^4 \]

b) Since \( f'(x) \) exists for all real numbers, the only critical points are where \( f'(x) = 0 \).

\[ 2x - 6x^2 + 4x^3 = 0 \]
\[ 2x(1 - 3x + 2x^2) = 0 \]
\[ 2x(1 - x)(1 - 2x) = 0 \]
\[ x = 0 \text{ or } 1 - x = 0 \text{ or } 1 - 2x = 0 \]
\[ x = 0 \text{ or } 1 = x \text{ or } 1 = 2x \]
\[ x = 0 \text{ or } 1 = x \text{ or } \frac{1}{2} = x \]

Critical points

\[ f(0) = 0^2(1 - 0)^2 = 0 \]
\[ f(1) = 1^2(1 - 1)^2 = 0 \]
\[ f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^2 = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

Thus, \((0, 0)\), \((1, 0)\), and \(\left(\frac{1}{2}, \frac{1}{16}\right)\) are on the graph.

c) Use the Second-Derivative Test:

\[ f''(0) = 2 - 12 \cdot 0 + 12 \cdot 0^2 = 2 > 0, \text{ so } (0, 0) \text{ is a relative minimum.} \]
\[ f''\left(\frac{1}{2}\right) = 2 - 12 \cdot \frac{1}{2} + 12 \left(\frac{1}{2}\right)^2 = 2 - 6 + 3 = -1 < 0, \text{ so } \left(\frac{1}{2}, \frac{1}{16}\right) \text{ is a relative maximum.} \]
\[ f''(1) = 2 - 12 \cdot 1 + 12 \cdot 1^2 = 2 > 0, \text{ so } (1, 0) \text{ is a relative minimum.} \]

Then if we use the points 0, \(\frac{1}{2}\), and 1 to divide the real number line into four intervals, \((-\infty, 0)\), \((0, \frac{1}{2})\), \(\left(\frac{1}{2}, 1\right)\), and \((1, \infty)\), we know that \( f \) is decreasing on \((-\infty, 0)\), \(\left(-\infty, \frac{1}{2}\right)\), \(\left(\frac{1}{2}, 1\right)\), and is increasing on \(\left(0, \frac{1}{2}\right)\) and on \((1, \infty)\).
d) $f''(x)$ exists for all real numbers, so the only possible inflection points are where $f''(x) = 0$.
\[
2 - 12x + 12x^2 = 0
\]
\[
2(1 - 6x + 6x^2) = 0
\]
Using the quadratic formula we find
\[
x = \frac{3 \pm \sqrt{3}}{6}
\]
$x \approx 0.21$ or $x \approx 0.79$ Possible inflection points
\[
f(0.21) \approx 0.03 \text{ and } f(0.79) \approx 0.03, \text{ so } (0.21, 0.03) \text{ and } (0.79, 0.03) \text{ are on the graph.}
\]

c) To determine the concavity we use 0.21 and 0.79 to divide the real number line into three intervals.
A: $(-\infty, 0.21)$, B: $(0.21, 0.79)$, and C: $(0.79, \infty)$.
A: Test 0, $f''(0) = 2 - 12 \cdot 0 + 12 \cdot 0^2 = 2 > 0$
B: Test 0.5, $f''(0.5) = 2 - 12(0.5) + 12(0.5)^2 = -1 < 0$
C: Test 0.7, $f''(0.7) = 2 - 12 \cdot 0.7 + 12 \cdot 0.7^2 = 2 > 0$
Then $f$ is concave up on $(-\infty, 0.21)$ and on $(0.79, \infty)$ and is concave down on $(0.21, 0.79)$, so $(0.21, 0.03)$ and $(0.79, 0.03)$ are both inflection points.

d) Sketch the graph using the preceding information.

23. $f(x) = 2x^3 - 3x^2$

a) $f'(x) = 6x^2 - 6x^1$
$f''(x) = 12x - 60x^3$

b) Since $f'(x)$ exists for all real numbers, the only critical points are where $f'(x) = 0$.
\[
60x^2 - 15x^1 = 0
\]
\[
15x^2(4 - x^2) = 0
\]
\[
15x^2 = 0 \text{ or } 4 - x^2 = 0
\]
\[
x = 0 \text{ or } x = 2 \text{ or } x = -2
\]
Critical points
\[
f(0) = 20 \cdot 0^3 - 3 \cdot 0^2 = 0
\]
\[
f(-2) = 20(-2)^3 - 3(-2)^2 = -160 + 96 = -64
\]
\[
f(2) = 20 \cdot 2^3 - 3 \cdot 2^2 = 160 - 96 + 64
\]
Thus, $(0, 0)$, $(-2, -64)$, and $(2, 64)$ are on the graph.

e) Use the Second-Derivative Test:
\[
f''(-2) = 120(-2) - 60(-2)^3 = -240 + 480 = 240 > 0, \text{ so } (-2, -64) \text{ is a relative minimum.}
\]
\[
f''(2) = 120 \cdot 2 - 60 \cdot 2^3 = 240 - 192 = 48 < 0, \text{ so } (2, 64) \text{ is a relative maximum.}
\]
\[
f''(0) = 120 \cdot 0 - 60 \cdot 0^3 = 0, \text{ so we will use the First-Derivative Test on } x = 0. \text{ Use } 0 \text{ to divide the intervals } (-2, 2) \text{ into two intervals, A: } (-2, 0). \text{ and B: } (0, 2). \text{ Test a point in each interval.}
\]
A: Test -1, $f'(-1) = 60(-1)^2 - 15(-1)^1 = 60 - 15 = 45 > 0$
B: Test 1, $f'(1) = 60 \cdot 1^2 - 15 \cdot 1^1 = 60 - 15 = 45 > 0$
Then $f$ is increasing on both intervals so $(0, 0)$ is not a relative extremum.
We use the points $-2$ and $2$ to divide the real number line into three intervals, $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$. We know that $f$ is decreasing on $(\infty, -2)$ and on $(2, \infty)$ and is increasing on $(-2, 2)$.

d) $f''(x)$ exists for all real numbers, so the only possible inflection points are where $f''(x) = 0$.
\[
120x - 60x^3 = 0
\]
\[
60x(2 - x^2) = 0
\]
\[
60x = 0 \text{ or } 2 - x^2 = 0
\]
\[
x = 0 \text{ or } x = \pm \sqrt{2}
\]
Possible inflection points
\[
f(-\sqrt{2}) = 20(-\sqrt{2})^3 - 3(-\sqrt{2})^2 = -40\sqrt{2} + 12\sqrt{2} = -28\sqrt{2}
\]
\[
f(0) = 0 \text{ from step (b)}
\]
\[
f(\sqrt{2}) = 2(\sqrt{2})^3 - 3(\sqrt{2})^2 = 4\sqrt{2} - 12\sqrt{2} = 28\sqrt{2}
\]
Thus, $(-\sqrt{2}, -28\sqrt{2})$ and $(\sqrt{2}, 28\sqrt{2})$ are also on the graph.

e) To determine the concavity we use $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the real number line into four intervals.
A: $(-\infty, -\sqrt{2})$, B: $(-\sqrt{2}, 0)$, C: $(0, \sqrt{2})$, and D: $(\sqrt{2}, \infty)$.
A: Test -2, $f''(-2) = 120(-2) - 60(-2)^3 = 240 > 0$
B: Test -1, $f''(-1) = 120(-1) - 60(-1)^3 = 60 > 0$
C: Test 0, $f''(0) = 120 \cdot 0 - 60 \cdot 0^3 = 60 > 0$
D: Test 2, $f''(2) = 120 \cdot 2 - 60 \cdot 2^3 = -480 = 240 < 0$
Thus $f$ is concave up on $(-\infty, -\sqrt{2})$ and on $(0, \sqrt{2})$ and is concave down on $(-\sqrt{2}, 0)$ and on $(\sqrt{2}, \infty)$, so $(-\sqrt{2}, -28\sqrt{2})$, $(0, 0)$, and $(\sqrt{2}, 28\sqrt{2})$ are all inflection points.
25. \( f(x) = x\sqrt{4 - x^2} = x(4 - x^2)^{1/2} \)

a) \( f'(x) = x \cdot \frac{1}{2}(4 - x^2)^{-1/2}(-2x) + 1 \cdot (4 - x^2)^{1/2} \)
\[ = \frac{-x^2}{\sqrt{4 - x^2}} \cdot \sqrt{4 - x^2} \]
\[ = \frac{-x^2}{4 - x^2} = \frac{4 - 2x^2}{4 - x^2}, \text{ or } (4 - 2x^2)(4 - x^2)^{-1/2} \]

\( f''(x) = -2x(4 - 2x^2) - 4x(4 - x^2)(4 - x^2)^{-1/2} \)
\[ = -2x(4 - 2x^2) - 4x(4 - x^2)(4 - x^2)^{-1/2} \]
\[ = -2x^3 + 4x^3 - 4x + 4x^3 \]
\[ = 2x^3 - 4x + 4x \]
\[ = 2x(x^2 - 2) \]
\[ = 2x(x^2 - 2)^{1/2} \]

b) \( f'(x) \) does not exist where \( 4 - x^2 = 0 \). Solve:
\[ 4 - x^2 = 0 \]
\[ 2 + x)(2 - x) = 0 \]
\[ 2 + x = 0 \text{ or } 2 - x = 0 \]
\[ x = -2 \text{ or } 2 = x \]

Note that \( f(x) \) is not defined for \( x < -2 \) or \( x > 2 \). (For these values \( 4 - x^2 < 0 \).) Therefore, relative extrema cannot occur at \( x = -2 \) or \( x = 2 \), because there is no open interval containing \( -2 \) or \( 2 \) on which the function is defined. For this reason, we do not consider \(-2 \) and \( 2 \) further in our discussion of relative extrema.

Critical points occur where \( f'(x) = 0 \). Solve:
\[ \frac{4 - 2x^2}{\sqrt{4 - x^2}} = 0 \]
\[ 4 - 2x^2 = 0 \]
\[ 1 = 2x^2 \]
\[ 2 = x^2 \]
\[ \pm \sqrt{2} = x \] (critical points)

\( f(-\sqrt{2}) = -\sqrt{2}\sqrt{4 - (-\sqrt{2})^2} = -\sqrt{2} \cdot \sqrt{2} = -2 \)

\( f(\sqrt{2}) = \sqrt{2}\sqrt{4 - (\sqrt{2})^2} = \sqrt{2} \cdot \sqrt{2} = 2 \)

Then \(-\sqrt{2}, -2\) and \((\sqrt{2}, 2)\) are on the graph.

c) Use the Second-Derivative Test:
\[ f''(-\sqrt{2}) = \frac{2(\sqrt{2})^3 - 12(-\sqrt{2})}{4(4 - (\sqrt{2})^2)^{3/2}} = \]
\[ = \frac{-4\sqrt{2} + 12\sqrt{2}}{2\sqrt{2}} = \frac{8\sqrt{2}}{2\sqrt{2}} = 4 > 0, \text{ so } (-\sqrt{2}, -2) \text{ is a relative minimum.} \]

\[ f''(\sqrt{2}) = \frac{2(\sqrt{2})^3 - 12(\sqrt{2})}{4(4 - (\sqrt{2})^2)^{3/2}} = \]
\[ = \frac{4\sqrt{2} - 12\sqrt{2}}{2\sqrt{2}} = \frac{-8\sqrt{2}}{2\sqrt{2}} = -4 < 0, \text{ so } (\sqrt{2}, 2) \text{ is a relative maximum.} \]

If we use the points \(-\sqrt{2}\) and \(\sqrt{2}\) to divide the interval \((-2, 2)\) into three intervals, \((-2, -\sqrt{2})\), \((-\sqrt{2}, \sqrt{2})\), and \((\sqrt{2}, 2)\), we know that \( f \) is decreasing on \((-2, -\sqrt{2})\) and is increasing on \((-\sqrt{2}, \sqrt{2})\).

Note that \( f(x) = 0 \) is not defined for \( x = \pm \sqrt{6} \). Therefore, the only possible inflection point is \( x = 0 \).
\[ f(0) = 0\sqrt{4 - 0^2} = 0 \cdot 2 = 0 \]

Then \((0, 0)\) is on the graph.
Exercise Set 3.2

27. \( f(x) = (x - 1)^{1/3} - 1 \)

a) \( f'(x) = \frac{1}{3} (x - 1)^{-2/3} \), or \( \frac{1}{3(x - 1)^{2/3}} \)

\( f''(x) = \frac{1}{3} \left( \frac{2}{3} \right) (x - 1)^{-5/3} \)

\( = \frac{2}{9} (x - 1)^{-5/3} \), or \( -\frac{2}{9(x - 1)^{5/3}} \)

b) \( f'(x) \) does not exist for \( x = 1 \). The equation \( f'(x) = 0 \) has no solution, so \( x = 1 \) is the only critical point. \( f'(1) = (1 - 1)^{1/3} = 0 \), \( -1 \), so \( (1, -1) \) is on the graph.

c) Use the First-Derivative Test: Use 1 to divide the real number line into two intervals. A: \((\infty, 1)\) and B: \((1, \infty)\). Test a point in each interval.

A: Test 0, \( f''(0) = \frac{2}{9(0 - 1)^{5/3}} = -\frac{2}{9} < 0 \)

B: Test 2, \( f''(2) = \frac{2}{9(2 - 1)^{5/3}} = \frac{2}{9} > 0 \)

Then \( f \) is concave up on \((-\infty, 1)\) and concave down on \((1, \infty)\), so \((1, -1)\) is an inflection point.

29. \( f(x) = x + \cos 2x \)

\( f'(x) = 1 - 2 \sin 2x \)

We solve \( f'(x) = 0 \)

\[ 1 - 2 \sin 2x = 0 \]
\[ \sin 2x = \frac{1}{2} \]
\[ 2x = \frac{\pi}{6}, \frac{5\pi}{6} \]
\[ x = \frac{\pi}{12}, \frac{5\pi}{12} \]

and

\[ 2x = \frac{13\pi}{6}, \frac{17\pi}{6} \]
\[ x = \frac{13\pi}{12}, \frac{17\pi}{12} \]
The values above divide the number line into five intervals. We apply test points to check the sign of the first derivative in each of those intervals.

\[ f'(0) = 1 - 2 \sin(2(0)) = 1 > 0 \]

\[ f'(\frac{\pi}{6}) = 1 - 2 \sin\left(\frac{\pi}{3}\right) = -0.7321 < 0 \]

\[ f'(\pi) = 1 - 2 \sin(2\pi) = 1 > 0 \]

\[ f'(\frac{5\pi}{6}) = 1 - 2 \sin\left(\frac{5\pi}{3}\right) = -1 < 0 \]

\[ f'(\frac{\pi}{4}) = 1 - 2 \sin\left(\frac{\pi}{4}\right) < 2.7321 > 0 \]

Thus, there is a relative maximum at \( x = \frac{\pi}{12} \) and \( x = \frac{11\pi}{12} \) and a relative minimum at \( x = \frac{5\pi}{12} \) and \( x = \frac{13\pi}{12} \).

\[ f\left(\frac{\pi}{12}\right) = \frac{\pi}{12} + \cos\left(\frac{\pi}{6}\right) = 1.128 \]

\[ f\left(\frac{11\pi}{12}\right) = \frac{11\pi}{12} + \cos\left(\frac{11\pi}{6}\right) = 1.209 \]

\[ f\left(\frac{5\pi}{12}\right) = \frac{5\pi}{12} + \cos\left(\frac{5\pi}{6}\right) = 0.434 \]

\[ f\left(\frac{13\pi}{12}\right) = \frac{13\pi}{12} + \cos\left(\frac{13\pi}{6}\right) = 3.585 \]

Thus, we sketch the graph.

31. \( f(x) = \frac{x}{3} - \sin\frac{2x}{3} \)

\[ f'(x) = \frac{1}{3} - \frac{2}{3} \cos\frac{2x}{3} \]

We solve \( f'(x) = 0 \):

\[ \cos\frac{2x}{3} = \frac{1}{2} \]

\[ x = \frac{\pi}{3} \]

The zero of the first derivative divides the number line into two intervals. We apply test points to check the sign of the first derivative in each of those intervals.

\[ f'(\frac{\pi}{4}) = \frac{1}{8} - \frac{2}{3} \sin\left(\frac{\pi}{3}\right) = -0.244 < 0 \]

\[ f'(x) = \frac{1}{3} - \frac{2}{3} \sin\left(\frac{2x}{3}\right) = 0.667 > 0 \]

Thus, there is a relative minimum at \( x = \frac{\pi}{2} \).

\[ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \sin\left(\frac{\pi}{3}\right) = -0.342 \]

\[ f''(x) = \frac{1}{9} \sin\frac{2x}{3} \]

\[ f''(x) = \frac{2}{3} \]

\[ \sin\frac{2x}{3} = 0 \]

\[ \sin\frac{2x}{3} = 0 \]

\[ \frac{2x}{3} = 0 \]
\[
\frac{2x}{3} = \pi \\
x = \frac{3\pi}{2}
\]

The second derivative zero divides the number line into two intervals, we determine the sign of the second derivative in each of these intervals:

\begin{align*}
f''(x) & = \frac{1}{6} \sin \left( \frac{3\pi x}{2} \right) - 0.365 > 0 \\
f''(2\pi) & = \frac{1}{6} \sin \left( \frac{3\pi \cdot 2\pi}{2} \right) \approx -0.385 < 0
\end{align*}

This means that we have an inflection point at \( x = \frac{3\pi}{2} \).

\[
f\left( \frac{3\pi}{2} \right) = \frac{3\pi}{2} - \sin \left( \frac{3\pi}{2} \right) = \frac{3\pi}{2} + 1.571
\]

We sketch the graph.

We have

\[
x = \frac{3\pi}{2}
\]

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 0 & 1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 0 & 1 & 2 & 3 & 4 \\
\hline
y & 0 & 1 & 0 & -1 & 0 \\
\end{array}
\]

33. \( f(x) = \sin x - \cos x \)

\[
f'(x) = \cos x - \sin x
\]

We solve \( f'(x) = 0 \)

\[
\begin{align*}
\cos x - \sin x &= 0 \\
\cos x &= \sin x \\
x &= \frac{\pi}{4}
\end{align*}
\]

and

\[
x = \frac{5\pi}{4}
\]

The zeros of the first derivative divide the number line into three intervals. We apply test points to check the sign of the first derivative in each of these intervals.

\begin{align*}
f'(0) = \cos(0) - \sin(0) &= 1 > 0 \\
f'(\frac{\pi}{4}) = \cos\left( \frac{\pi}{4} \right) - \sin(\frac{\pi}{4}) &= -1 < 0 \\
f'(\frac{5\pi}{4}) = \cos(\frac{5\pi}{4}) - \sin(\frac{5\pi}{4}) &= 1 > 0
\end{align*}

Thus, there is a relative maximum at \( x = \frac{\pi}{4} \) and a relative minimum at \( x = \frac{5\pi}{4} \).

\[
f\left( \frac{\pi}{4} \right) = \sin\left( \frac{\pi}{4} \right) + \cos\left( \frac{\pi}{4} \right) = 1.414 \\
f\left( \frac{5\pi}{4} \right) = \sin\left( \frac{5\pi}{4} \right) + \cos\left( \frac{5\pi}{4} \right) = -1.414
\]

\[
f''(x) = -\sin x - \cos x
\]

\[
f''(x) = 0
\]

35. \( f(x) = \sqrt{3} \sin x + \cos x \)

\[
f'(x) = \sqrt{3} \cos x - \sin x
\]

We solve \( f'(x) = 0 \)

\[
\begin{align*}
\sqrt{3} \cos x - \sin x &= 0 \\
\sqrt{3} \cos x &= \sin x \\
x &= \frac{\pi}{3}
\end{align*}
\]

and

\[
x = \frac{4\pi}{3}
\]

The zeros of the first derivative divide the number line into three intervals. We apply test points to check the sign of the first derivative in each of these intervals.

\begin{align*}
f'(0) = \sqrt{3} \cos(0) + \sin(0) &= 1.732 > 0 \\
f'(\frac{\pi}{3}) = \sqrt{3} \cos\left( \frac{\pi}{3} \right) + \sin\left( \frac{\pi}{3} \right) &= -1.732 < 0 \\
f'(\frac{4\pi}{3}) = \sqrt{3} \cos\left( \frac{4\pi}{3} \right) + \sin\left( \frac{4\pi}{3} \right) &= 1.732
\end{align*}

Thus, we have a relative maximum at \( x = \frac{\pi}{3} \) and a relative minimum at \( x = \frac{4\pi}{3} \).

\[
f\left( \frac{\pi}{3} \right) = \sqrt{3} \sin\left( \frac{\pi}{3} \right) + \cos\left( \frac{\pi}{3} \right) = 2
\]
\[ f\left(\frac{\pi}{4}\right) = \sqrt{3} \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = -2 \]

\[ f''(x) = -\sqrt{3} \sin x - \cos x \]
\[ f''(\pi) = -\sqrt{3} \sin \pi - \cos \pi = 0 \]
\[ x = \frac{5\pi}{6} \]
and
\[ x = \frac{11\pi}{6} \]

The second derivative zeros divide the number line into three intervals. We determine the sign of the second derivative in each of those intervals:

- At \( x = 0 \), \( f''(0) = -\sqrt{3} \sin(0) - \cos(0) = -1 < 0 \)
- At \( x = \pi \), \( f''(\pi) = -\sqrt{3} \sin(\pi) - \cos(\pi) = -0.845 < 0 \)
- The sign of the second derivative is positive between these two points.

Thus, we have a relative minimum at \( x = \frac{5\pi}{6} \) and a relative maximum at \( x = \frac{11\pi}{6} \).

\[ f\left(\frac{5\pi}{6}\right) = \frac{\sin\left(\frac{5\pi}{6}\right)}{2 - \cos\left(\frac{5\pi}{6}\right)} = 0.174 \]
\[ f\left(\frac{11\pi}{6}\right) = \frac{\sin\left(\frac{11\pi}{6}\right)}{2 - \cos\left(\frac{11\pi}{6}\right)} = 0.798 \]

We sketch the graph.

37. \( f(x) = \frac{\sin x}{2 - \cos x} \)
\[ f'(x) = \frac{2 \cos x - 1}{(2 - \cos x)^2} \]

We solve \( f'(x) = 0 \):

\[ \frac{2 \cos x - 1}{(2 - \cos x)^2} = 0 \]
\[ \cos x = \frac{1}{2} \]
\[ x = \frac{\pi}{3} \]

The zeros of the first derivative divide the number line into three intervals. We apply test points to check the sign of the first derivative in each of those intervals:

- At \( x = 0 \), \( f'(0) = \frac{2 \sin(0)(2 - \cos(0))}{(2 - \cos(0))^2} = 1 > 0 \)
- At \( x = \pi \), \( f'(\pi) = \frac{2 \sin(\pi)(2 - \cos(\pi))}{(2 - \cos(\pi))^2} = -1 > 0 \)
- At \( x = 2\pi \), \( f'(2\pi) = \frac{2 \sin(2\pi)(2 - \cos(2\pi))}{(2 - \cos(2\pi))^2} = 1 \)

This means that we have an inflection point at \( x = \frac{\pi}{3} \) and at \( x = \frac{2\pi}{3} \).

\[ f\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{2 - \cos\left(\frac{\pi}{3}\right)} = 0.577 \]
\[ f\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{2 - \cos\left(\frac{2\pi}{3}\right)} = 0.174 \]

We sketch the graph.
39. \( f(x) = \cos^3 x \)

\[ f'(x) = -2 \cos x \sin x = -\sin 2x \]

We solve \( f'(x) = 0 \)

\[ -\sin 2x = 0 \]
\[ x = 0 \]
\[ x = \frac{\pi}{2} \]
\[ x = \pi \]
\[ x = \frac{3\pi}{2} \]
\[ x = 2\pi \]

The zeros of the first derivative divide the number line into four intervals. We apply test points to check the sign of the first derivative in each of these intervals. \( f'(\frac{\pi}{4}) = -\sin(\frac{3\pi}{2}) = -1 < 0 \)

\( f'(\frac{3\pi}{4}) = -\sin(\frac{5\pi}{4}) = -1 > 0 \)

\( f'(\frac{5\pi}{4}) = -\sin(\frac{7\pi}{4}) = -1 > 0 \)

Thus, we have a relative maximum at \( x = \pi \) and a relative minimum at \( x = \frac{\pi}{4} \), and \( x = \frac{3\pi}{4} \).

\( f(\pi) = \cos^3(\pi) = -1 \)

\( f(\frac{\pi}{4}) = \cos^3(\frac{\pi}{4}) = 0 \)

\( f(\frac{3\pi}{4}) = \cos^3(\frac{3\pi}{4}) = 0 \)

\( f(0) = \cos^3(0) - 1 \) and \( f(2\pi) = \cos^3(2\pi) - 1 \) which means that there is a relative maximum at \( x = 0 \) and \( x = 2\pi \) as well.

\( f''(x) = -2 \cos 2x \)

\( f''(\frac{\pi}{4}) = 0 \)
\[ x = \frac{\pi}{4} \]
\[ x = \frac{3\pi}{4} \]
\[ x = \frac{5\pi}{4} \]

The second derivative zeros divide the number line into five intervals. We determine the sign of the second derivative in each of those intervals:

\( f''(0) = -2 \cos(0) = -2 < 0 \)

\( f''(\frac{\pi}{4}) = -2 \cos(\frac{\pi}{4}) = 2 > 0 \)

\( f''(\pi) = -2 \cos(2\pi) = -2 < 0 \)

\( f''(\frac{3\pi}{4}) = -2 \cos(\frac{3\pi}{4}) = 2 > 0 \)

\( f''(2\pi) = -2 \cos(4\pi) = -2 < 0 \)

This means that we have an inflection point at \( x = \frac{\pi}{4} \), \( x = \frac{3\pi}{4} \), and at \( x = \frac{5\pi}{4} \).

\( f(\frac{\pi}{4}) = -2 \cos^3(\frac{\pi}{4}) = 0.5 \)

\( f(\frac{3\pi}{4}) = -2 \cos^3(\frac{3\pi}{4}) = 0.5 \)

\( f(\frac{5\pi}{4}) = -2 \cos^3(\frac{5\pi}{4}) = 0.5 \)

\( f(\frac{\pi}{4}) = -2 \cos^3(\frac{3\pi}{4}) = 0.5 \)

We sketch the graph.

41. \( f(x) = \cos^4 x \)

\[ f'(x) = -4 \cos^3 x \sin x \]

We solve \( f'(x) = 0 \)

\[ -4 \cos^3 x \sin x = 0 \]
\[ x = 0 \]
\[ x = \frac{\pi}{2} \]
\[ x = \pi \]
\[ x = \frac{3\pi}{2} \]
\[ x = 2\pi \]

The zeros of the first derivative divide the number line into four intervals. We apply test
points to check the sign of the first derivative in each of those intervals: 

\[
f'(\frac{3\pi}{4}) = -4 \cos^3\left(\frac{3\pi}{4}\right) \sin^2\left(\frac{3\pi}{4}\right) = -1 < 0
\]

\[
f'(\frac{3\pi}{4}) = -4 \cos^3\left(\frac{3\pi}{4}\right) \sin\left(\frac{3\pi}{4}\right) = 1 > 0
\]

\[
f'(\frac{\pi}{2}) = -4 \cos^3\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) = 1 > 0
\]

Thus, we have a relative minimum at \(x = \frac{\pi}{2}\), and at \(x = \frac{3\pi}{4}\) and a relative maximum at \(x = \pi\)

\[
f\left(\frac{3\pi}{4}\right) = \cos^4\left(\frac{3\pi}{4}\right) = 0
\]

\[
f\left(\frac{\pi}{2}\right) = \cos^4\left(\frac{\pi}{2}\right) = 0
\]

\[
f(\pi) = \cos^4(\pi) = 1
\]

\(f(0) = \cos^4(0) = 1\) and \(f(2\pi) = \cos^4(2\pi) = 1\) which means that there is a relative maximum at \(x = 0\) and \(x = 2\pi\) as well.

\[
f''(x) = 12 \sin^2x \cos^2x - 4 \cos^4x
\]

\[
f''(x) = 0
\]

either

\[4 \cos^2x = 0\]

\[x = \frac{\pi}{2}, \frac{3\pi}{2}\]

or

\[3 \sin^2x - \cos^2x = 0\]

\[\tan^2x = \frac{1}{2}\]

\[x = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}\]

\[x = \pm \frac{7\pi}{6}, \pm \frac{11\pi}{6}\]

\[x = \pm \frac{13\pi}{6}\]

\(f''(x)\) exists for all values of \(x\), so we solve \(f''(x) = 0\).

\[6x = 0\]

\[x = 0\] Possible inflection point

We use the possible inflection point, 0, to divide the real number line into two intervals: A: \((-\infty, 0)\) and B: \((0, \infty)\). Test a point in each interval.

A: Test \(-1\), \(f''(-1) = 6(-1) = -6 < 0\)

B: Test 1, \(f''(1) = 6 \cdot 1 - 6 > 0\)

Then \(f\) is concave down on \((-\infty, 0)\) and concave up on \((0, \infty)\). We find that \(f(0) = 0^3 - 3 \cdot 0 + 1 = 1\), so \((0, 1)\) is an inflection point.

43. \(f(x) = x^3 + 3x + 1\)

\[f'(x) = 3x^2 + 3\]

\[f''(x) = 6x\]

\(f''(x)\) exists for all values of \(x\), so we solve \(f''(x) = 0\).

\[6x = 0\]

\[x = 0\] Possible inflection point

The second derivative zeros divide the number line into seven intervals, we determine the sign of the second derivative in each of those intervals:

\[f''(0) = 4 \cos^2(0)(3 \sin^2(0) - \cos^2(0)) = -4 < 0\]

\[f''\left(\frac{\pi}{2}\right) = 4 \cos^2\left(\frac{\pi}{2}\right)(3 \sin^2\left(\frac{\pi}{2}\right) - \cos^2\left(\frac{\pi}{2}\right)) = 2 > 0\]

\[f''\left(\frac{\pi}{4}\right) = 4 \cos^2\left(\frac{\pi}{4}\right)(3 \sin^2\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{2}\right)) = 2 > 0\]
8x = -4
8x = 1
x = 1/2  Possible inflection point

We use the possible inflection point, 1/2, to divide the real number line into two intervals, A: (-∞, 1/2) and B: (1/2, ∞). Test a point in each interval.

A: Test 0, f''(0) = 8(0) - 1 = -4 < 0
B: Test 1, f''(1) = 8(1) - 1 = 4 > 0

Then f is concave down on (-∞, 1/2) and concave up on (1/2, ∞). We find that f(1/2) = \(\frac{4}{3} \left(\frac{1}{2}\right)^3 - 2 \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \frac{1}{6}\)

so (1/2, 1/6) is an inflection point.

47. f(x) = -x - sin x

f'(x) = 1 - cos x
f''(x) = sin x

f''(x) exists for all real numbers. Solve

sin x = 0

x = nπ  Possible inflection points

Use -2π, -π, 0, π, and 2π to divide the real number line.

A: Test -π/2, f''(-π/2) = sin(-π/2) = -1 < 0
B: Test π/2, f''(π/2) = sin(π/2) = 1 > 0
C: Test π, f''(π) = sin(π) = 0
D: Test 2π, f''(2π) = sin(2π) = 0

Note that since sin x is a periodic function this pattern of changing signs will continue indefinitely. Therefore, the inflection points will occur at nπ.

49. f(x) = tan x

f'(x) = sec^2 x
f''(x) = 2 sec^2 x tan x

f''(x) does not exist for x = -π/2 + nπ.

f''(x) = 0
2 sec^2 x tan x = 0

x = nπ

Use -2π, -π, 0, π, and 2π to divide the real number line.

A: Test -π/4, f''(-π/4) = 2 sec^2(-π/4) tan(-π/4) = 1 > 0
B: Test π/4, f''(π/4) = 2 sec^2(π/4) tan(π/4) = 1 < 0
C: Test π, f''(π) = 2 sec^2(π) tan(π) = 0
D: Test 2π, f''(2π) = 2 sec^2(2π) tan(2π) = 0

Note that since 2 sec^2 x tan x is a periodic function this pattern of changing signs will continue indefinitely. Therefore, the inflection points will occur at nπ.

51. f(x) = tan x + sec x

f'(x) = sec^2 x + sec x tan x

f''(x) = 2 sec^2 x tan x + sec^3 x + tan^2 x sec x

\[f''(x) = \frac{\sec^3 x + \sin^2 x - 2 \sin x + 2}{\cos^3 x} \]

The values that make f''(x) = 0 are not in the domain of the second derivative, therefore there are no inflection points.

53. - 103 Left to the student.

105. V(r) = k(2r^2 - r^3) - 2kr^2 - kr^3, 0 ≤ r ≤ 20

The maximum occurs at the critical value of the function, since the endpoints result in a value of zero.

2kr - 3kr^2 = 0

k(40 - 3r) = 0

r = 40/3

r = 0 not acceptable

107. T(x) = 0.0338x^3 - 0.996x^2 + 8.57x^2 - 18.5x + 13.5

a)

b)

Using the quadratic formula, we solve T''(x) = 0 to get.

x = 10.833 ± 11
x = 3.001 ± 7
109. \( N(x) = -0.00096x^3 + 0.006x^2 - 0.1x + 1.0 \)

a)

\[-0.0018x^2 + 0.012x - 0.1 = \frac{dN}{dx} \]

\[N'(x) = 0\]

\[-0.0018x^2 + 0.012x - 0.1 = 0\]

Using the quadratic formula

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Thus, we have a relative maximum at \( x = 56.904 \) and a relative minimum at \( x = 9.763 \).

\[N(9.763) = -0.00006(9.763)^3 + 0.006(9.763)^2 - 0.1(9.763) + 1.0 = 1.410\]

\[N(56.904) = -0.00006(56.904)^3 + 0.006(56.904)^2 - 0.1(56.904) + 1.0 = 4.582\]

b)

\[N''(x) = -0.00036x + 0.012\]

\[N''(x) = 0\]

\[-0.00036x + 0.012 = 0\]

\[x = \frac{100}{3}\]

Thus, the zero of the second derivative divides the number line into two intervals. We check the sign of the second derivative in each of those intervals:

\[N''(10) = -0.00036(10) + 0.012 = 0.0084 > 0\]

\[N''(30) = -0.00036(30) + 0.012 = -0.006 < 0\]

Thus, there is an inflection point at \( x = \frac{100}{3} \) that has a value of

\[N \left( \frac{100}{3} \right) = -0.00006 \left( \frac{100}{3} \right)^3 + 0.006 \left( \frac{100}{3} \right)^2 - 0.1 \left( \frac{100}{3} \right) + 1.0 = 3.011\]

111. \( h \) is the graph of the derivative of \( g \) since the maximums and minimums occur where \( h \) is zero.

113. a) Relative maximum at approximately (2.7)
   b) Relative minimum at (8,0)
   c) Inflection points at (1,1)
   d) Increasing on (0,2) and (8,12)
   e) Decreasing on (2,8)
   f) Concave up on (4,12)
   g) Concave down on (0.4)

115. Left to the student. (Answers vary)

117. Relative minimum at (0,0), and a relative maximum at (1,1)

119. Relative maximum at (0,0) and a relative minimum at (0.8, -1.1)
121. Relative minimum at \((0.25, -0.25)\)

\[
\begin{array}{c|c|c|c|c|c}
& 0.5 & 1 & 1.5 & 2 & 2.5 \\
\hline
x & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Exercise Set 3.3

1. Find \( \lim_{x \to \infty} \frac{2x - 1}{5x} \).

We will use some algebra and the fact that as \( x \to \infty \).

\[
\lim_{x \to \infty} \frac{2x - 1}{5x} = \lim_{x \to \infty} \frac{2x}{5x} \cdot \frac{1/x}{1/x} = \frac{2}{5} \]

2. Find \( \lim_{x \to \infty} \frac{2x - 5}{4x + 3} \).

Multiplying by a form of 1

\[
\lim_{x \to \infty} \frac{2x - 5}{4x + 3} = \lim_{x \to \infty} \frac{2x}{4x} \cdot \frac{1}{1 + \frac{3}{4} \cdot \frac{1}{x}} = \frac{2}{4} = \frac{1}{2}
\]

3. Find \( \lim_{x \to \infty} \left( 5 - \frac{2}{x} \right) \).

We will use the fact that as \( x \to \infty \), \( \frac{b}{ax^n} \to 0 \), for any positive integer \( n \).

\[
\lim_{x \to \infty} \left( 5 - \frac{2}{x} \right) = \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{2}{x} = 5 - 0 = 5
\]

4. Find \( \lim_{x \to \infty} \frac{4 - 3x}{5 - 2x^2} \).

We divide the numerator and the denominator by \( x^2 \), the highest power of \( x \) in the denominator.

\[
\lim_{x \to \infty} \frac{4 - 3x}{5 - 2x^2} = \lim_{x \to \infty} \frac{4}{x^2} = \frac{4}{\infty} = 0
\]
Chapter 3: Application of Differentiation

11. Find \( \lim_{x \to \infty} \frac{8x^4 - 3x^2}{5x^2 + 6x} \).

We divide the numerator and the denominator by \( x^2 \), the highest power of \( x \) in the denominator.

\[
\lim_{x \to \infty} \frac{8x^4 - 3x^2}{5x^2 + 6x} = \lim_{x \to \infty} \frac{8x^2 - 3}{5 + \frac{6}{x}} = \lim_{x \to \infty} \frac{8x^2 - 3}{5} = \infty
\]

13. Find \( \lim_{x \to \infty} \frac{6x^4 - 5x^2 + 7}{8x^6 + 4x^3 - 8x} \).

We divide the numerator and the denominator by \( x^6 \), the highest power of \( x \) in the denominator.

\[
\lim_{x \to \infty} \frac{6x^4 - 5x^2 + 7}{8x^6 + 4x^3 - 8x} = \lim_{x \to \infty} \frac{0 - \frac{5}{x^2} + \frac{7}{x^6}}{8 + \frac{4}{x^3} - \frac{8}{x^7}} = 0
\]

15. Find \( \lim_{x \to \infty} \frac{11x^2 + 4x - 6x^2 + 2}{6x^3 + 5x^2 + 3x - 1} \).

We divide the numerator and the denominator by \( x \), the highest power of \( x \) in the denominator.

\[
\lim_{x \to \infty} \frac{11x^2 + 4x - 6x^2 + 2}{6x^3 + 5x^2 + 3x - 1} = \lim_{x \to \infty} \frac{11x^2 + 4x - 6x^2 + 2}{6x^3 + 5x^2 + 3x - 1} = \lim_{x \to \infty} \frac{11x^2 + 4x - 6x^2 + 2}{6x^3 + 5x^2 + 3x - 1} = \lim_{x \to \infty} \frac{0 + 0 - 0}{0 + 0 - 0} = \infty
\]

d) Critical points. The number 0 is not in the domain of \( f \). Now \( f'(x) \) exists for all values of \( x \) except 0. The equation \( f'(x) = 0 \) has no solution, so there are no critical points.

c) Increasing, decreasing, relative extrema. Use 0 to divide the real number line into two intervals, A: \((-\infty, 0)\) and B: \((0, \infty)\). Test a point in each interval.

A: Test 1, \( f'(x) = -\frac{4}{x^2} < 0 \)

B: Test 1, \( f'(x) = -\frac{4}{x^2} < 0 \)

Then \( f \) is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) Inflection points. \( f''(x) \) does not exist, but because \( f'(x) \) does not exist there cannot be an inflection point at 0. The equation \( f''(x) = 0 \) has no solution, so there are no inflection points.

g) Concavity. Use 0 to divide the real number line as in step (e). Note that for any \( x < 0 \), \( x^3 < 0 \), so

\[
f''(x) = \frac{8}{x^5} < 0
\]

and for any \( x > 0 \), \( x^3 > 0 \), so

\[
f''(x) = \frac{8}{x^5} > 0
\]

Then \( f \) is concave down on \((-\infty, 0)\) and concave up on \((0, \infty)\).

h) Sketch. Use the preceding information to sketch the graph. Compute function values as needed.

17. \( f(x) = \frac{4}{x} \), or \( 4x^{-1} \)

a) Intercepts. Since the numerator is the constant 4, there are no \( x \)-intercepts. The number 0 is not in the domain of the function, so there are no \( y \)-intercepts.

b) Asymptotes.

Vertical. The denominator is 0 for \( x = 0 \), so the line \( x = 0 \) is a vertical asymptote.

Horizontal. The degree of the numerator is less than the degree of the denominator, so \( y = 0 \) is a horizontal asymptote.

Oblique. There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) Derivatives.

\( f'(x) = -4x^{-2} = -\frac{4}{x^2} \)

\( f''(x) = 8x^{-1} = \frac{8}{x} \)

19. \( f(x) = -\frac{2}{x-5} \)

a) Intercepts. Since the numerator is the constant \(-2\), there are no \( x \)-intercepts. To find the \( y \)-intercepts we compute \( f(0) \):

\[
f(0) = -\frac{2}{0-5} = \frac{-2}{-5} = \frac{2}{5}
\]

The \( y \)-intercept \( \left(0, \frac{2}{5}\right) \) is the \( y \)-intercept.
b) Asymptotes.

Vertical. The denominator is 0 for $x = -5$, so the line $x = -5$ is a vertical asymptote.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is a horizontal asymptote.

Oblique. There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) Derivatives.

$$f'(x) = 2(x - 5)^{-2} - \frac{2}{(x - 5)^2}$$

$$f''(x) = -4(x - 5)^{-3} + \frac{4}{(x - 5)^3}$$

d) Critical Points. $f'(5)$ does not exist, but because $f(5)$ does not exist, $x = 5$ is not a critical point. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) Increasing, Decreasing, Relative Extrema. Use 5 to divide the real number line into two intervals, $A: (-\infty, 5)$ and $B: (5, \infty)$. Test a point in each interval.

$A$: Test 0, $f'(0) = \frac{2}{(0 - 5)^2} = \frac{2}{25} > 0$

$B$: Test 6, $f'(6) = \frac{2}{(6 - 5)^2} = 2 > 0$

Then $f$ is increasing on both intervals, since there are no critical points, there are no relative extrema.

I) Inflection Points. $f''(5)$ does not exist, but because $f'(5)$ does not exist there cannot be an inflection point at 5. The equation $f''(x) = 0$ has no solution, so there are no inflection points.

g) Continuity. Use 5 to divide the real number line as in step (c). Note that for any $x < 5$, $(x - 5)^3 < 0$, so

$$f''(x) = -\frac{4}{(x - 5)^3} > 0$$

and for any $x > 5$, $(x - 5)^3 > 0$, so

$$f''(x) = -\frac{4}{(x - 5)^3} < 0$$

Then $f$ is concave up on $(-\infty, 5)$ and concave down on $(5, \infty)$.

h) Sketch. Use the preceding information to sketch the graph. Compute function values as needed.

21. $f(x) = \frac{1}{x - 3}$

a) Intercepts. Since the numerator is the constant 1, there are no $x$-intercepts.

$$f(0) = \frac{1}{0 - 3} = -\frac{1}{3}$$

so $(0, -\frac{1}{3})$ is the $y$-intercept.

b) Asymptotes.

Vertical. The denominator is 0 for $x = 3$, so the line $x = 3$ is a vertical asymptote.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is a horizontal asymptote.

Oblique. There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) Derivatives.

$$f'(x) = -(x - 3)^{-2} - \frac{1}{(x - 3)^2}$$

$$f''(x) = 2(x - 3)^{-3} - \frac{2}{(x - 3)^3}$$

d) Critical Points. $f'(3)$ does not exist, but because $f(3)$ does not exist, $x = 3$ is not a critical point. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) Increasing, Decreasing, Relative Extrema. Use 3 to divide the real number line into two intervals, $A: (-\infty, 3)$ and $B: (3, \infty)$. Test a point in each interval.

$A$: Test 0, $f'(0) = \frac{1}{(0 - 3)^2} = \frac{1}{9} < 0$

$B$: Test 4, $f'(4) = \frac{1}{(4 - 3)^2} = 1 < 0$

Then $f$ is decreasing on both intervals, since there are no critical points, there are no relative extrema.

I) Inflection Points. $f''(3)$ does not exist, but because $f'(3)$ does not exist there cannot be an inflection point at 3. The equation $f''(x) = 0$ has no solution, so there are no inflection points.

g) Continuity. Use 3 to divide the real number line as in step (e). Note that for any $x < 3$, $(x - 3)^3 < 0$, so

$$f''(x) = \frac{2}{(x - 3)^2} < 0$$

and for any $x > 3$, $(x - 3)^3 > 0$, so

$$f''(x) = \frac{2}{(x - 3)^2} > 0$$

Then $f$ is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$.

h) Sketch. Use the preceding information to sketch the graph. Compute function values as needed.
23. \( f(x) = \frac{-2}{x+5} \)

a) **Intercepts.** Since the numerator is the constant \(-2\), there are no \(x\)-intercepts.

\[ f(0) = \frac{-2}{0+5} = \frac{-2}{5} \quad \text{so} \quad \left(0, \frac{-2}{5}\right) \text{ is the } y\text{-intercept.} \]

b) **Asymptotes.**

**Vertical.** The denominator is \(0\) for \(x = -5\), so the line \(x = -5\) is a vertical asymptote.

**Horizontal.** The degree of the numerator is less than the degree of the denominator, so \(y = 0\) is a horizontal asymptote.

**Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

\[ f'(x) = 2(x+5)^{-2} \quad \frac{2}{(x+5)^2} \]

\[ f''(x) = -4(x+5)^{-3} = \frac{-4}{(x+5)^3} \]

d) **Critical points.** \(f'(-5)\) does not exist, but because \(f(-5)\) does not exist, \(x = -5\) is not a critical point.

The equation \(f'(x) = 0\) has no solution, so there are no critical points.

e) **Increasing, decreasing, relative extrema.** Use \(-5\) to divide the real number line into two intervals, \(A: (-\infty, -5)\) and \(B: (-5, \infty)\). Test a point in each interval.

\[ A: \text{Test } -6, \quad f'(-6) = \frac{-1}{(-6+5)^2} = \frac{-1}{1} > 0 \]

\[ B: \text{Test } 0, \quad f'(0) = \frac{-2}{(0+5)^2} = \frac{-2}{25} > 0 \]

Then \(f\) is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) **Inflection points.** \(f''(-5)\) does not exist, but because \(f(-5)\) does not exist, there cannot be an inflection point at \(-5\). The equation \(f''(x) = 0\) has no solution, so there are no inflection points.

g) **Concavity.** Use \(-5\) to divide the real number line as in step (e). Test a point in each interval.

\[ A: \text{Test } -6, \quad f''(-6) = \frac{-1}{(-6+5)^2} = \frac{-1}{1} > 0 \]

\[ B: \text{Test } 0, \quad f''(0) = \frac{-2}{(0+5)^2} = \frac{-2}{25} < 0 \]

Then \(f\) is concave up on \((-\infty, -5)\) and concave down on \((-5, \infty)\).

h) **Sketch.** Use the preceding information to sketch the graph. Compute function values as needed.

25. \( f(x) = \frac{2x+1}{x} \)

a) **Intercepts.** To find the \(x\)-intercepts, solve \(f(x) = 0\).

\[ \frac{2x+1}{x} = 0 \]

\[ 2x + 1 = 0 \]

\[ 2x = -1 \]

\[ x = -\frac{1}{2} \]

Since \(x = -\frac{1}{2}\) does not make the denominator 0, the \(x\)-intercept is \((-\frac{1}{2}, 0)\). The number 0 is not in the domain of \(f\), so there are no \(y\)-intercepts.

b) **Asymptotes.**

**Vertical.** The denominator is \(0\) for \(x = 0\), so the line \(x = 0\) is a vertical asymptote.

**Horizontal.** The numerator and denominator have the same degree, so \(y = \frac{2}{1}\) or \(y = 2\), is a horizontal asymptote.

**Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

\[ f'(x) = \frac{1}{x^2} \]

\[ f''(x) = 2x^{-3}, \quad \text{or} \quad \frac{2}{x^3} \]

d) **Critical points.** \(f'(0)\) does not exist, but because \(f'(0)\) does not exist \(x = 0\) is not a critical point. The equation \(f''(x) = 0\) has no solution, so there are no critical points.
e) Increasing, decreasing, relative extrema. Use 0 to divide the real number line into two intervals. A: \((-\infty, 0)\) and B: \((0, \infty)\). Test a point in each interval.

A: Test -1. \(f'(-1) = \frac{1}{(-1)^2} = 1 < 0\)

B: Test 1. \(f'(1) = \frac{1}{1^2} = 1 > 0\)

Then \(f\) is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

g) Concavity. Use 0 to divide the real number line as in step (e). Test a point in each interval.

A: Test -1. \(f''(-1) = \frac{2}{(-1)^3} = -2 < 0\)

B: Test 1. \(f''(1) = \frac{2}{1^3} = 2 > 0\)

Then \(f\) is concave down on \((\infty, 0)\) and concave up on \((0, \infty)\).

h) Sketch. Use the preceding information to sketch the graph. Compute function values as needed.

d) Critical points. \(f'(0)\) does not exist, but because \(f'(x)\) does not exist \(x = 0\) is not a critical point. Solve \(f'(x) = 0\).

\[
1 - \frac{9}{x^2} = 0
\]

\[
1 = \frac{9}{x^2}
\]

\[
x^2 = 9
\]

\[
x = \pm 3
\]

Thus, -3 and 3 are critical points. \(f(-3) = -6\) and \(f(3) = 6\), so \((-3, -6)\) and \((3, 6)\) are on the graph.

e) Increasing, decreasing, relative extrema. Use -3, 0, and 3 to divide the real number line into four intervals. A: \((-\infty, -3)\), B: \((-3, 0)\), C: \((0, 3)\), and D: \((3, \infty)\). Test a point in each interval.

A: Test -4. \(f'(-4) = 1 - \frac{9}{(-4)^2} = \frac{7}{16} > 0\)

B: Test -1. \(f'(-1) = 1 - \frac{9}{(-1)^2} = -8 < 0\)

C: Test 1. \(f'(1) = 1 - \frac{9}{1^2} = -8 < 0\)

D: Test 4. \(f'(4) = 1 - \frac{9}{4^2} = \frac{7}{16} > 0\)

Then \(f\) is increasing on \((-\infty, -3)\) and on \((3, \infty)\) and is decreasing on \((-3, 0)\) and on \((0, 3)\). Thus, there is a relative maximum at \((-3, -6)\) and a relative minimum at \((3, 6)\).

f) Inflection points. \(f''(0)\) does not exist, but because \(f'(x)\) does not exist there cannot be an inflection point at 0. The equation \(f''(x) = 0\) has no solution, so there are no inflection points.

g) Concavity. Use 0 to divide the real number line into two intervals. A: \((-\infty, 0)\) and B: \((0, \infty)\). Test a point in each interval.

A: Test -1. \(f''(-1) = \frac{18}{(-1)^3} = -18 < 0\)

B: Test 1. \(f''(1) = \frac{18}{1^3} = 18 > 0\)

Thus \(f\) is concave down on \((-\infty, 0)\) and concave up on \((0, \infty)\).

h) Sketch. Use the preceding information to sketch the graph. Compute other function values as needed.
29. \( f(x) = \frac{2}{x^2} \)

a) **Intercepts.** Since the numerator is the constant 2, there are no \( x \)-intercepts. The number 0 is not in the domain of the function, so there are no \( y \)-intercepts.

b) **Asymptotes.**

   *Vertical.* The denominator is 0 for \( x = 0 \), so the line \( x = 0 \) is a vertical asymptote.

   *Horizontal.* The degree of the numerator is less than the degree of the denominator, so \( y = 0 \) is a horizontal asymptote.

   *Oblique.* There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

   \[
   f'(x) = -4x^{-3} = -\frac{4}{x^3} \\
   f''(x) = 12x^{-4} = \frac{12}{x^4}
   \]

d) **Critical points.** \( f'(0) \) does not exist, but because \( f'(x) \) does not exist \( x = 0 \) is not a critical point. The equation \( f'(x) = 0 \) has no solution, so there are no critical points.

e) **Increasing, decreasing, relative extrema.** Use 0 to divide the real number line into two intervals, A: \((-\infty, 0)\) and B: \((0, \infty)\).

   A: Test -1, \( f'(-1) = -\frac{4}{(-1)^3} = -4 < 0 \)

   B: Test 1, \( f'(1) = -\frac{4}{1^3} = -4 < 0 \)

   Then \( f \) is increasing on \((-\infty, 0)\) and decreasing on \([0, \infty)\). Since there are no critical points, there are no relative extrema.

f) **Inflection points.** \( f''(0) \) does not exist, but because \( f'(x) \) does not exist there cannot be an inflection point at 0. The equation \( f''(x) = 0 \) has no solution, so there are no inflection points.

g) **Concavity.** Use 0 to divide the real number line as in step (e).

   A: Test -1, \( f''(-1) = -\frac{12}{(-1)^2} = 12 > 0 \)

   B: Test 1, \( f''(1) = -\frac{12}{1^1} = 12 > 0 \)

   Then \( f \) is concave up on both intervals.

h) **Sketch.** Use the preceding information to sketch the graph. Compute function values as needed.

---

31. \( f(x) = \frac{x}{x^2} - 3 \)

a) **Intercepts.** The numerator is 0 for \( x = 0 \) and this value of \( x \) does not make the denominator 0, so \((0, 0)\) is the \( x \)-intercept. \( f(0) = 0 \) is not a \( y \)-intercept.

b) **Asymptotes.**

   *Vertical.* The denominator is 0 for \( x = \infty \), so the line \( x = \infty \) is a vertical asymptote.

   *Horizontal.* The numerator and the denominator have the same degree, so \( y = \frac{1}{1} = 1 \), is a horizontal asymptote.

   *Oblique.* There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

   \[
   f'(x) = -\frac{3}{(x - 3)^2} \\
   f''(x) = 6(x - 3)^{-3} = \frac{6}{(x - 3)^3}
   \]

d) **Critical points.** \( f'(3) \) does not exist, but because \( f'(x) \) does not exist \( x = 3 \) is not a critical point. The equation \( f'(x) = 0 \) has no solution, so there are no critical points.

e) **Increasing, decreasing, relative extrema.** Use 3 to divide the real number line into two intervals, A: \((-\infty, 3)\) and B: \((3, \infty)\). Test a point in each interval.

   A: Test 0, \( f'(0) = -\frac{1}{3} < 0 \)

   B: Test 4, \( f'(4) = -3 < 0 \)

   Then \( f \) is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) **Inflection points.** \( f''(3) \) does not exist, but because \( f'(x) \) does not exist there cannot be an inflection point at 3. The equation \( f''(x) = 0 \) has no solution, so there are no inflection points.

g) **Concavity.** Use 3 to divide the real number line as in step (e).

   A: Test 0, \( f''(0) = -\frac{2}{9} < 0 \)

   B: Test 4, \( f''(4) = 6 > 0 \)

   Then \( f \) is concave down on \((-\infty, 3)\) and concave up on \((3, \infty)\).
33. \( f(x) = \frac{1}{x^2 + 3} \)

a) **Intercepts.** Since the numerator is the constant 1, there are no \( x \)-intercepts.

\[ f(0) = \frac{1}{3} \quad \text{so} \quad \left(0, \frac{1}{3}\right) \text{ is the } y\text{-intercept}. \]

b) **Asymptotes.**

- **Vertical.** \( x^2 + 3 = 0 \) has no real number solutions, so there are no vertical asymptotes.
- **Horizontal.** The degree of the numerator is less than the degree of the denominator, so \( y = 0 \) is a horizontal asymptote.
- **Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

\[
\begin{align*}
\frac{df}{dx} &= -\frac{2x}{(x^2 + 3)^2} \\
\frac{d^2f}{dx^2} &= \frac{6x^2 - 6}{(x^2 + 3)^3}
\end{align*}
\]

d) **Critical points.** \( f'(x) \) exists for all real numbers.

Solve \( f'(x) = 0 \).

\[
\begin{align*}
&-\frac{2x}{(x^2 + 3)^2} = 0 \\
&-2x = 0 \\
&x = 0 \quad \text{Critical point}
\end{align*}
\]

From step (a) we already know \( \left(0, \frac{1}{3}\right) \) is on the graph.

e) **Increasing, decreasing, relative extrema.** Use 0 to divide the real number line into two intervals, \( A: (-\infty, 0) \) and \( B: (0, \infty) \). Test a point in each interval.

A: Test \( x = -1 \), \( f'(-1) = \frac{1}{8} > 0 \)

B: Test \( x = 1 \), \( f'(1) = -\frac{1}{8} < 0 \)

Then \( f \) is increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\). Thus, \( \left(0, \frac{1}{3}\right) \) is a relative maximum.

35. \( f(x) = \frac{x - 1}{x + 2} \)

a) **Intercepts.** The numerator is 0 for \( x = 1 \) and this value of \( x \) does not make the denominator 0, so \( (1, 0) \) is the \( x \)-intercept.

\[ f(0) = \frac{-1}{2} = -\frac{1}{2} \quad \text{so} \quad \left(0, -\frac{1}{2}\right) \text{ is the } y\text{-intercept}. \]

b) **Asymptotes.**

- **Vertical.** The denominator is 0 for \( x = -2 \), so the line \( x = -2 \) is a vertical asymptote.
- **Horizontal.** The numerator and the denominator have the same degree, so \( y = \frac{1}{1} \) or \( y = 1 \), is a horizontal asymptote.
- **Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.
c) Derivatives.
\[ f'(x) = \frac{3}{(x + 2)^2} \]
\[ f''(x) = \frac{-6}{(x + 2)^3} \]

d) Critical points. \( f'(-2) \) does not exist, but because \( f(-2) \) does not exist, \( x = -2 \) is not a critical point. The equation \( f''(x) = 0 \) has no solution, so there are no critical points.

e) Increasing, decreasing, relative extrema. Use -2 to divide the real number line into two intervals, A: \((-\infty, -2)\) and B: \((-2, \infty)\). Test a point in each interval.

A: Test -3, \( f'(-3) = 3 > 0 \)
B: Test 1, \( f'(-1) = 3 > 0 \)

Then \( f \) is increasing on both intervals. Since there are no critical points, there are relative extrema.

f) Inflection points. \( f''(-2) \) does not exist, but because \( f(-2) \) does not exist, there cannot be an inflection point at -2. The equation \( f''(x) = 0 \) has no solution, so there are no inflection points.

g) Concavity. Use -2 to divide the real number line as in step (e). Test a point in each interval.

A: Test -3, \( f''(-3) = 6 > 0 \)
B: Test 1, \( f''(-1) = -6 < 0 \)

Then \( f \) is concave up on \((-\infty, -2)\) and concave down on \((-2, \infty)\).

h) Sketch. Use the preceding information to sketch the graph. Compute function values as needed.

![Graph of f(x) = x^2 - 4 / (x + 3)]

37. \( f(x) = \frac{x^2 - 4}{x + 3} \)

a) Intercepts. The numerator \( x^2 - 4 = (x + 2)(x - 2) \) is 0 for \( x = -2 \) and \( x = 2 \), and neither of these values makes the denominator 0. Thus, the \( x \)-intercepts are \((-2, 0)\) and \((2, 0)\).

\[ f(0) = \frac{0^2 - 4}{0 + 3} = -\frac{4}{3} \]

b) Asymptotes.

Vertical. The denominator is 0 for \( x = -3 \), so the line \( x = -3 \) is a vertical asymptote.

Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.

Oblique.

\[ f(x) = x - 3 + \frac{5}{x + 3} \]

As \(|x|\) gets very large, \( f(x) \) approaches \( x - 3 \), so \( y = x - 3 \) is an oblique asymptote.

d) Derivatives.

\[ f'(x) = \frac{x^2 + 6x + 4}{(x + 3)^2} \]
\[ f''(x) = \frac{10}{(x + 3)^2} \]

f) Critical points. \( f'(-3) \) does not exist, but because \( f(-3) \) does not exist, \( x = -3 \) is not a critical point. Solve \( f'(x) = 0 \).

\[ \frac{x^2 + 6x + 4}{(x + 3)^2} = 0 \]
\[ x^2 + 6x + 4 = 0 \]
\[ x = -3 \pm \sqrt{5} \]

Using the quadratic formula \( x \approx -5.24 \) or \( x \approx -0.76 \) Critical points

\[ f(-5.24) = 10.17 \] and \( f(-0.76) = -1.53 \), so \((-5.24, -10.17)\) and \((-0.76, -1.53)\) are on the graph.

g) Increasing, decreasing, relative extrema. Use -5.24, -3, and -0.76 to divide the real number line into four intervals, A: \((-\infty, -5.24)\), B: \((-5.24, -3)\), C: \((-3, -0.76)\), and D: \((-0.76, \infty)\). Test a point in each interval.

A: Test -6, \( f'(-6) = \frac{4}{9} > 0 \)
B: Test -4, \( f'(-4) = -4 < 0 \)
C: Test -2, \( f'(-2) = -4 < 0 \)
D: Test 0, \( f'(0) = \frac{4}{9} > 0 \)

Then \( f \) is increasing on \((-\infty, -5.24)\) and on \((-0.76, \infty)\) and is decreasing on \((-5.24, -3)\) and on \((-3, -0.76)\). Thus, \((-5.24, -10.17)\) is a relative maximum and \((-0.76, -1.53)\) is a relative minimum.

h) Inflection points. \( f''(x) \) does not exist, but because \( f(-3) \) does not exist, there cannot be an inflection point at -3. The equation \( f''(x) = 0 \) has no solution, so there are no inflection points.

g) Concavity. Use -3 to divide the real number line into two intervals, A: \((-\infty, -3)\) and B: \((-3, \infty)\). Test a point in each interval.

A: Test -4, \( f''(-4) = -10 < 0 \)
B: Test -2, \( f''(-2) = 10 > 0 \)
Then $f$ is concave down on $(-\infty, -3)$ and concave up on $(3, \infty)$.

b) Sketch. Use the preceding information to sketch the graph. Compute other function values as needed.

![Graph](image)

39. $f(x) = \frac{x - 1}{x^2 - 2x - 3}$

a) **Integrals.** The numerator is 0 for $x = 1$, and this value of $x$ does not make the denominator 0. Then $(1, 0)$ is the $x$-intercept.

$$f(0) = \frac{0 - 1}{0^2 - 2 \cdot 0 - 3} = -\frac{1}{3}, \text{ so } (0, \frac{1}{3}) \text{ is the } x\text{-intercept.}$$

b) **Asymptotes.**

- **Vertical.** The denominator $x^2 - 2x - 3 = (x + 1)(x - 3)$ is 0 for $x = -1$ or $x = 3$. Then the lines $x = -1$ and $x = 3$ are vertical asymptotes.

- **Horizontal.** The degree of the numerator is less than the degree of the denominator, so $y = 0$ is a horizontal asymptote.

- **Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) **Derivatives.**

$$f'(x) = \frac{-2x + 5}{(x^2 - 2x - 3)^2}$$

$$f''(x) = \frac{2x^3 - 6x^2 + 30x - 26}{(x^2 - 2x - 3)^3}$$

d) **Critical points.** $f'(1)$ and $f'(3)$ do not exist, but because $f'(1)$ and $f'(3)$ do not exist $x = -1$ and $x = 3$ are not critical points. The equation $f''(x) = 0$ has no real number solution, so there are no critical points.

e) **Increasing, decreasing, relative extrema.** Use -1 and 3 to divide the real number line into three intervals. A: $(-\infty, -1)$, B: $(-1, 3)$, and C: $(3, \infty)$. Test a point in each interval.

A: Test -2, $f'(-2) = -\frac{13}{25} < 0$

B: Test 0, $f'(0) = \frac{5}{9} < 0$

C: Test 1, $f'(1) = -\frac{13}{25} < 0$

Then $f$ is decreasing on all three intervals. Since there are no critical points, there are no relative extrema.

f) **Inflection points.** $f''(-1)$ and $f''(3)$ do not exist, but because $f'(-1)$ and $f'(3)$ do not exist there cannot be an inflection point at -1 or at 3. Solve $f''(x) = 0$.

$$\frac{2x^3 - 6x^2 + 30x - 26}{(x^2 - 2x - 3)^3} = 0$$

$$\frac{2x^3 - 6x^2 + 30x - 26}{(x - 1)(x^2 - 4x - 26)} = 0$$

$x = 1$ No real number solution

$f(1) = 0$, so $(1, 0)$ is on the graph and is a possible inflection point.

g) **Concavity.** Use -1, 1, and 3 to divide the real number line into four intervals. A: $(-\infty, -1)$, B: $(-1, 1)$, C: $(1, 3)$, and D: $(3, \infty)$. Test a point in each interval.

A: Test $-2$, $f''(-2) = -\frac{216}{125} < 0$

B: Test 0, $f''(0) = \frac{26}{25} > 0$

C: Test 2, $f''(2) = -\frac{26}{25} < 0$

D: Test 4, $f''(4) = \frac{126}{125} > 0$

Then $f$ is concave down on $(-\infty, -1)$ and on $(1, 3)$ and is concave up on $(-1, 1)$ and on $(3, \infty)$. Thus, $(1, 0)$ is an inflection point.

h) Sketch. Use the preceding information to sketch the graph. Compute other function values as needed.

![Graph](image)

41. $f(x) = \frac{2x^2}{x^2 - 16}$

a) **Integrals.** The numerator is 0 for $x = 0$, and this value of $x$ does not make the denominator 0, so $(0, 0)$ is the $x$-intercept.

$$f(0) = \frac{0}{16} = 0$$

so the $y$-intercept is the $x$-intercept $(0, 0)$.

b) **Asymptotes.**

- **Vertical.** The denominator $x^2 - 16 \cdot (x + 4)(x - 4)$ is 0 for $x = -4$ or $x = 4$, so the lines $x = -4$ and $x = 4$ are vertical asymptotes.

- **Horizontal.** The numerator and denominator have the same degree, so $y = \frac{2}{1} = 2$ is a horizontal asymptote.

- **Oblique.** There is no oblique asymptote since the degree of the numerator is not one more than the degree of the denominator.
c) Derivatives.
\[ f'(x) = \frac{-64x}{(x^2 - 16)^2} \]
\[ f''(x) = \frac{192x^2 + 1024}{(x^2 - 16)^3} \]
d) Critical points. \( f'(-4) \) and \( f'(4) \) do not exist, but because \( f'(-4) \) and \( f'(4) \) do not exist \( x = -4 \) and \( x = 4 \) are not critical points. Solve \( f'(x) = 0 \).
\[ \frac{-64x}{(x^2 - 16)^2} = 0 \]
\[ -64x = 0 \]
\[ x = 0 \quad \text{Critical point} \]
From step (a) we already know that \((0, 0)\) is on the graph.

c) Increasing, decreasing, relative extrema. Use \(-4, 0, \) and \(4\) to divide the real number line into four intervals, \( A: (-\infty, -4) \), \( B: (-4, 0) \), \( C: (0, 4) \), and \( D: (4, \infty) \).
- **A:** Test \(-5, f'(-5) = \frac{-326}{81} > 0 \)
- **B:** Test \(-1, f'(-1) = \frac{64}{225} > 0 \)
- **C:** Test \(1, f'(1) = \frac{-64}{225} < 0 \)
- **D:** Test \(5, f'(5) = \frac{320}{81} < 0 \)
Then \( f \) is increasing on \((-\infty, -4)\) and on \((-4, 0)\) and is decreasing on \((0, 4)\) and on \((4, \infty)\). Thus, there is a relative maximum at \((0, 0)\).

d) Inflection points. \( f''(-4) \) and \( f''(4) \) do not exist, but because \( f'(-4) \) and \( f'(4) \) do not exist there cannot be an inflection point at \(-4 \) or \(4 \). The equation \( f''(x) = 0 \) has no real-number solution, so there are no inflection points.

g) Concavity. Use \(-4 \) and \(4 \) to divide the real number line into three intervals, \( A: (-\infty, -4) \), \( B: (-4, 4) \), and \( C: (4, \infty) \). Test a point in each interval.
- **A:** Test \(-5, f''(-5) = \frac{-5824}{729} > 0 \)
- **B:** Test \(0, f''(0) = \frac{-2}{4} < 0 \)
- **C:** Test \(5, f''(5) = \frac{-5821}{729} > 0 \)
Then \( f \) is concave up on \((-\infty, -4)\) and on \((4, \infty)\) and is concave down on \((-4, 4)\).

h) Sketch. Use the preceding information to sketch the graph. Compute other function values as needed.
e) **Increasing, decreasing, relative extrema.** Use \(-1, 0, 1\) to divide the real number line into four intervals. A: \((-\infty, -1]\), B: \((-1, 0]\), C: \((0, 1]\), and D: \((1, \infty)\). Test a point in each interval.

\[
A: \text{Test } -2, f'(-2) = \frac{4}{9} > 0 \\
B: \text{Test } -\frac{1}{2}, f'\left(-\frac{1}{2}\right) = -\frac{16}{9} < 0 \\
C: \text{Test } \frac{1}{2}, f'\left(\frac{1}{2}\right) = -\frac{16}{9} < 0 \\
D: \text{Test } 2, f'(2) = -\frac{4}{9} < 0 
\]

Then \(f\) is increasing on \((-\infty, -1] and on \((-1, 0]\) and is decreasing on \((0, 1]\) and on \((1, \infty)\). Thus, there is a relative maximum at \((-1, 1)\).

f) **Infection points.** \(f''(-1)\) and \(f''(0)\) do not exist, but because \(f'(-1)\) and \(f'(0)\) do not exist there cannot be an infection point at \(-1\) or at \(0\). The equation \(f''(x) = 0\) has no real-number solution, so there are no infection points.

g) **Concavity.** Use \(-1, 0, 1\) to divide the real number line into three intervals. A: \((-\infty, -1]\), B: \((-1, 1]\), and C: \((1, \infty)\). Test a point in each interval.

\[
A: \text{Test } -2, f''(-2) = \frac{25}{9} > 0 \\
B: \text{Test } 0, f''(0) = -2 < 0 \\
C: \text{Test } 2, f''(2) = \frac{25}{9} > 0 
\]

Then \(f\) is concave up on \((-\infty, -1]\) and on \((1, \infty)\) and is concave down on \((-1, 1]\).

h) **Sketch.** Use the preceding information to sketch the graph. Compute other function values as needed.

![Graph of f(x) = \(\frac{x^2 + 1}{x}\)](image)

45. **f(x) = \(\frac{x^2 + 1}{x}\)**

a) **Intercepts.** Since the numerator has no real-number solutions, there are no \(x\)-intercepts.

\(f(0)\) does not exist, so there is no \(y\)-intercept.

b) **Asymptotes.**

- **Vertical.** The denominator \(x\) is 0 for \(x = 0\), so the line \(x = 0\) is a vertical asymptote.

- **Horizontal.** The degree of the numerator is not less than or equal to the degree of the denominator, so there is no horizontal asymptote.

- **Oblique.** The degree of the numerator is one more than the degree of the denominator, so there is an oblique asymptote. When we divide \(x^2 + 1\) by \(x\) we have \(f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}\). As \(x\) gets very large, \(\frac{1}{x}\) approaches 0. Thus, \(y = x\) is an oblique asymptote.

c) **Derivatives.**

\[
f'(x) = \frac{x^2 - 1}{x^2} \\
f''(x) = \frac{2}{x^3}
\]

d) **Critical points.** \(f'(0)\) does not exist, but because \(f(0)\) does not exist 0 is not a critical point. Solve \(f'(x) = 0\).

\[
x^2 - 1 = 0 \\
x = 1 \text{ or } x = -1 
\]

Critical points

\(f(-1) = -2\) and \(f(1) = 2\), so \((-1, -2)\) and \((1, 2)\) are on the graph.

g) **Increasing, decreasing, relative extrema.** Use \(-1, 0, 1\) to divide the real number line into four intervals. A: \((-\infty, 0]\), B: \((-1, 0]\), C: \((0, 1]\), and D: \((1, \infty)\). Test a point in each interval.

\[
A: \text{Test } -2, f''(-2) = \frac{3}{4} > 0 \\
B: \text{Test } -\frac{1}{2}, f'\left(-\frac{1}{2}\right) = -3 < 0 \\
C: \text{Test } \frac{1}{2}, f'\left(\frac{1}{2}\right) = -3 < 0 \\
D: \text{Test } 2, f'\left(\frac{1}{2}\right) = \frac{3}{4} > 0 
\]

Then \(f\) is increasing on \((-\infty, -1]\) and on \((1, \infty)\) and is decreasing on \((-1, 0]\) and \((0, 1]\). Thus, there is a relative maximum at \((-1, -2)\) and a relative minimum at \((1, 2)\).

f) **Infection points.** \(f''(0)\) does not exist, but because \(f(0)\) does not exist there cannot be an infection point at 0. The equation \(f''(x) = 0\) has no solution, so there are no infection points.

g) **Concavity.** Use \(0\) to divide the real number line into two intervals. A: \((-\infty, 0]\) and B: \((0, \infty)\). Test a point in each interval.

\[
A: \text{Test } -1, f''(-1) = 2 < 0 \\
B: \text{Test } 1, f''(1) = 2 > 0 
\]

Then \(f\) is concave down on \((-\infty, 0]\) and is concave up on \((0, \infty)\).
47. $C(p) = \frac{\$48,000}{100 - p}$

We will only consider the interval $[0, 100]$ since it is not possible to remove less than 0% or more than 100% of the pollutants and $C(p)$ is not defined for $p = 100$.

\[
\begin{align*}
    C(0) &= \frac{\$48,000}{100 - 0} = \$480,000 \\
    C(20) &= \frac{\$48,000}{100 - 20} = \$600,000 \\
    C(80) &= \frac{\$48,000}{100 - 80} = \$240,000 \\
    C(90) &= \frac{\$48,000}{100 - 90} = \$8000
\end{align*}
\]

\[\lim_{p \to 100^-} C(p) = \lim_{p \to 100^-} \frac{\$48,000}{100 - p} = \infty\]

c) The cost of removing 100% of the pollutants is infinitely high.

d) Using the techniques of this section we find the following additional information.

**Intercepts.** No x-intercept; $(0, 480)$ is the $C$-intercept.

**Asymptotes.** Vertical, $p = 100$

**Horizontal.** $C = 0$

**Oblique.** None

*Increasing, decreasing, relative extrema.** $C(p)$ is increasing on $[0, 100)$. There are no relative extrema.

*Reflection points, convexity.** $C(p)$ is concave up on $[0, 100)$. There is no inflection point.

We use this information and compute other function values as needed to sketch the graph.

49. $T(t) = \frac{6t}{t^2 + 1} + 98.6$

\[T'(t) = \frac{6(t^2 + 1) - 6t(2t)}{(t^2 + 1)^2} + 98.6 = \frac{6t^2 - 6t^2}{(t^2 + 1)^2} + 98.6 = \frac{6t^2}{(t^2 + 1)^2} + 98.6 = 98.6\]

\[T(0) = \frac{6(0)}{0^2 + 1} + 98.6 = 98.6\]

\[T(1) = \frac{6(1)}{1^2 + 1} + 98.6 = 101.6\]

\[T(2) = \frac{6(2)}{4 + 1} + 98.6 = 101\]

\[T(5) = \frac{6(5)}{25 + 1} + 98.6 = 99.8\]

\[T(10) = \frac{6(10)}{100 + 1} + 98.6 = 99.2\]

\[\lim_{t \to \infty} T'(t) = \lim_{t \to \infty} \frac{6t}{t^2 + 1} + 98.6 = \lim_{t \to \infty} \frac{6t}{t^2 + 1} + 98.6 = 98.6\]

\[= 98.6\]

\[= 98.6\]

\[= 98.6\]

\[= 98.6\]

*The maximum occurs at the critical value of the function.*

\[T''(t) = \frac{6(t^2 + 1) - 6t(2t)}{(t^2 + 1)^2} + 98.6 = \frac{6t^2 - 6t^2}{(t^2 + 1)^2} + 98.6 = \frac{6t^2}{(t^2 + 1)^2} + 98.6 = 98.6\]

\[6 = 6t^2\]

\[\frac{6t^2}{(t^2 + 1)^2} = 0\]

\[-6t^2 = 0\]

\[t = 1\]

Note we ignore the negative option since the question involves the interval $[0, \infty)$.

From part a) we know that $T'(1) = 101.6$
e) According to this model the temperature does not return to 98.6° since that temperature is reached only when \( t \) approaches infinity.

51. a) \( E(9) = 9 \cdot \frac{4}{9} = 4.00 \)
\( E(8) = 9 \cdot \frac{4}{8} = 4.50 \)
\( E(7) = 9 \cdot \frac{4}{7} \approx 5.14 \)
\( E(6) = 9 \cdot \frac{4}{6} = 6.00 \)
\( E(5) = 9 \cdot \frac{4}{5} = 7.20 \)
\( E(4) = 9 \cdot \frac{4}{4} = 9.00 \)
\( E(3) = 9 \cdot \frac{4}{3} = 12.00 \)
\( E(2) = 9 \cdot \frac{4}{2} = 18.00 \)
\( E(1) = 9 \cdot \frac{4}{1} = 36.00 \)
\( E\left(\frac{2}{3}\right) = 9 \cdot \frac{4}{\frac{3}{2}} \cdot \frac{3}{2} = 54.00 \)
\( E\left(\frac{1}{3}\right) = 9 \cdot \frac{4}{1} \cdot \frac{3}{1} = 108.00 \)

We complete the table.

<table>
<thead>
<tr>
<th>Innings pitched (( i ))</th>
<th>Earned-run average (( E' ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1.00</td>
</tr>
<tr>
<td>8</td>
<td>4.50</td>
</tr>
<tr>
<td>7</td>
<td>5.14</td>
</tr>
<tr>
<td>6</td>
<td>6.00</td>
</tr>
<tr>
<td>5</td>
<td>7.20</td>
</tr>
<tr>
<td>4</td>
<td>9.00</td>
</tr>
<tr>
<td>3</td>
<td>12.00</td>
</tr>
<tr>
<td>2</td>
<td>18.00</td>
</tr>
<tr>
<td>1</td>
<td>36.00</td>
</tr>
<tr>
<td>2/3</td>
<td>51.00</td>
</tr>
<tr>
<td>1/3</td>
<td>108.00</td>
</tr>
</tbody>
</table>

b) As \( i \) approaches 0 from the right, the values of \( E' \) increase without bound, so
\[ \lim_{i \to 0^+} E' = \infty. \]

c) Since \( \lim_{i \to 1^+} E' = \infty \), the earned run average would be \( \infty \).

53. Asymptotes can be thought of as “limiting lines” for graphs of functions. The graphs and limits on pages 201, 202, and 203 of the text illustrate vertical, horizontal, and oblique asymptotes.

55.
\[
\lim_{x \to 5} \frac{x^2 - 6x + 5}{x^2 - 3x - 10} = \lim_{x \to 5} \frac{(x - 1)(x - 5)}{(x + 2)(x - 5)} = \lim_{x \to 5} \frac{x - 1}{x + 2}
\]
\[= \frac{5 - 1}{5 + 2} = \frac{4}{7} \]

57. We divide the numerator and the denominator by \( x^2 \), the highest power of \( x \) in the denominator.
\[
\lim_{x \to \infty} \frac{-6x^3 + 7x}{2x^3 - 3x - 10} = \lim_{x \to \infty} \frac{-6x + \frac{7}{x}}{2 - \frac{3}{x} - \frac{10}{x^2}}
\]
\[= \lim_{x \to \infty} -6x + 0 = -\infty \]

(The numerator increases without bound negatively while the denominator approaches 2.)

59.
\[
\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2} \]

61.
\[
\lim_{x \to \infty} \frac{2x^2 + x}{x + 1} = \lim_{x \to \infty} \frac{2x^2 + \frac{x}{x}}{1 + \frac{1}{x}} = \lim_{x \to \infty} \frac{2x^2 + 1}{1 + \frac{1}{x}} = -\infty \]

63. Undefined

65. Since the numerator is bounded by \( \pm 1 \) and the denominator grows indefinitely, then
\[
\lim_{x \to \infty} \frac{\cos x}{x} = 0 \]

see figure below
67. \( \lim_{r \to 1^-} f(r) = \infty \)
69. \( \lim_{r \to -\infty} f(x) = -\infty \)
71. \( \lim_{r \to \infty} f(x) = 1 \)
73. \( \lim_{r \to 2^-} f(x) = -\infty \)
75. \( \lim_{r \to 1^-} f(x) = -\infty \)
77. \( \lim_{r \to -\infty} f(x) = 0 \)

83.

85.

a) \( \lim_{r \to \infty} f(r) = 1 \)
\( \lim_{r \to -\infty} f(x) = -1 \)

b) The function is undefined over the interval \((-2, -1)\).

c) The domain of the function is given by
\[ x < -2, \quad -1 < x < 3, \quad \text{and} \quad x > 3 \]
The function is undefined on the interval \((-2, -1)\) since the interval the radicant is negative.

\[ \lim_{r \to -2^-} f(x) = 0 \]
\[ \lim_{r \to -2^+} f(x) = 0 \]

Exercise Set 3.4

1. a) \( x = 85 \)
    b) \( x = 0 \)
    c) \( y = 150 \)
    d) \( y = 210 \)

3. \( f(x) = x^3 - x^2 - x + 2 \)
   \( f'(x) = \frac{3x^2 - 2x - (3x + 1)(x - 1)}{(3x + 1)(x + 1)} = 0 \)
   \( x = \frac{1}{2} \) not in the interval
   \( x = 1 \)
   \( f(1) = 1 - 1 - 1 + 2 = 1 \)
Exercise Set 3.4

Next, we check the endpoints of the interval
\[ f(0) = 0 - 0 - 2 \cdot 2 \]
\[ f(2) = 8 - 4 - 2 \cdot 2 \]
Thus we have an absolute maximum at (2, 4) and an absolute minimum at (1, 1).

5. \( f(x) = 3x^2 - 2 \) \( f'(x) = 6x \), which means the function does not have critical values. Next, we check the endpoints of the interval
\[ f(-1) = -3 - 2 = -5 \]
\[ f(1) = 3 - 2 = 1 \]
Thus we have an absolute maximum at (1, 1) and an absolute minimum at (-1, -5).

7. \( f(x) = 3 - 2x - 5x^2 \)
\[ f'(x) = -2 - 10x \]
\[ f'(x) = 0 \]
\[-2 - 10x = 0 \]
\[ x = \frac{-1}{5} \]
\[ f\left(\frac{-1}{5}\right) = 3 \left(\frac{2}{5}\right) - \frac{1}{5} \]
\[ = \frac{16}{5} \]

Next, we check the endpoints of the interval
\[ f(-3) = 3 - 6 - 15 = -36 \]
\[ f(3) = 3 - 6 - 15 = -36 \]
Thus, we have an absolute maximum at \( \left(-3, \frac{16}{5}\right) \) and an absolute minimum at (3, -18).

9. \( f(x) = x^3 - 1 \)
\[ f'(x) = 3x^2 \]
\[ f'(x) = 0 \]
\[-3x^2 = 0 \]
\[ x = 0 \]
\[ f(0) = 0 - 1 \]
\[ = -1 \]

Next, we check the endpoints of the interval
\[ f(-8) = -1 - (-512) = 513 \]
\[ f(8) = 1 - 512 = -511 \]
Thus, we have an absolute maximum at (-8, 513) and an absolute minimum at (8, -511).

11. \( f(x) = 12 + 9x - 3x^2 - x^3 \)
\[ f'(x) = 9 - 6x - 3x^2 \]
\[ = 3(3 + x)(1 - x) \]
\[ f'(x) = 0 \]
\[ 3(3 + x)(1 - x) = 0 \]
\[ x = -3 \]
\[ x = 1 \]
\[ f(-3) = 12 - 27 - 27 - 15 \]
\[ f(1) = 12 + 9 - 4 - 1 - 17 \]
Note that the critical values are the same as the endpoints of the interval.
Thus, we have an absolute maximum at (1, 17) and an absolute minimum at (-3, -15).

13. \( f(x) = x^4 - 2x^3 \)
\[ f'(x) = 4x^3 - 6x^2 \]
\[ = 2x^2(2x - 3) \]
\[ f'(x) = 0 \]
\[ 2x^2(2x - 3) = 0 \]
\[ x = 0 \]
\[ x = \frac{3}{2} \]
\[ f(0) = 0 - 0 = 0 \]
\[ f\left(\frac{3}{2}\right) = \frac{81}{16} - \frac{27}{4} = -\frac{27}{16} \]

Next, we check the endpoints of the interval
\[ f(-2) = 16 - 2(-8) = 32 \]
\[ f(2) = 16 - 2(8) = 0 \]
Thus, we have an absolute maximum at (-2, 32) and an absolute minimum at \( \left(2, -\frac{27}{16}\right) \).

15. \( f(x) = 4x^3 - 2x^3 + 5 \)
\[ f'(x) = 12x^2 - 6x \]
\[ = 4x(x^2 - 1) \]
\[ f'(x) = 0 \]
\[ 4x(x^2 - 1) = 0 \]
\[ x = 0 \]
\[ x = 1 \]
\[ x = -2 \cdot 5 = 4 \]
\[ f(-1) = 1 - 2 \cdot 5 = 4 \]

Next, we check the endpoints of the interval
\[ f(-2) = 16 - 2(-5) = 15 \]
\[ f(2) = 16 - 2(5) = 13 \]
Thus, we have an absolute maximum at (-2, 15) and an absolute minimum at (1, 13) and an absolute minimum at (-1, 4) and (1, 4).

17. \( f(x) = \frac{x + 3}{3^{3/2} - 5} \)
\[ f'(x) = \frac{3^{3/2} - 5}{3^{3/2} - 5} \]
which is undefined at \( x = -\frac{3}{2} \) (the only critical value).

19. \( f(x) = x + \frac{1}{x} \)
\[ f'(x) = 1 - \frac{1}{x^2} \]
\[ f'(x) = 0 \]
\[ 1 - \frac{1}{x^2} = 0 \]
\[ x^2 - 1 = 0 \]
\[ x = -1 \text{ not acceptable} \]
\[ x = 1 \]
\[ f(1) = 1 + 1 = 2 \]

Next, we check the endpoint of the interval 
\[ f(20) = 20 + 0.05 = 20.05 \]
Thus, we have an absolute maximum at (20, 20.05) and an absolute minimum at (1, 2).

21. \[ f(x) = \frac{x^2}{x^2 + 1} \]
\[ f'(x) = \frac{2x(x^2 + 1) - 2x(x^2)}{(x^2 + 1)^2} \]
\[ = \frac{2x}{(x^2 + 1)^2} \]
\[ f'(x) = 0 \]
\[ f(0) = 0 \]

Next, we check the endpoints of the interval 
\[ f(-2) = \frac{4}{4 + 1} = 0.8 \]
\[ f(2) = \frac{4}{4 + 1} = 0.8 \]
Thus, we have an absolute maximum at (-2, 0.8) and at (2, 0.8) and an absolute minimum at (0, 0).

23. \[ f(x) = \frac{1}{3}(x + 1)^{-1/3}, \text{ which is undefined at } x = -1 (\text{ the only critical value}) \]
\[ f'(-1) = 0 \]
Next, we check the endpoints of the interval 
\[ f(-2) = (-1)^{-1/3} = -1 \]
\[ f(2) = (2)^{1/3} = 3 \]
Thus, we have an absolute maximum at (26, 3) and an absolute minimum at (-2, -1).

25. \[ f(x) = \frac{x + 2}{x^2 + 5} \]
\[ f'(x) = \frac{(x^2 + 5) - 2x(x + 2)}{(x^2 + 5)^2} \]
\[ = \frac{5 - 4x - x^2}{(x^2 + 5)^2} \]
\[ f'(x) = 0 \]
\[ f(-5) = \frac{-3}{25 + 5} = -0.1 \]
\[ f(1) = \frac{3}{1 + 5} = 0.5 \]

Next, we check the endpoints of the interval 
\[ f(-6) = \frac{-1}{36 + 5} = -0.0976 \]
\[ f(0) = \frac{2}{5 + 5} = 0.19612 \]
Thus, we have an absolute maximum at (1, 0.5) and an absolute minimum at (-5, -0.1).

27. \[ f(x) = x(x - x^2)^{1/2} \]
\[ f'(x) = (x - x^2)^{1/2} + \frac{x(1 - 2x)}{2(x - x^2)^{1/2}} \]
\[ f'(x) = 0 \]
\[ \\
\[ \rightarrow \]
\[ 2(x - x^2) = x(2x - 1) \]
\[ 4x^3 - 3x \neq 0 \]
\[ x = 0 \]
\[ x = \frac{3}{4} \]
\[ f(0) = 0 \]
\[ f \left( \frac{3}{4} \right) = \frac{3}{4} \left( \frac{3}{4} - 9 \right)^{1/2} + 0.32476 \]

Next, we check the endpoint of the interval 
\[ f(1) = 1(0) = 0 \]
Thus, we have an absolute maximum at \(\left( \frac{3}{4}, 0.32476 \right)\), and an absolute minimum at (0, 0) and at (1, 0).

29. \[ f(x) = x(x + 3)^{1/2} \]
\[ f'(x) = (x + 3)^{1/2} + \frac{x}{2(x + 3)^{1/2}} \]
\[ f'(x) = 0 \]
\[ \rightarrow \]
\[ 2(x + 3) = -x \]
\[ 3x + 6 = 0 \]
\[ x = -2 \]
\[ f(-2) = -2(-2 + 3)^{1/2} = -2 \]

Next, we check the endpoints of the interval 
\[ f(-3) = -3(0) = 0 \]
\[ f(6) = 6(9)^{1/2} = 18 \]
Thus, we have an absolute maximum at (6, 18), and an absolute minimum at (-2, -2).

31. \[ f(x) = x + 2 \sin x \]
\[ f'(x) = 1 + 2 \cos x \]
\[ f'(x) = 0 \]
\[ 1 + 2 \cos x = 0 \]
\[ \cos x = -\frac{1}{2} \]
\[ x = \frac{2\pi}{3} \]
\[ x = \frac{4\pi}{3} \]
\[ f \left( \frac{2\pi}{3} \right) = \frac{2\pi}{3} \left( 2 \sin \left( \frac{2\pi}{3} \right) \right) = 3.8264 \]
\[
\begin{align*}
\sin \left( \frac{4\pi}{3} \right) &= -\frac{\sqrt{3}}{2} \\
\frac{4\pi}{3} &= \frac{2\pi}{3} + \frac{\pi}{2} \\
&= \frac{2\pi}{3} \\
&= \frac{\pi}{3}
\end{align*}
\]

Next, we check the endpoints of the interval
\[
\begin{align*}
f(0) &= 0 \\
0 &= 0 \\
f(2\pi) &= 2\pi + 2 = 2\pi \\
Thus, we have an absolute maximum at \((2\pi, 2\pi)\) and an absolute minimum at \((0, 0)\).
\]

33. \( f(x) = \frac{\sin x}{1 + \sin x} \)

\[
\begin{align*}
f'(x) &= \frac{(2 + \sin x \cos x - \sin x \cos x)(2 + \sin x)^2 - 2\cos x(2 + \sin x)^2}{(2 + \sin x)^4} \\
&= \frac{2\cos x}{(2 + \sin x)^2} \\
f'(x) &= 0 \\
2\cos x &= 0 \\
2\cos x &= 0 \\
x &= \frac{\pi}{2} \\
x &= \frac{3\pi}{2} \\
\frac{\pi}{2} &= -1 \\
\frac{3\pi}{2} &= -1
\end{align*}
\]

Next, we check the endpoints of the interval
\[
\begin{align*}
f(0) &= 0 \\
0 &= 0 \\
f(\frac{\pi}{2}) &= \frac{1}{1 + 1} = 1 \\
Thus, we have an absolute maximum at \(\left( \frac{\pi}{2}, \frac{1}{2} \right) \) and an absolute minimum at \((0, 0)\).
\]

35. \( f(x) = \frac{\sin x}{1 + \sin x} \)

\[
\begin{align*}
(1 + \sin x)^2 \cos x - 2\sin x(1 + \sin x)\cos x &\quad (1 + \sin x)^4 - f'(x) \\
\cos x &\quad (1 + \sin x)^3 \\
f'(x) &= 0 \\
\cos x &= 0 \\
(1 + \sin x)^4 &\quad \cos x = 0 \\
x &= \frac{\pi}{2} \\
\frac{\pi}{2} &= 1 \\
\frac{1}{1 + 1} &= 1
\end{align*}
\]

Note that \( f'(x) \) is undefined at \( x = -\frac{\pi}{2} \) but the value does not belong to the given interval.

Next, we check the endpoints of the interval
\[
\begin{align*}
f(0) &= 0 \\
0 &= 0 \\
f(\frac{\pi}{2}) &= \frac{1}{1 + 1} = 1 \\
Thus, we have an absolute maximum at \(\left( \frac{\pi}{2}, \frac{1}{2} \right) \) and an absolute minimum at \((0, 0)\) and at \((\pi, 0)\).
Next, we check the endpoints of the interval

\[ f(0) = 0 - 0 = 0 \]
\[ f(2\pi) = 0 - 0 = 0 \]

Thus, we have an absolute maximum at \( \left( \frac{\pi}{3}, 1.4142 \right) \) and an absolute minimum at \( \left( \frac{5\pi}{3}, -1.4142 \right) \) and \( \left( \frac{7\pi}{3}, -1.4142 \right) \).

41. \( f(x) = \frac{-3}{4}x^3 \)

\[
f'(x) = 1 - x^2
\]
\[
f''(x) = 2x
\]
\[
f'''(1) = -2 < 0
\]

The function has an absolute maximum of \(-\frac{1}{3}\), no absolute minimum.

43. \( f(x) = -0.001x^2 + 4.8x - 60 \)

\[
f'(x) = -0.002x + 4.8
\]
\[
f''(x) = 0
\]
\[
-0.002x + 4.8 = 0
\]
\[
x = 2400
\]
\[
f(2400) = -5760 + 11520 - 60 - 5700
\]
\[
f'''(x) = -0.002 < 0
\]

The function has an absolute maximum of 5700, no absolute minimum.

45. \( f(x) = 2x + \frac{72}{x} \)

\[
f'(x) = 2 - \frac{72}{x^2}
\]
\[
f''(x) = 0
\]
\[
2 - \frac{72}{x^2} = 0
\]
\[
x^2 = 36
\]
\[
x = 6
\]
\[
f'(6) = 12 + 12 = 24
\]
\[
f''(x) = \frac{72}{x^3}
\]
\[
f'''(6) = \frac{1}{3} > 0
\]

The function has an absolute minimum of 24, no absolute maximum.

47. \( f(x) = x^2 + \frac{432}{x} \)

\[
f'(x) = 2x - \frac{432}{x^2}
\]
\[
f''(x) = 0
\]
\[
2x - \frac{432}{x^2} = 0
\]
\[
x^4 = 216
\]
\[
x = 6
\]
\[
f(6) = 36 + 72 = 108
\]
\[
f'''(x) = 2 + \frac{432}{x^3}
\]
\[
f'''(6) = 2 + \frac{2}{4} < 0
\]

The function has an absolute minimum of 108, no absolute maximum.

49. \( f(x) = (x + 1)^3 \)

\[
f'(x) = 3(x + 1)^2
\]
\[
f''(x) = 0
\]
\[
3(x + 1)^2 = 0
\]
\[
x = -1
\]
\[
f'(1) = 0
\]
\[
f''(x) = 6(x + 1)
\]
\[
f''(-1) = 0
\]

The function has no absolute maximum or minimum.

51. \( f(x) = -2x - 3 \)

The function is linear, which means that on the interval \((-\infty, \infty)\) the function has no absolute maximum or minimum.

53. \( f(x) = x^{3/4} \)

\[
f'(x) = \frac{3}{4}x^{-1/4}
\]
\[
f''(x) = -\frac{3}{4}x^{-1/4} \quad f'(x) \text{ and } f''(x) \text{ are undefined at } x = 0
\]
\[
f(1) = (1)^{3/4} = 1
\]
\[
f(0) = (0)^{3/4} = 0
\]
\[
f''(1) = -\frac{2}{9}
\]
\[
f''(-1) = -\frac{2}{9}
\]

The function has an absolute maximum of 1 and an absolute minimum of 0.
55. \( f(x) = \frac{1}{3}x^3 - x + \frac{2}{3} \)

The function grows indefinitely over \((-\infty, \infty)\), which means it has no absolute maximum or minimum.

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -100 & -80 & -60 & -40 & -20 \\
\hline
f(x) & 300000 & 200000 & 100000 & -200000 & -300000 \\
\hline
\end{array}
\]

57. \( f(x) = x^3 - 2x^2 \)

\[
f'(x) = 3x^2 - 4x \\
f''(x) = 6x - 4 \\
f'''(x) = 6
\]

\[
f(-1) = -1 - 2 - 1 = -4 \\
f(0) = 0 - 0 = 0 \\
f(1) = 1 - 2 = -1
\]

The function has an absolute minimum of \(-1\) and no absolute maximum.

59. \( f(x) = \tan x \cot x \)

\[
f'(x) = \sec^2 x \cot x - \csc^2 x \\
f''(x) = 0
\]

\[
f\left( \frac{\pi}{3} \right) = 1 + 1 = 2
\]

The function is undefined at \(x = 0\) and \(x = \frac{\pi}{2}\).

61. \( f(x) = \frac{1}{x + \cos x} \)

\[
f'(x) = \frac{\cos x - \sin x}{(\sin x + \cos x)^2} \\
f''(x) = 0
\]

\[
\begin{array}{c}
\cos x - \sin x \hfill \\
\hfill (\sin x + \cos x)^2
\end{array}
\]

\[
f\left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}
\]

\[
f''(x) = -2 \sin x \cos x - 1 \\
= \frac{1}{(\sin x + \cos x)^2}
\]

\[
f''\left( \frac{\pi}{4} \right) = \frac{1}{(\sqrt{2} + \sqrt{2})^2} = 0
\]

The function has an absolute minimum of \(\frac{1}{2}\) and no absolute maximum.

63. \( f(x) = \frac{1}{x - 2 \sin x} \)

\[
f'(x) = \frac{2 \cos x - 1}{(x - 2 \sin x)^2} \\
f''(x) = 0
\]

\[
f\left( \frac{\pi}{3} \right) = -1.46
\]

\[
f''(x) = -2 \sin x(x - 2 \sin x) - \frac{2(2 \cos x - 1)(1 - \cos x)}{(x - 2 \sin x)^3}
\]

\[
f''\left( \frac{\pi}{3} \right) = -3.693 < 0
\]

The function has an absolute maximum of \(-1.46\) and no absolute minimum.

65. \( f(x) = 2 \csc x + \cot x \)

\[
f'(x) = -2 \csc x \cot x - \csc^2 x \\
f''(x) = 0
\]

\[
-2 \csc x \cot x - \csc^2 x \hfill \\
\hfill 0
\]

\[
f\left( \frac{\pi}{3} \right) = -1.7321
\]

\[
f''(x) = -2 \csc x \cot x + 2 \csc^2 x \\
= 2 \csc^2 x \cot x
\]

\[
f''\left( \frac{\pi}{3} \right) = -0.7098 < 0
\]

The function has an absolute maximum of \(-1.7321\) and no absolute minimum.
67. \( f(x) = \tan x - 2 \sec x \)

\[
f'(x) = \sec^2 x - 2 \sec x \tan x
\]

\[
f'(x) = 0
\]

\[\frac{\sec^2 x - 2 \sec x \tan x}{x} = 0 \]

\[\frac{5\pi}{6} = \frac{1.7321}{6}
\]

\[f''(x) = 2 \tan x \sec^2 x - 2 \sec^3 x - 2 \sec x \tan^2 x
\]

\[f''(\frac{5\pi}{6}) = 2.309 > 0
\]

The function has an absolute minimum of 1.7321 and no absolute maximum.

69. \( f(x) = \frac{1}{1 - 2 \sin x} \)

\[
f'(x) = \frac{\frac{2 \cos x}{(1 - 2 \sin x)^2}}{1 - 2 \sin x}
\]

\[f'(x) = 0
\]

\[\cos x = 0
\]

\[x = \frac{\pi}{2}
\]

\[f(\frac{\pi}{2}) = -1
\]

\[f''(x) = \frac{8 \cos^2 x}{(1 - 2 \sin x)^3} - \frac{2 \sin x}{(1 - 2 \sin x)^2}
\]

\[f''(\frac{\pi}{2}) = -2 < 0
\]

The function has an absolute maximum of -1 and no absolute minimum.

71. \[y = -6.1x^2 + 752x + 22620
\]

\[y' = -12.2x + 752
\]

\[-12.2x + 752 = 0
\]

\[x = \frac{752}{12.2} = 61.64
\]

\[y'' = -12.2 < 0
\]

The maximum number of accidents occurs at \(x = 61.64\) mph.

73. \( r(x) = 104.5x^2 - 1501.5x + 6016 \)

\[r'(x) = 209x - 1501.5
\]

\[209x - 1501.5 = 0
\]

\[x = \frac{1501.5}{209} = 7.18
\]

\[r(7.18) = 104.5(7.18)^2 - 1501.5(7.18) + 6016
\]

\[= 15434.75
\]

\[r''(x) = 209 > 0
\]

The death rate is minimized at 7.18 hours of sleep per night.

75. a) \( D(h) = 0.139443 \sqrt{h} - 0.338382 h^{0.3064} \)

b) \( D(h) = 0.139443 \sqrt{h} - 0.239382 h^{0.3064} \)

\[D'(h) = \frac{0.697215}{\sqrt{h}} - \frac{0.09449624}{h^{1.3064}}
\]

\[0.697215 - \frac{0.09449624}{h^{1.3064}} = 0
\]

\[h^{1.3064} = 0.0069830098
\]

\[h = 18.816
\]

\[D(18.816) = 0.139443 \sqrt{18.816} - 0.0044964(18.816)^{0.3064}
\]

\[= -0.1581
\]

We check the endpoints of the interval

\[D(0) = 0.139443 \sqrt{0} - 0.0044964(0)^{0.3064} = 0
\]

\[D(215) = 0.139443 \sqrt{215} - 0.0044964(215)^{0.3064} = 0.0048
\]

The function has an absolute maximum of 0.0048 and an absolute minimum of -0.1581.

77. Left to the student. (answers vary.)

79. a) \( \cos \theta = -\delta \overline{BYP} = -\frac{1}{\sqrt{3}} \)

b) \( \sin \theta = \overline{Opp} \overline{BYP} \frac{\sqrt{3}}{\sqrt{3}} \)

c) From Example 4

\[S''(\theta) = \frac{3}{2} a^2 \left[ \frac{(\sqrt{3} \sin(\theta))(\sin^2(\theta))}{\sin^4(\theta)} \right]
\]

\[= \frac{3}{2} a^2 \left[ \frac{(1 - \sqrt{3} \cos(\theta))(2 \sin(\theta) \cos(\theta))}{\sin^4(\theta)} \right]
\]

\[S''(\theta) = \frac{3}{2} a^2 \left[ \frac{\sqrt{3}(\frac{\overline{BYP}}{\sqrt{3}})^2}{(\frac{\overline{BYP}}{\sqrt{3}})^4} \right]
\]

\[= \frac{3}{2} a^2 \left[ \frac{\frac{\overline{BYP}}{\sqrt{3}}}{\frac{\overline{BYP}}{\sqrt{3}}} \right]
\]

\[= \frac{3}{2} a^2 \left[ \frac{\overline{BYP}}{\sqrt{3}} \right]
\]

81. Absolute maximum at \( x = 4 \) and absolute minimum at \( x = 2 \)
83. No absolute maximum and no absolute minimum

9. \( Q = 2e^2 + 3y^2 \) and \( x + y = 5 \)

\[
Q = 2e^2 + 3(5 - x)^2 \\
= 2e^2 + 75 - 30x + 3x^2 \\
= 5x^2 - 30x + 75 \\
Q' = 10x - 30 \\
10x - 30 \quad 0 \\
x = 3 \\
y = 5 - 3 - 2 \\

When \( x = 3 \) and \( y = 2 \)

\[
Q = \left(2\right)^2 + \left(3\right)^2 \\
= 12 \\
= 30
\]

11. \( Q = e^2 + g^2 \) and \( x + y = 20 \)

\[
Q = e^2 + (20 - x)^2 \\
= e^2 + 400 - 40x + x^2 \\
= 2x^2 - 40x + 400 \\
Q' = 4x - 40 \\
4x - 40 \quad 0 \\
x = 10 \\
y = 20 - 10 - 10 \\

When \( x = 10 \) and \( y = 10 \)

\[
Q = \left(10\right)^2 + \left(10\right)^2 \\
= 100 + 100 \\
= 200
\]

13. \( Q = xy \) and \( \frac{1}{2}e^2 + y = 16 \)

\[
Q = \left(16 - \frac{1}{3}e^2\right) \\
= 16\frac{4}{3}x^2 \\
Q' = 16 - 4x^2 \\
= \left(4 - x^2\right) \\
4(4 - x^2) \quad 0 \\
x = 2 \\
y = 16 - \frac{4}{3}(1) \frac{32}{3} \\
\]

and

When \( x = 2 \) and \( y = \frac{32}{3} \)

\[
Q = \frac{2}{3} \left(\frac{32}{3}\right) \\
= \frac{64}{9}
\]

Exercise Set 3.5

1. \( Q = xy \) and \( x + y = 50 \)

\[
Q = x(50 - x) \\
= 50x - x^2 \\
Q' = 50 - 2x \\
50 - 2x = 0 \\
x = 25 \\
y = 50 - 25 = 25 \\
The two numbers are \( x = 25 \) and \( y = 25 \)

3. There cannot be a minimum product since there is only one critical value for the function and the second derivative is positive for values of \( x \).

5. \( Q = xy \) and \( y - x = 1 \)

\[
Q = x(x + 1) \\
= x^2 + x \\
Q' = 2x + 1 \\
2x + 1 = 0 \\
x = -\frac{1}{2} \\
y = -\frac{1}{2} + \frac{1}{2} \\
The two numbers are \( x = 2 \) and \( y = 2 \)

7. \( Q = xy^2 \) and \( x + y^2 = 1 \)

\[
Q = x(1 - x) \\
= x - x^2 \\
Q' = 1 - 2x \\
1 - 2x = 0 \\
x = \frac{1}{2} \\
y = \frac{1}{2} + \frac{1}{2} \\
\]

When \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \)

\[
Q = \frac{1}{2} \left(\frac{1}{\sqrt{2}}\right)^2 \\
= \frac{1}{4}
\]
15. \( Q = \sqrt{x} + \sqrt{y} \) and \( x + y = 1 \)

\[
Q = \sqrt{x} + \sqrt{1-x} \\
= x^{1/2} + (1-x)^{1/2} \\
Q' = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}} \\
\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}} = 0 \\
\sqrt{x} = \sqrt{1-x} \\
x = 1 - x \\
x = \frac{1}{2} \\
y = 1 - \frac{1}{2} = \frac{1}{2}
\]

When \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \)

\[
Q = \frac{\sqrt{1}}{2} + \frac{\sqrt{1}}{2} = \frac{1}{\sqrt{2}} = \sqrt{2} \\

17. \( A = lw \) and \( 2l + w = 20 \)

\[
A = l(20 - 2l) \\
= 20l - 2l^2 \\
A' = 20 - 4l \\
20 - 4l = 0 \\
l = 5 \\
w = 20 - 2(5) = 10 \\

When \( l = 5 \) and \( w = 10 \)

\[
A = 5(10) = 50 \\

The rectangular fence is 5 yards by 10 yards with a maximum area of 50 squared yards.
\]

19. \( A = lw \) and \( 2l + 2w = 54 \rightarrow l + w = 27 \)

\[
A = l(27 - l) \\
= 27l - l^2 \\
A' = 27 - 2l \\
27 - 2l = 0 \\
l = 13.5 \\
w = 27 - 13.5 = 13.5 \\

When \( l = 13.5 \) and \( w = 13.5 \)

\[
A = 13.5(13.5) = 182.25 \\

The room is 13.5 feet by 13.5 feet yards with a maximum area of 182.25 squared feet.
\]

21. Length: \( l = 30 - 2x \), Width: \( 30 - 2x \), and Height: \( h = x \)

\[
V = (30 - 2x)(30 - 2x)x \\
= 900x - 120x^2 + 4x^3 \\
V' = 900 - 240x + 12x^2 \\
= 12(15 - x)(5 - x) \\
12(15 - x)(5 - x) = 0 \\
x = 15 \text{ not acceptable} \\
x = 5 \\
l = 30 - 2(5) = 20 \\
w = 30 - 2(5) = 20 \\
h = 5 \\
V = 20(20)5 = 2000 \\

The box has a length and width of 20 inches and a height of 5 inches with a maximum volume of 2000 cubic inches.
\]

23. Length: \( x \), Width: \( x \), and Height: \( y \)

\[
SA = x^2 + 4xy + x^2y = 62.5 \\
SA = x^2 + 4x \left( \frac{62.5}{x^2} \right) \\
= x^2 + 250 \\
SA' = 2x - \frac{250}{x^2} \\
2x = \frac{250}{x^2} = 0 \\
x = 5 \\
y = \frac{62.5}{5^2} = 2.5 \\

When \( x = 5 \) and \( y = 2.5 \)

\[
SA = 5^2 - 4(5)(2.5) = 75 \\

The square based box has dimensions 5 x 5 x 2.5 inches and a maximum surface area of 75 square inches.
\]

25. \( A = \frac{1}{2}(2 \sin \frac{\theta}{2})(\cos \frac{\theta}{2}) \)

\[
A = \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
A' = -\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \\
-\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 0 \\
\sin \frac{\theta}{2} = \cos \frac{\theta}{2} \\
\frac{\theta}{2} = \frac{\pi}{4} \\
\frac{\theta}{2} = \frac{\pi}{2} \\
A = \sin \frac{\pi}{4} \cos \frac{\pi}{4} \\
= \frac{1}{2}
\]
27. Price of ticket: \( x \)

Number of people at the game: \( N = -10000x + 130000 \)

\[
x(-10000x + 130000) + 1.50(-10000x + 130000) = R
\]

\[
-10000x^2 + 130000x - 15000x + 19500
-10000x^2 + 115000x + 19500 = 0
-20000x + 115000 = N
-20000x + 115000 = 0
x = 5.75
-10000(5.75) + 130000 = N
72500
\]

The maximum revenue occurs when the price of the ticket is $5.75 and 72500 people attend the game.

29. Number of trees per acre: \( N \)

Number of bushels per tree: \( B = -N + 50 \)

\[
\begin{align*}
Y & = N(-N + 50) \\
Y' & = -2N + 50 \\
N & = 25
\end{align*}
\]

The farmer should plant 25 trees per acre to maximize bushel yields per tree.

31. \( x^2 y = 320 \Rightarrow y = \frac{320}{x^2} \)

\[
C' = 0.15x^2 + 0.10x^2 + 0.025(4xy)
= 0.25x^2 + 0.1x \left( \frac{320}{x^2} \right)
= 0.25x^2 + \frac{32}{x}
C'' = 0.5x - \frac{32}{x^2}
0.5x - \frac{32}{x^2} = 0
x^3 = 64
x = 4
y = \frac{320}{16} = 20
\]

The dimensions of the box that minimize the cost are 4 x 4 x 20 feet.

33. Amount invested: \( x \), Rate: \( r \) and \( x = kr \)

\[
P = 0.18kr - kr^2
P' = 0.18k - 2kr
\]

\[
r = 0.09
\]

Interest rate to maximize profit should be 9%.

35. Price: \( p \), Percentage: \( \epsilon \), Ordered pair \((p, r)\)

\[
(25, 2.13) \text{ and } (26, 2.49)
\]

\[
r - 2.13 = -0.04(p - 25)
-0.04p + 3.13
\]

b) \( R = p(42 + 31.33) \)

\[
R' = -4p^2 + 313p
R'' = -8p + 313
-8p + 313 = 0
p = 39.13
\]

The price that maximizes revenue is $39.13.

37. Length: \( y \), width: \( 4x \), height: \( 4x \) \( y = 84 \)

\[
V = x^2y
= x^2(84 - 4x)
= 84x^2 - 4x^3
V' = 168x - 12x^2
168x - 12x^2 = 0
12x(x - 14) = 0
x = 0 \text{ not acceptable}
14
y = 84 - 4(14) = 28
\]

The maximum box has dimensions 14 x 14 x 28 inches.

39. \( 2g + 2x + \pi x = 24 \)

\[
A = 2xy + \frac{\pi x^2}{2}
= 2r(-x - \frac{\pi}{2} + 12) + \frac{\pi x^2}{2}
= -2x^2 - \frac{\pi x^2}{2} + 24r
A' = -4x - \pi x + 24
-4x - \pi x + 24 = 0
x(4 + \pi) = 24
x = \frac{24}{4 + \pi}
y = \frac{24}{4 + \pi} + \frac{\pi}{2} \left( \frac{24}{4 + \pi} \right) \times 12
= \frac{24}{4 + \pi}
\]

41. Let \( x \) be the number

\[
S = \frac{1}{x} - \frac{1}{x^2}
S' = -\frac{1}{x^2} + \frac{10}{x}
\]

\[
\frac{1}{x^2} + 10x = 0
x^3 = \frac{1}{10}
x = \sqrt[3]{\frac{1}{10}}
43. 
\[ S = \pi x^2 + (24 - x)^2 \]
\[ = \pi x^2 + 576 - 48x + x^2 \]
\[ = (\pi + 1)x^2 - 48x + 576 \]
\[ S' = 2(\pi + 1)x - 48 \]
\[ 2(\pi + 1)x - 48 = 0 \]
\[ x = \frac{24}{\pi + 1} \]
\[ 24 - x = \frac{24 - 24}{\pi + 1} = \frac{24\pi}{\pi + 1} \]

There is no maximum if the string is cut. A maximum could occur if the string is not cut and the whole string is used to make the circle.

45. left to the student

47. \( Q = x^3 + 2y^3 \) and \( x + y = 1 \)
\[ Q = (1 - y)^3 + 2y^3 \]
\[ = 1 - 3y + 3y^2 - y^3 + 2y^3 \]
\[ = 3y^2 + 6y - 3 \]
\[ Q' = 3y^2 + 6y - 3 \]
\[ y = \frac{-6 \pm \sqrt{36 + 36}}{6} \]
\[ = -1 \pm \sqrt{2} \]
\[ x = 1 - (-1 \pm \sqrt{2}) \]
\[ = 2 - \sqrt{2} \]

When \( x = 2 - \sqrt{2} \) and \( y = -1 + \sqrt{2} \)
\[ Q = (2 - \sqrt{2})^3 + 2(-1 + \sqrt{2})^3 \]
\[ = 8 - 12\sqrt{2} + 12 - 2\sqrt{2} \cdot 2(-1 + 3\sqrt{2} - 6 + 2\sqrt{2}) \]
\[ = 8 - 4\sqrt{2} \]

Exercise Set 3.6

1. \( f(x) = x^2, a = 3 \)
\[ f'(x) = 2x \]
\[ L(x) = f(3) + f'(3)(x - 3) \]
\[ = 9 + 6(x - 3) \]
\[ = 6x - 9 \]

3. \( f(x) = \frac{1}{x^2}, a = 4 \)
\[ f'(x) = -\frac{1}{x^2} \]
\[ L(x) = f(4) + f'(4)(x - 4) \]
\[ = \frac{1}{4} - \frac{1}{16}(x - 4) \]
\[ = -\frac{1}{16}x + \frac{1}{2} \]

5. \( f(x) = x^{1/2}, a = 4 \)
\[ f'(x) = \frac{3}{2}x^{1/2} \]
\[ L(x) = f(4) + f'(4)(x - 4) \]
\[ = 8 + 3(x - 4) \]
\[ = 3x - 4 \]

7. \( f(x) = \cos x, a = 0 \)
\[ f'(x) = -\sin x \]
\[ L(x) = f(0) + f'(0)(x - 0) \]
\[ = 1 + 0(x - 0) \]
\[ = 1 \]

9. \( f(x) = x \cos x, a = 0 \)
\[ f'(x) = -x \sin x + \cos x \]
\[ L(x) = f(0) + f'(0)(x - 0) \]
\[ = 0 + 1(x - 0) \]
\[ = x \]

11. \( f(x) = \sqrt{x}, a = 10 \)
\[ f'(x) = \frac{1}{2\sqrt{x}} \]
\[ L(10) = f(10) + f'(10)(10 - 16) \]
\[ = 4 \]

13. \( f(x) = \sqrt{3}, a = 100 \)
\[ f'(x) = \frac{1}{2\sqrt{x}} \]
\[ L(100) = f(100) + f'(100)(100 - 100) \]
\[ = 101 \]

15. \( f(x) = \sqrt{x}, a = 8 \)
\[ f'(x) = \frac{1}{3\sqrt{x^2}} \]
\[ L(8) = f(8) + f'(8)(8 - 8) \]
\[ = 2 \]

17. \( f(x) = \sqrt{x}, a = 10 \)
\[ f'(x) = \frac{1}{2\sqrt{x}} \]
\[ L(10) = f(10) + f'(10)(10 - 9) \]
\[ = 2 \]

19. \( f(x) = \sin x, a = 0 \)
\[ f'(x) = \cos x \]
\[ L(0.1) = f(0) + f'(0)(0.1 - 0) \]
\[ = 0 \]

= 0.1
21. \( f(x) = \tan x, \quad a = 0 \)
\[ f'(x) = \sec^2 x \]
\[ f'(-0.04) = f(0) + f'(0)(-0.04 - 0) = 0 + 1(-0.04 - 0) = -0.04 \]

23. \( f(x) = \frac{1}{3}x^3 - x + 1 \), lower 3
\[ f'(x) = \frac{1}{3}x^2 - 1 \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_1 = 0 \]
\[ x_2 = 0 - \frac{0.33333}{0.33333} = -1 \]
\[ x_3 = \frac{0.33333}{-0.33333} - \frac{0.38095}{-0.38095} = 0.38197 \]
\[ x_4 = \frac{0.38095}{0.74003} = 0.38197 \]
\[ x = 0.38197 \]

25. \( f(x) = x^3 - 3x + 3 \)
\[ f'(x) = \frac{3x^2}{3x^2 - 3} \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_1 = -3 \]
\[ x_2 = -3 - \frac{-15}{24} = -2.375 \]
\[ x_3 = -2.375 - \frac{-3.27148}{13.92188} = -2.1901 \]
\[ x_4 = -2.1901 - \frac{-0.58045}{10.73892} = -2.11158 \]
\[ x_5 = -2.11158 - \frac{-0.59099}{10.28777} = -2.10384 \]
\[ x_6 = -2.10384 - \frac{-0.56009}{10.27805} = -2.10380 \]
\[ x_7 = -2.10380 - \frac{-0.56009}{10.27805} = -2.10380 \]
\[ x = -2.10380 \]

27. \( f(x) = \sqrt{x^2 + 1} \), lower 4
\[ f'(x) = \frac{x}{\sqrt{x^2 + 1}} + \frac{1}{\sqrt{x^3 + 1}} \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_1 = -2 \]
\[ x_2 = -2 - \frac{0.5459}{2.4774} = 2.2079 \]
\[ x_3 = 2.2079 - \frac{0.0455}{2.0077} = 2.2250 \]
\[ x_4 = 2.2250 - \frac{0.0013}{2.0077} = 2.2266 \]
\[ x_5 = 2.2266 - \frac{0.0004}{2.0077} = 2.2267 \]
\[ x_6 = 2.2267 - \frac{0.0002}{2.0077} = 2.2268 \]
\[ x_7 = 2.2268 - \frac{0.0001}{2.0077} = 2.2268 \]
\[ x = 2.2268 \]

29. \( f(x) = \cos 2x - x \)
\[ f'(x) = -2 \sin 2x - 1 \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_1 = 0 \]
\[ x_2 = 0 - \frac{1}{2} = 1 \]
\[ x_3 = 1 - \frac{1.416}{-2.819} = 0.49769 \]
\[ x_4 = 0.49769 - \frac{0.0468}{-0.2678} = 0.51505 \]
\[ x_5 = 0.51505 - \frac{0.0003}{-0.2715} = 0.51494 \]
\[ x_6 = 0.51494 - \frac{0}{-0.2714} = 0.51494 \]
\[ x = 0.51494 \]

31. \( f(x) = \sin x - \cos x + x \)
\[ f'(x) = \cos x + \sin x + 1 \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_1 = 0 \]
\[ x_2 = 0 - \frac{1}{2} = 0.5 \]
\[ x_3 = 0.5 - \frac{0.10181}{2.3385} = 0.15679 \]
\[ x_4 = 0.15679 - \frac{0.0039}{2.3385} = 0.15663 \]
\[ x_5 = 0.15663 - \frac{0}{2.3385} = 0.15663 \]
\[ x = 0.15663 \]

33. Initial guess \( x = -1.5 \) leads to solution \( x = -1.142895 \)
Initial guess \( x = 0.25 \) leads to solution \( x = 0.176245 \)
Initial guess \( x = 4.9 \) leads to solution \( x = 0.06625 \)

35. Initial guess \( x = -3 \) leads to solution \( x = -2.86610 \)
Initial guess \( x = -0.5 \) leads to solution \( x = -0.56062 \)
Initial guess \( x = 0.5 \) leads to solution \( x = 0.10863 \)

37. Use Newton’s method on \( 0.05x^2 - 0.3x + 0.0001 \)
Initial guess \( x = 0 \) leads to solution \( x = 0.05452 \)
Initial guess \( x = 0.1 \) leads to solution \( x = 0.15229 \)
The two dosages are \( x = 0.055 \) and \( x = 0.15 \) cubic centimeters.

39. Use Newton’s method on \( -0.85 + 1.85x - 0.65x^2 + 0.00075x^3 \)
Initial guess \( x = 4 \) leads to solution \( x = 4.34980 \)
The age at which the median weight of boys is 15 pounds is \( t = 4.35 \) months.
41. Use Newton's method on

\[-0.000775 x^3 + 0.0696e^2 - 0.209x - 35.32\]

Initial guess \( x = 30 \) leads to solution \( x = 29.93332 \)

The rate is 40 per 100000 in the end of 1959.

43. Use Newton’s method on

\[-0.000654 x^4 + 0.0067 e^2 - 0.9997 x^2 - 0.84 e - 300.25\]

Initial guess \( x = 60 \) leads to solution \( x = 57.09821 \)
Initial guess \( x = 95 \) leads to solution \( x = 97.50401 \)

At the age of 57 years and 97.5 years 300 out of 100000 women will have breast cancer.

45. \( f(r) = r^{16} - 0.99 r^{14} - 0.0858 \left[ 1 - \left( \frac{0.99}{r} \right)^{15} \right] \)

\[
f'(r) = 15r^{14} - 13.86 r^{13} + \left( \frac{2.151081148}{r^{27}} \right)
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
x_1 = 1.1
\]

\[
x_2 = 1.1 - \frac{0.33386}{9.0507} = 1.0631
\]

\[
x_3 = 1.0631 - \frac{0.00303}{4.1082} = 1.042
\]

\[
x_4 = 1.042 - \frac{0.02021}{2.5499} = 1.0340
\]

\[
x_5 = 1.0340 - \frac{0.00248}{1.9274} = 1.03276
\]

\[
x_6 = 1.03276 - \frac{0}{1.8279} = 1.03276
\]

\[
r = 1.03276
\]

47. \( f(r) = r^4 - 0.98 r^3 - 0.1764 \left[ 1 - \left( \frac{0.98}{r} \right)^{15} \right] \)

\[
f'(r) = 4r^3 - 2.94 r^2 - \frac{1.05423846}{r^{16}}
\]

After 7 iterations of Newton’s method we reach \( r = 1.08846 \)

49. a) \[
v = \frac{77000 \cdot 100 \cdot \sec \frac{\pi}{9}}{4000000} = \frac{11.386}{s}
\]

b) \( f(t) = 1.925 \sec t, f'(t) = 11.086 \sec t \tan t \)

\[
L(0.01) = f\left( \frac{4\pi}{9} \right) + f'\left( \frac{4\pi}{9} \right) \{0.01\}
\]

\[
= 11.7147
\]

Difference in measurement = 11.7147 - 11.086 = 0.6287 \( \text{cm} \)

c) The difference in measurement is more sensitive to angle measurement when the frequency \( f \) gets smaller.

51. a) \( x_1 = 3 \)

\[
x_2 = 3 - \frac{3^3}{3} = 0 \]

\[
x_3 = 0 - \frac{0^3}{3} = 0 \]

\[
x = 0
\]

b) The tangent line at \( x_1 = 3 \) (which intersects \( x_{n+1} \)) intersected \( x = 0 \) instead of the closer solutions.

53. For \( x = 0.0469, f = 239.674 \) Hz

For \( x = 0.3118, f = 1953.96 \) Hz

For \( x = 0.5236, f = 2950.99 \) Hz

55. For \( x = 0.1162, f = 653.67 \) Hz

For \( x = 0.3283, f = 1850.12 \) Hz

For \( x = 0.4234, f = 2385.47 \) Hz

Exercise Set 3.7

1. \( xy - x + 2y = 3 \)

\[
x \frac{dy}{dx} + y - 1 + 2 \frac{dy}{dx} = 0
\]

\[
(x + 2) \frac{dy}{dx} = 1 - y
\]

\[
\frac{dy}{dx} = \frac{1 - y}{x + 2}
\]

For \( (-5, \frac{3}{2}) \)

\[
\frac{dy}{dx} = \frac{1 - \frac{3}{2}}{-5 + 2} = \frac{1}{9}
\]

3. \( x^2 + y^2 = 1 \)

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{-x}{y}
\]

For \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \)

\[
\frac{dy}{dx} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}
\]

5. \( x^2 y - 2x^3 - y^3 + 1 = 0 \)

\[
x^2 \frac{dy}{dx} + 2xy - 6x^2 - 3y^2 \frac{dy}{dx} = 0
\]

\[
(x^2 - 3y^2) \frac{dy}{dx} = 6x^2 - 2xy
\]

\[
\frac{dy}{dx} = \frac{6x^2 - 2xy}{x^2 - 3y^2}
\]

For \( (2, -3) \)

\[
\frac{dy}{dx} = \frac{6(2)^2 - 2(2)(-3)}{(2)^2 - 3(-3)^2} = \frac{-36}{23}
\]
Exercise Set 3.7

7. \( \sin y + x^2 - \cos y \)

\[
\frac{dy}{dx} \cos y + 2x - \frac{dy}{dx} \sin x
\]

\[
(\cos y + \sin y) \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = \frac{-2x}{\cos y + \sin y}
\]

For \( \frac{\pi}{2}, \frac{3\pi}{2} \)

\[
\frac{dy}{dx} = \frac{-2y}{\cos \frac{\pi}{2} + \sin \frac{\pi}{2}}
\]

= \( -\frac{2}{9} \)

9. \( x \sin x = y(1 + \cos y) \)

\[
x \cos x + \sin x - y(-\sin y) \frac{dy}{dx} + (1 + \cos y) \frac{dy}{dx}
\]

\[
x \cos x + \sin x - y(-\sin y) \cos y + (1 + \cos y) \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = \frac{x \cos x + \sin x}{-y \sin y + \cos y + 1}
\]

For \( \frac{\pi}{2}, \frac{3\pi}{2} \)

\[
\frac{dy}{dx} = \frac{\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}}{-\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} + 1}
\]

= \( \frac{6}{9 - x\sqrt{3}} \)

11. \( 2xy + 3 = 0 \)

\[
2x \frac{dy}{dx} + 2y = 0
\]

\[
\frac{dy}{dx} = \frac{-y}{x}
\]

13. \( x^2 - y^2 = 16 \)

\[
2x - 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{x}{y}
\]

15. \( y^5 = x^5 \)

\[
5y^4 \frac{dy}{dx} = 3x^5
\]

\[
\frac{dy}{dx} = \frac{3x^2}{5y^4}
\]

17. \( x^2y^4 + x^3y^3 = 11 \)

\[
3x^2y^4 \frac{dy}{dx} + 2x^3y^3 \frac{dy}{dx} + 3x^2y^3 = 0
\]

\[
(3x^2y^4 + 4x^2y^3) \frac{dy}{dx} = -2x^3y^3 - 3x^2y^4
\]

\[
\frac{dy}{dx} = \frac{-2x^3y^4 - 3x^2y^4}{3x^2y^2 + 4x^2y^3}
\]

19. \( \sqrt{x} + \sqrt{y} = 1 \)

\[
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}
\]

21. \( y^4 - \frac{x - 1}{x + 1} \)

\[
3y^2 \frac{dy}{dx} = \frac{x + 1 - (x - 1)}{(x + 1)^2}
\]

\[
\frac{dy}{dx} = \frac{-2}{3(x + 1)^2 y^3}
\]

23. \( x^{3/2} + y^{2/3} = 1 \)

\[
\frac{3}{2} x^{1/2} \frac{dy}{dx} \frac{3}{2} y^{1/3} \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}
\]

25. \( \frac{x^2y + xy + 1}{2x + y} = 1 \rightarrow x^2y + xy + 1 = 2x + y \)

\[
x^2 \frac{dy}{dx} + 2xy + x \frac{dy}{dx} + y = 2 + \frac{dy}{dx}
\]

\[
(x^2 + 1) \frac{dy}{dx} = 2 - 2xy - y
\]

\[
\frac{dy}{dx} = \frac{2 - 2xy - y}{x^2 + 1}
\]

27. \( 4 \sin x \cos y = 3 \)

\[
-4 \sin x \cos y \frac{dy}{dx} + 4 \cos x \cos y = 0
\]

\[
\frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \cos y}
\]

\[
= \frac{\cos x \cdot \text{cot} y}{\sin x \cdot \text{cot} y}
\]

29. \( x + y = \sin(\sqrt{y - x}) \)

\[
1 + \frac{dy}{dx} = \cos(\sqrt{y - x}) \frac{dy}{dx}
\]

\[
2\sqrt{y - x} + 2y - x \frac{dy}{dx} = \cos(\sqrt{y - x}) \frac{dy}{dx} - \cos(\sqrt{y - x})
\]

\[
(2\sqrt{y - x} - \cos(\sqrt{y - x})) \frac{dy}{dx} = -2\sqrt{y - x} + \cos(\sqrt{y - x})
\]

\[
\frac{dy}{dx} = \frac{-2\sqrt{y - x} + \cos(\sqrt{y - x})}{2\sqrt{y - x} - \cos(\sqrt{y - x})}
\]

31. \( A^3 + B^3 = 9 \) When \( A = 2, B = \sqrt[3]{9 - 8} = 1 \)

\[
3A^3 \frac{dA}{dt} + 3B^3 \frac{dB}{dt} = 0
\]

\[
3(2)^3 \frac{dA}{dt} + 3(1)^3 \frac{dA}{dt} = 0
\]

\[
\frac{dA}{dt} = \frac{-9}{12} = -\frac{3}{4}
\]

\[
\frac{dA}{dt} = \frac{-9}{12}
\]

\[
= -\frac{3}{4}
\]
33. \( V = \frac{4}{3}\pi r^3 \)

\[
\frac{dV}{dt} = \frac{4}{3} \pi r^3 \frac{dr}{dt} = \frac{4}{3} \pi (1.2)^2 (0.03) = 0.54287 \text{ m/s} \\
\]

35. \( V = \frac{\rho}{4Le} (R^2 - r^2) \)

\( a) \)

\[
\frac{dV}{dt} = \frac{2R\rho}{4Le} \frac{dR}{dt} = \frac{2R(100)}{4(1)(0.05)} \frac{dR}{dt} = 1000R \frac{dR}{dt} \\
\]

\( b) \)

\[
\frac{dV}{dt} = 1000R \frac{dR}{dt} = 1000(0.0075)(-0.0015) = -0.01125 \text{ m/s} \\
\]

37. \( S = \frac{\sqrt{kW}}{60} \)

\[
\frac{dS}{dt} = \frac{h}{120\sqrt{kW}} \frac{dw}{dt} = \frac{180}{120\sqrt{180}(63)} = (-4) \\
\]

\[= -0.0485 \text{ m/s} \\
\]

39. \( \frac{D^2 - x^2 + y^2}{D} = \frac{25^2 + 60^2}{65} = 65 \)

\[
2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\
\frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{D} = \frac{25(25) + 60(60)}{65} = 65 \text{ m/s} \\
\]

41. \( \theta = \frac{h}{100} \)

\[
\sec^2 \theta \frac{d\theta}{dx} = \frac{1}{100} \frac{dh}{dt} \\
\frac{dh}{dt} = 100 \sec^2 \theta \frac{d\theta}{dx} = 100 \sec^2 \left( \frac{\pi}{6}\right)(0.1) = 13.3333 \text{ m/s} \\
\]

43. \( a) \) \( R^2 : x^2 + y = 9 \)

When \( R = 10, x = \sqrt{100 - 9} = \sqrt{91} \)

\[
2R \frac{dR}{dt} = 2x \frac{dx}{dt} \\
\frac{dx}{dt} = \frac{R \frac{dR}{dt}}{x} = \frac{10}{\sqrt{91}}(2) = 2.0966 \frac{f}{sec} \\
\]

\( b) \) \( \sin \theta = \frac{3}{R} \)

When \( R = 10, \theta = \sin^{-1} \left( \frac{3}{10} \right) = 0.30469 \text{ rad} \)

\[
\cos \theta \frac{d\theta}{dt} = -\frac{3}{R^2} \frac{dR}{dt} = -\frac{3}{100 \cos(0.30469)}(2) = \frac{3}{3 \sqrt{91}} \\
-\theta \frac{d\theta}{dt} = \frac{3}{3 \sqrt{91}} \frac{rad}{sec} \\
\]

45. \( x^2 + x - 2y = 4 \)

\[
x \frac{dy}{dx} + y + 1 = -\frac{2}{2} \frac{dy}{dx} = 0 \\
\]

\[
(x - 2) \frac{dy}{dx} = -1 - y \\
\frac{dy}{dx} = \frac{y + 1}{2 - x} \\
\frac{d^2 y}{dx^2} = \frac{(2 - x) (\frac{dy}{dx})^2 + (y + 1)}{(2 - x)^2} = \frac{(2 - x)^3 + y + 1}{(2 - x)^2} \\
= \frac{(2 - x) + y + 1}{(2 - x)^2} \\
= \frac{y + 1}{(2 - x)^2} \\
= \frac{2y + 1}{(2 - x)^2} \\
\]

47. \( x^2 - y^2 = 5 \)

\[
2x - 2y \frac{dy}{dx} = 0 \\
\frac{dy}{dx} = x \\
\frac{d^2 y}{dx^2} = \frac{y - x \frac{dy}{dx}}{y^2} = \frac{y - x}{y^2} \frac{y^2}{y^2} = \frac{y^2 - x^2}{y^2} \\
\]

100 

Chapter 3: Application of Differentiation
49. Left to the student (answers vary)

51.

53.

55.
Chapter 4
Exponential and Logarithmic Functions

Exercise Set 4.1

1. Graph: $y = 4^x$
First we find some function values.

Note: For
\[
\begin{array}{ccc}
\text{x} & \text{y} = 4^x & \text{Value} \\
-2 & 0.0625 & \frac{1}{16} \\
-1 & 0.25 & \frac{1}{4} \\
0 & 1 & 1 \\
1 & 4 & 4 \\
2 & 16 & 16 \\
\end{array}
\]

Plot these points and connect them with a smooth curve.

5. Graph: $x = 4^y$
First we find some function values.

Note: For
\[
\begin{array}{ccc}
\text{x} & \text{y} & \text{Value} \\
2 & -2 & \frac{1}{16} \\
2 & -1 & \frac{1}{4} \\
2 & 0 & 1 \\
2 & 1 & 4 \\
2 & 2 & 16 \\
\end{array}
\]

Plot these points and connect them with a smooth curve.

3. Graph: $y = (0.4)^x$
First we find some function values.

Note: For
\[
\begin{array}{ccc}
\text{x} & \text{y} = (0.4)^x & \text{Value} \\
-2 & 6.25 & \frac{1}{(0.4)^2} \\
-1 & 2.5 & \frac{1}{0.4} \\
0 & 1 & 1 \\
1 & 0.4 & 0.4 \\
2 & 0.16 & 0.16 \\
\end{array}
\]

Plot these points and connect them with a smooth curve.

7. $N = 1000(1 + 0.20)^t - 1000(1.2) = 1200$
$N = 1000(1 + 0.20)^2 - 1000(1.44) = 1440$
$N = 1000(1 + 0.20)^5 - 1000(2.488) = 2488$

9. $N = 286000000(1 + 0.021)^t - 286000000(1.021) = 298138126$
$N = 286000000(1 + 0.021)^2 - 286000000(1.0424) = 3173188026$

11. $f(x) = e^{3x}$
$f'(x) = 3e^{3x}$
$\frac{d}{dx} f(x) = f'(x) \cdot e^{3x}$
13. \( f(x) = 5e^{-2x} \)
   \[ f'(x) = 5 \cdot \left( -2 \right) e^{-2x} \]
   \[ = -10e^{-2x} \]

15. \( f(x) = 3 - e^{-x} \)
   \[ f'(x) = -e^{-x} \]
   \[ = e^{-x} \]

17. \( f(x) = -7e^x \)
   \[ f'(x) = -7e^x \]
   \[ = e^{-x} \]

19. \( f(x) = \frac{1}{2} e^{2x} \)
   \[ f'(x) = \frac{1}{2} \cdot 2e^{2x} \]
   \[ = e^{2x} \]

21. \( f(x) = x^3 e^x \)
   \[ f'(x) = x^3 e^x + 3x^2 e^x \]

23. \( f(x) = (x^2 + 3x - 9)e^x \)
   \[ f'(x) = (2x + 3)e^x + (x^2 + 3x - 9)e^x \]

25. \( f(x) = (\sin x)e^x \)
   \[ f'(x) = (\sin x)e^x + (\cos x)e^x \]

27. \( f(x) = \frac{x^3}{e^x} \)
   \[ f'(x) = \frac{3x^2 e^x - x^3 e^x}{e^{2x}} \]
   \[ = \frac{x^2 e^x}{x^2} \]
   \[ = \frac{e^x(x-4)}{x^2} \]

31. \( f(x) = e^{-x^2/2} \)
   \[ f'(x) = \frac{1}{2} e^{-x^2/2} \]
   \[ = e^{-x^2/2} \]

33. \( y = e^{\sqrt{x} - 1} \)
   \[ \frac{dy}{dx} = \frac{1}{2} (\sqrt{x} - 1)^{-1/2} \cdot e^{(\sqrt{x} - 1)^{3/2}} \]
   \[ = \frac{2}{\sqrt{x} - 1} \]

35. \( y = \sqrt[4]{x^2 - 1} \)
   \[ \frac{dy}{dx} = \frac{1}{2} (x^2 - 1)^{-3/2} \cdot e^{(x^2 - 1)^{1/2}} \]
   \[ = \frac{e^{x^2 - 1}}{2} \]

37. \( y = \tan(e^x + 1) \)
   \[ \frac{dy}{dx} = \sec^2(e^x + 1) \cdot e^x \]

39. \( y = e^{\pi x} \cdot x^3 \)
   \[ \frac{dy}{dx} = e^{\pi x} \cdot 3x^2 \cdot e^x \]

41. \( y = (2x + \cos x)e^{3x+1} \)
   \[ \frac{dy}{dx} = (2x + \cos x) \cdot e^{3x+1} + (1 + 3x)e^{3x+1} \]
   \[ = e^{3x+1}(2x + \cos x - 3 \sin x + 2) \]

43. \( y = xe^{-2x} + e^{-x} + x^3 \)
   \[ \frac{dy}{dx} = x \cdot (-2) e^{-2x} + 1 \cdot e^{-x} + (1) \cdot e^{-x} + 3x^2 \]
   \[ = (1 + 2x)e^{-2x} + e^{-x} + 3x^2 \]

45. \( y > 1 - e^{-x} \)
   \[ \frac{dy}{dx} = 0 - (-1)e^{-x} \]
   \[ = e^{-x} \]

47. \( y = \frac{1}{x} \)
   \[ \frac{dy}{dx} = 0 - \left( \frac{1}{x^2} \right) \]
   \[ = -\frac{1}{x^2} \]

49. \( y = (e^{2x} + 1)^5 \)
   \[ \frac{dy}{dx} = 5(e^{2x} + 1)^4 \cdot 2xe^{2x} \]
   \[ = 10xe^{2x}(e^{2x} + 1)^4 \]
51. \( \begin{align*}
\frac{dy}{dx} &= \frac{e^0(1 - e^{i \theta})}{e^{i \theta}} \quad \text{Factoring} \\
&= \frac{1 - e^0}{e^i} \quad \text{Simplifying} \\
&= \frac{e^i(-4e^{4\theta} - e^{1 - e^{i \theta}})}{4e^{4\theta}^2} \quad \text{Using the Quotient Rule} \\
&= \frac{e^{i(-4e^{4\theta} - 1 + e^{i \theta})}}{e^i} \\
&= \frac{-3e^{4\theta} - 1}{e^i} \\
&= -3e^{i \theta} - e^{-i}
\end{align*} \)

53. \( \begin{align*}
\frac{dy}{dx} &= \frac{(x^2 + 1)e^x - 2x \cdot e^x}{(x^2 + 1)^2} \\
&= \frac{e^x(x^2 - 2x + 1)}{(x^2 + 1)^2} \\
&= \frac{e^x(x - 1)^2}{(x^2 + 1)^2}
\end{align*} \)

55. \( \begin{align*}
f(x) &= e^{x^2} + \sqrt{x} \\
f'(x) &= 2xe^{x^2} + \frac{1}{2\sqrt{x}}
\end{align*} \)

57. \( \begin{align*}
f(x) &= e^{x^2} - x - 1 \\
f'(x) &= 2xe^{x^2} - \frac{1}{2\sqrt{x}} + x - 1
\end{align*} \)

61. Graph: \( f(x) = e^{2x} \)

Using a calculator we first find some function values.

Note: For
\( x = -2, f(-2) = e^{2(-2)} = e^{-4} = 0.0183 \)
\( x = -1, f(-1) = e^{2(-1)} = e^{-2} = 0.1353 \)
\( x = 0, f(0) = e^{2(0)} = 1 \)
\( x = 1, f(1) = e^{2(1)} = e^2 = 7.3891 \)
\( x = 2, f(2) = e^{2(2)} = e^4 = 54.598 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.0183</td>
</tr>
<tr>
<td>-1</td>
<td>0.1353</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>7.3891</td>
</tr>
<tr>
<td>2</td>
<td>54.598</td>
</tr>
</tbody>
</table>

Plot these points and connect them with a smooth curve.

Derivatives. \( f'(x) = 2e^{2x} \) and \( f''(x) = 4e^{2x} \).

Critical points of \( f \). Since \( f'(x) > 0 \) for all real numbers \( x \), we know that the derivative exists for all real numbers and there is no solution of the equation \( f'(x) = 0 \). There are no critical points and therefore no maximum or minimum values.

Increasing. Since \( f'(x) > 0 \) for all real numbers \( x \), the function \( f \) is increasing over the entire real line, \( (-\infty, \infty) \).

Inflection points. Since \( f''(x) > 0 \) for all real numbers \( x \), the equation \( f''(x) = 0 \) has no solution and there are no points of inflection.

Concavity. Since \( f''(x) > 0 \) for all real numbers \( x \), the function \( f' \) is increasing and the graph is concave up over the entire real line.

63. Graph: \( f(x) = e^{-2x} \)

Using a calculator we first find some function values.

Note: For
\( x = -2, f(-2) = e^{-2(-2)} = e^4 = 54.598 \)
\( x = -1, f(-1) = e^{-2(-1)} = e^2 = 7.3891 \)
\( x = 0, f(0) = e^{-2(0)} = e^0 = 1 \)
\( x = 1, f(1) = e^{-2(1)} = e^{-2} = 0.1353 \)
\( x = 2, f(2) = e^{-2(2)} = e^{-4} = 0.0183 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>54.598</td>
</tr>
<tr>
<td>-1</td>
<td>7.3891</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.1353</td>
</tr>
<tr>
<td>2</td>
<td>0.0183</td>
</tr>
<tr>
<td>$x$</td>
<td>$f(x)$</td>
</tr>
<tr>
<td>-----</td>
<td>-------</td>
</tr>
<tr>
<td>-2</td>
<td>54.598</td>
</tr>
<tr>
<td>-1</td>
<td>7.3891</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.1353</td>
</tr>
<tr>
<td>2</td>
<td>0.0183</td>
</tr>
</tbody>
</table>

Plot these points and connect them with a smooth curve.

**Derivatives.** $f'(x) = -2e^{-2x}$ and $f''(x) = 4e^{-2x}$.

**Critical points of $f$.** Since $f'(x) < 0$ for all real numbers $x$, we know that the derivative exists for all real numbers and there is no solution of the equation $f'(x) = 0$. There are no critical points and therefore no maximum or minimum values.

**Decreasing.** Since $f'(x) < 0$ for all real numbers $x$, the function $f$ is decreasing over the entire real line, $(-\infty, \infty)$.

**Inflection points.** Since $f''(x) > 0$ for all real numbers $x$, the equation $f''(x) = 0$ has no solution and there are no points of inflection.

**Concavity.** Since $f''(x) > 0$ for all real numbers $x$, the function $f'$ is increasing and the graph is concave up over the entire real line.

**65.** Graph: $f(x) = 3 - e^{-x}$, for nonnegative values of $x$.

Using a calculator we first find some function values.

Note: For
$x = 0, f(0) = 3 - e^{-0} = 3 - 1 = 2$
$x = 1, f(1) = 3 - e^{-1} = 3 - 0.3679 = 2.6321$
$x = 2, f(2) = 3 - e^{-2} = 3 - 0.1353 = 2.8647$
$x = 3, f(3) = 3 - e^{-3} = 3 - 0.0498 = 2.9502$
$x = 4, f(4) = 3 - e^{-4} = 3 - 0.0183 = 2.9817$
$x = 6, f(6) = 3 - e^{-6} = 3 - 0.0025 = 2.9975$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2.6321</td>
</tr>
<tr>
<td>3</td>
<td>2.8647</td>
</tr>
<tr>
<td>4</td>
<td>2.9502</td>
</tr>
<tr>
<td>6</td>
<td>2.9975</td>
</tr>
</tbody>
</table>

Plot these points and connect them with a smooth curve.

**Derivatives.** $f'(x) = -e^{-x}$ and $f''(x) = -e^{-x}$.

**Critical points of $f$.** Since $f'(x) > 0$ for all real numbers $x$, we know that the derivative exists for all real numbers and there is no solution of the equation $f''(x) = 0$. There are no critical points and therefore no maximum or minimum values.

**Increasing.** Since $f'(x) > 0$ for all real numbers $x$, the function $f'$ is increasing over the entire real line, $(-\infty, \infty)$.

**Inflection points.** Since $f''(x) < 0$ for all real numbers $x$, the equation $f''(x) = 0$ has no solution and there are no points of inflection.

**Concavity.** Since $f''(x) < 0$ for all real numbers $x$, the function $f''$ is decreasing and the graph is concave down over the entire real line.

**67. - 71.** Left to the student

**73.** We first find the slope of the tangent line at $(0, 1)$, $f'(0)$:

$f(x) = e^x$

$f'(x) = e^x$

$f'(0) = e^0 = 1$

Then we find the equation of the line with slope 1 and containing the point (0, 1):

$y - y_1 = m(x - x_1)$

Point-slope equation

$y - 1 = 1(x - 0)$

$y - 1 = x$

$y = x + 1$

**75.** Left to the student

**77.** $C(t) = 100^2 e^{-t}$

a) $C(0) = 10 \cdot 0^2 \cdot e^{0} = 0$ ppm

$C'(1) = 10 \cdot 1^2 \cdot e^{-1}$

$\approx 10(0.367879)$

$\approx 3.7$ ppm
Exercise Set 4.1

\[ C(2) = 10 \cdot 2^2 \cdot e^{-2} \]
\[ \approx 10(0.135335) \]
\[ \approx 5.4 \text{ ppm} \]
\[ C(3) = 10 \cdot 3^2 \cdot e^{-3} \]
\[ \approx 90(0.040967) \]
\[ \approx 4.18 \text{ ppm} \]
\[ C(10) = 10 \cdot 10^2 \cdot e^{-10} \]
\[ \approx 1000(0.000015) \]
\[ \approx 0.05 \text{ ppm} \]

b) We plot the points (0,0), (1,3.7), (2,5.1), (3,1.88), and (10,0.05) and other points as needed. Then we connect the points with a smooth curve.

\[ C''(t) = 10t^2(-1)e^{-t} + 20te^{-t} \]
\[ = 10te^{-t}(t - 2), \text{ or } 10te^{-t}(2 - t) \]

\[ C''(t) \text{ exists for all } t \in [0, \infty). \text{ We solve } C''(t) = 0, \]
\[ 10te^{-t}(2 - t) \]
\[ t(2 - t) = 0 \]
\[ (10t^2 / 0) \]
\[ t < 0 \text{ or } 2 - t = 0 \]
\[ t = 0 \text{ or } 2 - t = 0 \]

Since the function has two critical points, we analyze the graph to find the maximum value. We see that the maximum value of the concentration is about 5.4 ppm. It occurs at \( t = 2 \) hr.

e) The derivative represents the rate of change of the concentration of the medication with respect to the time \( t \).

79. a) \( r(t) = 0.153 \left( e^{(0.1417+0.47^t)} - e^{-4.3150+0.47^t} \right) \)
\[ = 0.153 \left( e^{(0.1417-1.8523)} - e^{(0.1417-1.8523)} \right) \]

b) \( r'(t) = 0.153 \left( e^{(0.1417+1.8523)} \right) (0.141) - \]
\[ = 0.153 \left( e^{(0.1417+1.8523)} \right) (0.153) \]
\[ + 0.021573 \left( e^{(0.1417-1.8523)} \right) - \]
\[ 0.023509 \left( e^{(0.1417-1.8523)} \right) \]

81. \( g = e^{0.5x} e^{3x^2+2x-1} \)
\[ \frac{dy}{dx} = \frac{dy}{dx}\left( e^{3x^2+2x-1} \right) + e^{3x^2+2x-1} \left( \frac{3}{2} \right) \]
\[ \frac{dy}{dx} = \frac{1}{2} \sqrt{e} e^{3x^2+2x-1} \left( 2x(9e^2 + 2) \right) \]
\[ \frac{dy}{dx} = \frac{1}{2} \sqrt{e} e^{3x^2+2x-1} \left( 18e^2 + 4e^2 + 3 \right) \]

80. \( y = r^{1/2} \cdot (e^r)^{1/2} \)
\[ \frac{dy}{dx} = \frac{1}{2} \sqrt{r} \cdot (e^r)^{1/2} \cdot e^{1/2} \cdot (e^r) \]
\[ = \frac{1}{2} \sqrt{r} \cdot e^{1/2} \cdot (1 + \frac{1}{\sqrt{r}}) \]

85. \( g = \sin(e^r) \)
\[ \frac{dy}{dx} = \cos(e^r) \cdot \sin(e^r) \cdot e^r \]
\[ = e^r \sin(e^r) \cdot \cos(e^r) \]

87. \( y = 1 + e^{4x^2+1} \)
\[ \frac{dy}{dx} = e^{4x^2+1} \cdot e^{1+e^x} - e^{1+e^x} \cdot e^x \]

89. \( f(t) = (1 + t)^{1/2} \)
\[ f(1) = 2 \]
\[ f(0.5) = 1.5^2 = 2.25 \]
\[ f(0.2) = 1.2^2 = 1.4432 \]
\[ f(0.1) = 1.1^2 = 1.21974 \]
\[ f(0.01) = 1.001^{100} = 2.71692 \]

91. \( f(x) = x^2 e^{-x} \)
\[ f'(x) = x^2(-e^{-x}) + 2xe^{-x} \]
\[ = e^{-x}(2x - x^2) \]

Solve \( f'(x) = 0 \)
\[ e^{-x}(2x - x^2) = 0 \]
\[ 2x - x^2 = 0 \]
\[ x(2-x) = 0 \]
\[ x = 0 \]
\[ x = 2 \]

We use test values to determine the nature of the critical values we found.
\[ f'(1) = e^{-1}(2-1) = e^{-1} > 0 \]
\[ f'(3) = e^{-3}(6-9) = -3e^{-3} < 0 \]

Therefore, the function has a maximum at \( x = 2 \). We find \( f(2) \)
\[ f(2) = 2^2 e^{-2} \]
\[ = \frac{4}{e^2} \]

The maximum occurs at \( (2, \frac{4}{e^2}) \)
93. \( F(t) = e^{-0.104(1-15) + 0.56(35 - t)} \)
   a) \( \lim_{t \to 15^+} F(t) = 0 \)
   b) \( \lim_{t \to 35^-} F(t) = 0 \)
   c) \[ F''(t) = e^{-0.104(1-15) + 0.56(35 - t)} \cdot \frac{-0.56}{(35-t)^2} - 0.9(1-15)^2 + 0.56(35-t)^2 \]

Solve for \( F''(t) = 0 \)

\[ \frac{9}{(t-15)^2} - \frac{0.56}{(35-t)^2} = 0 \]
\[ \frac{9}{(t-15)^2} = \frac{0.56}{(35-t)^2} \]
\[ 9(35-t)^2 = 0.56(t-15)^2 \]
\[ 11025 - 630t + 9t^2 - 0.56t^2 + 168t - 126 = 0 \]
\[ 8.44t^2 - 613.2t + 10890 = 0 \]

Using the quadratic formula, we get: \( x = 31.0071 \).

The other zero of the function falls outside the interval \( 15 < t < 35 \). Using test values to determine the sign of the first derivative on either side of \( x = 31.0071 \) gives \( F''(20) > 0 \) and \( F''(33) < 0 \). Therefore \( F''(t) \) has a maximum at \( x = 31.0071 \). Find \( F(31.0071) \)

\[ F(31.0071) = e^{-0.104(1-31.0071) + 0.56(35-31.0071)} \]
\[ = e^{-0.7625} \]
\[ = 0.48535 \]

95. \( D(t) = 34.4 \cdot 14.8 \cdot e^{-0.542t/14.8} \)
   a) Find \( D(110) \) and \( D(150) \)

\[ D(110) = 34.4 - \frac{30.48}{1 + 29.4e^{-0.542(110)}} = 34.4 - 30.157 = 4.253 \]
\[ D(150) = 34.4 - \frac{30.48}{1 + 29.4e^{-0.542(150)}} = 34.4 - 30.462 = 3.938 \]

b) \( \lim_{t \to \infty} D(t) = 3.92 \) Which is to say that the annual death rate in Mexico will not go below 3.92 per 1000 citizens.

97. Rewrite as follows \( f(x) = 3.26e^{0.97x} - 5 \), which gives \( f'(x) = 3.424e^{1.07x} \), and now try to find the zero of \( f(x) \) using Newton's method starting with a guess of \( x_0 = 1.5 \)

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{f(x_n)}{3.424e^{1.07x}} \]
\[ = 1.5 - \frac{3.26e^{0.97x} - 5}{3.424e^{1.07x}} \]
\[ = 1.5 - \frac{3.26 - 5}{3.424e^{0.97x}} \]
\[ = 1.5 - \frac{3.26 - 5}{3.424e^{0.97x}} \]
\[ = 1.5 - \frac{2.74}{3.424e^{0.97x}} \]
\[ = 1.5 - \frac{2.74}{3.424e^{0.97x}} \]
\[ = 1.5 - \frac{0.8588}{3.424e^{0.97x}} \]
\[ = 0.3508 - \frac{0.8588}{3.424e^{0.97x}} \]
\[ = 0.3508 - \frac{0.8588}{3.424e^{0.97x}} \]
\[ = 0.3508 - \frac{0.8588}{3.424e^{0.97x}} \]
Exercise Set 4.2

1. \( \log_2 8 = 3 \) Logarithmic equation
   \[ 2^3 = 8 \] 8 is the base, 3 is the exponent.

2. \( \log_b 2 = \frac{1}{3} \) Logarithmic equation
   \[ 8^{1/3} = 2 \] 8 is the base, 1/3 is the exponent.

3. \( \log_a K = J \) Logarithmic equation
   \[ a^J = K \] Exponential equation; a is the base, J is the exponent.

4. \( -\log_{10} b = -p \) Logarithmic equation
   \[ \log_{10} b = p \] Multiplying by -1
   \[ 10^{-p} \] b Exponential equation; 10 is the base, -p is the exponent.

5. \( e^M b \) Exponential equation; e is the base, M is the exponent
   \[ \log_{10} b = M \] Logarithmic equation
   or \( \ln b = M \) ln is the abbreviation for log. b

6. \( l0^2 \) 100 Exponential equation; 10 is the base, 2 is the exponent
   \[ \log_{10} 100 = 2 \] Logarithmic equation

7. \( 10^{-1} = 0.1 \) Exponential equation; 10 is the base, -1 is the exponent
   \[ \log_{10} 0.1 = -1 \] Logarithmic equation

8. \( M^p = V \) Exponential equation; M is the base, p is the exponent
   \[ \log_M V = p \] Logarithmic equation

9. \( \log_5 15 = \log_5 3 \cdot 5 \)
   \[ = \log_5 3 + \log_5 5 \] (P1)
   \[ = 1.099 + 1.000 \]
   \[ = 2.098 \]

10. \( \log_5 \frac{1}{5} = \log_5 1 - \log_5 5 \) (P2)
    \[ = 0 - \log_5 5 \] (P1)
    \[ = -\log_5 5 \]
    \[ = -1.000 \]

11. \( \ln 20 \ln 1.5 \)
    \[ = \ln 4 \ln 5 \] (P1)
    \[ = 1.3863 \times 1.6094 \]
    \[ = 2.29957 \]

12. \( \ln \frac{1}{4} = \ln 1 - \ln 4 \) (P2)
    \[ = 0 - 1.3863 \] (P6)
    \[ = -1.3863 \]

13. \( \ln \sqrt{e} = \ln e^{1/2} \)
    \[ = \ln e \] (P5)
    \[ = 1 \]

14. \( \ln 3927 = 8.275731 \) Using a calculator and rounding to six decimal places

15. \( \ln 0.0182 = -4.896334 \)

16. \( \ln 8.995619 = 1 \)

17. \( e^t = 100 \)
    \[ \ln e^t = \ln 100 \] Taking the natural logarithm on both sides
    \[ t = \ln 100 \] (P5)
    \[ t = 4.605170 \] Using a calculator
    \[ t \approx 4.6 \]

18. \( e^t = 60 \)
    \[ \ln e^t = \ln 60 \] Taking the natural logarithm on both sides
    \[ t = \ln 60 \] (P5)
    \[ t = 4.094345 \] Using a calculator
    \[ t \approx 4.1 \]

19. \( e^{-t} = 0.1 \)
    \[ \ln e^{-t} = \ln 0.1 \] Taking the natural logarithm on both sides
    \[ -t = \ln 0.1 \] (P5)
    \[ t = -\ln 0.1 \]
    \[ t = -(-2.302585) \] Using a calculator
    \[ t = 2.302585 \]
    \[ t \approx 2.3 \]

20. \( e^{-0.02t} = 0.06 \)
    \[ \ln e^{-0.02t} = \ln 0.06 \] Taking the natural logarithm on both sides
    \[ -0.02t = \ln 0.06 \] (P5)
    \[ t = \frac{\ln 0.06}{-0.02} \]
    \[ t = \frac{-2.302585}{-0.02} \] Using a calculator
    \[ t \approx 114.1 \]
43. \( y = -6 \ln x \)

\[
\frac{dy}{dx} = -6 \cdot \frac{1}{x} = -\frac{6}{x}
\]

45. \( y = x^4 \ln x - \frac{1}{2} x^2 \)

\[
\frac{dy}{dx} = x^4 \cdot \frac{1}{x} + 4x^3 \cdot \ln x - \frac{1}{2} \cdot 2x = x^3(1 + 4 \ln x) - x
\]

47. \( y = \ln \frac{x}{x^e} \)

\[
\frac{dy}{dx} = \frac{x^e \cdot 1 - x^3 \cdot \ln x}{x^8} = \frac{x^2(1 - 4 \ln x)}{x^3} = \frac{f - 4 \ln x}{x^5}
\]

49. \( y = \ln \frac{x}{4} \)

\( y = \ln x - \ln 4 \) \( \text{(P2)} \)

\[
\frac{dy}{dx} = -\frac{1}{x} = \frac{1}{x}
\]

51. \( y = \ln \cos x \)

\[
\frac{dy}{dx} = -\sin x \cdot \frac{1}{\cos x} = -\tan x
\]

53. \( f(x) = \ln(4x) \)

\[
f'(x) = 4 \cdot \frac{1}{4x} \cdot \frac{1}{4x} = \frac{1}{x} \cdot \frac{1}{\ln 4x} = \frac{1}{x \ln 4x}
\]

55. \( f(x) = \ln \left( \frac{x^2 - 7}{x} \right) \)

\[
f'(x) = \frac{x^2 - 7}{x^2} \cdot \frac{1}{x^2 - 7} \]

\[
\left( \frac{d}{dx} \ln g(x) - g'(x) \cdot \frac{1}{g(x)} \right)
\]

\[
\frac{d}{dx} \ln g(x) = g'(x) \cdot \frac{1}{g(x)}
\]

\[
\text{Using Quotient Rule to find}
\]

\[
\frac{2x^2 - x^2 + 7}{x^2} \cdot \frac{x}{x^2 - 7} = \frac{x^2}{x^2 - 7}
\]

57. \( f(x) = e^x \ln x \)

\[
f'(x) = e^x \cdot \frac{1}{x} + e^x \cdot \ln x \) \text{ Using the Product Rule}
\]

\[
e^x \left( \frac{1}{x} + \ln x \right)
\]

59. \( f(x) = e^x \sec x \)

\[
f'(x) = e^x \cdot \sec x \cdot \tan x \cdot v(x + 1)
\]

\[
e^x \sec x \cdot \tan x \cdot v(x + 1)
\]

61. \( f(x) = \ln(e^x + 1) \)

\[
f'(x) = e^x \cdot \frac{1}{e^x + 1} \]

\[
\left( \frac{d}{dx} (e^x + 1) = e^x \right)
\]

\[
e^x \cdot \frac{1}{e^x + 1}
\]

63. \( f(x) = (\ln x)^2 \)

\[
f'(x) = 2(\ln x) \cdot \frac{1}{x} \) \text{ Extended Power Rule}
\]

\[
\frac{2 \ln x}{x}
\]
65. \[ g = (\ln x)^{-2} \]
\[
\frac{dy}{dx} = -4(\ln x)^{-5} \cdot \frac{1}{x}
\]
\[ = -4(\ln x)^{-5} \cdot \frac{1}{x} \]

67. \[ f(t) = \ln(1 + t^2)^3 \]
\[ f'(t) = 6t^2 \cdot \frac{1}{1 + t^2} \]
\[ = \frac{15t^2}{1 + t^2} \]

69. \[ f(x) = |\ln(x + 5)|^3 \]
\[ f'(x) = 3|\ln(x + 5)|^2 \cdot \frac{1}{x + 5} \]
\[ = \frac{3|\ln(x + 5)|^2}{x + 5} \]

71. \[ f(t) = |(t^2 + 3)(t^2 - 1)| \]
\[ f'(t) = \frac{1}{(t^2 + 3)(t^2 - 1)} \cdot \frac{2t^2 + 6t + 3t^2 - 2}{(t^2 + 3)(t^2 - 1)} \]
\[ = \frac{5t^2 - 3t + 6t}{(t^2 + 3)(t^2 - 1)} \]

73. \[ y = -\frac{x^5}{(8x + 5)^2} \]
\[ y' = -5x^4 \cdot \frac{1}{8x + 5} - 2 \cdot 8x + 5 \cdot 8x + 5 \cdot \frac{1}{(8x + 5)^2} \]
\[ = -5 \cdot \frac{16x + 25}{8x + 5} \cdot \frac{1}{x + 5} \]

75. \[ y = \ln \left( \frac{-\sin x}{\sin x} \right) \]
\[ dy \frac{dx}{dx} = -\frac{\cos x (1 + \ln \sin x)}{\sin^2 x} \]
\[ = -\frac{\cos x (1 + \ln \sin x)}{\sin^2 x} \]

77. \[ y = \frac{1}{n+1} \left( \ln x - \frac{1}{n+1} \right) \]
\[ dy \frac{dx}{dx} = \frac{1}{n+1} \left( \frac{1}{x} - 0 \right) + x^n \left( \ln x - \frac{1}{n+1} \right) \]
\[ = \frac{1}{n+1} x^n \ln x - \frac{x^n}{n+1} \]
\[ = x^n \ln x \]

79. \[ y = \ln \left( \frac{t + \sqrt{1 + t^2}}{1 + \sqrt{1 + t^2}} \right) \]
\[ \frac{dy}{dx} = \frac{1}{t + \sqrt{1 + t^2}} \cdot \frac{1}{1 + \sqrt{1 + t^2}} \]
\[ = \frac{1}{t + \sqrt{1 + t^2}} \cdot \frac{1}{1 + \sqrt{1 + t^2}} \]

81. \[ y = \sin(\ln x) \]
\[ \frac{dy}{dx} = \cos(\ln x) \cdot \frac{1}{x} \]
\[ = \frac{\cos(\ln x)}{x} \]

83. \[ y = (\sin x) \ln(\tan x) \]
\[ \frac{dy}{dx} = (\sin x) \left( \sec^2 x \cdot \frac{1}{\tan x} \right) + (\cos x) \frac{\ln(\tan x)}{\sec x} \]
\[ = \cos x \ln(\tan x) \cdot \sec x \]

85. \[ y = \ln \left( \frac{sec 2x + \tan 2x}{sec 2x + \tan 2x} \right) \]
\[ \frac{dy}{dx} = \frac{1}{2 \sec 2x + \tan 2x} \cdot \left( 2 \sec 2x \tan 2x + 2 \sec^2 2x \right) \]
\[ = \frac{2 \sec 2x}{sec 2x + \tan 2x} \]

87. \[ y = \ln \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right) \]
\[ \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \left( e^{2x} + e^{-2x} \right) \]
\[ = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \left( e^{2x} - e^{-2x} \right) \]
\[ = \frac{e^{2x} - e^{-2x}}{e^x + e^{-x}} \cdot \left( e^x - e^{-x} \right) \]
\[ = \frac{1}{e^2 - e^{-2}} \]
\[ = \frac{1}{e^{2x} - 1} \]

89. Using the points (10, 164) and (70, 3015) we get
\[ 164 \rightarrow A(10) \]
\[ 3015 \rightarrow A(70) \]
93. Using the point (25000, 1079) and (500000, 29) we get
\[ \frac{1079}{29} = \frac{A_{25000}}{A_{500000}} \]
\[ 29 \cdot A_{500000} = 1079 \cdot A_{25000} \]
\[ \log_{10} \left( \frac{29}{1079} \right) = \log_{10} \left( \frac{A_{500000}}{A_{25000}} \right) \]
\[ c = \log_{10} \left( \frac{29}{1079} \right) \]
\[ A = \frac{1079}{29} \]
\[ A = 219449722.1 \]
Therefore, \( y = 219449722.1 \cdot 10^{-1.207} \)

91. Using the point (30, 25) and (100000000, 250) we get
\[ 250 = A_{100000000} \]
\[ 25 = A_{30} \]
\[ \frac{25}{250} = \left( \frac{3}{100000000} \right)^c \]
\[ c = \log_{10} \left( \frac{3}{100000000} \right) \]
\[ A = \frac{250}{100000000} \]
\[ A = 15.907 \]
Therefore, \( y = 15.907 \cdot 10^{0.333} \)
Exercise Set 4.2

113

c)  

\[ P(t) = 100(1 - e^{-0.2t}) \]

90 = 100(1 - e^{-0.2t})  
Replacing \( P(t) \) by \( 90 \)

0.9 = 1 - e^{-0.2t}

0.1 = -e^{-0.2t}

\[ \ln 0.1 = \ln e^{-0.2t} \]  
Taking the natural logarithm on both sides

\[ -0.2t = \ln 0.1 \]

\[ t = \frac{-\ln 0.1}{-0.2} \]

\[ t = \frac{2.3026}{0.2} \]

\[ t = 11.5 \]

Using a calculator

Thus it will take approximately 11.5 months for 90% of the doctors to become aware of the new medicine.

d)  

\[ \lim_{t \to \infty} P(t) = \lim_{t \to \infty} 100(1 - e^{-0.2t}) = 100. \]

This indicates that 100% of doctors will eventually accept the new medicine.

97)  

a)  

\[ S(t) = 68 - 20\ln(t + 1), \quad t \geq 0 \]

\[ S(0) = 68 - 20\ln(0 + 1) \]  
Substituting \( 0 \) for \( t \)

\[ = 68 - 20\ln 1 \]

\[ = 68 - 20 \cdot 0 \]

\[ = 68 \]

Thus the average score when they initially took the test was 68%.

b)  

\[ S(4) = 68 - 20\ln(4 + 1) \]  
Substituting 4 for \( t \)

\[ = 68 - 20\ln 5 \]

\[ = 68 - 20(1.609438) \]  
Using a calculator

\[ = 68 - 32.18876 \]

\[ = 35.81124 \]

\[ \approx 36\% \]

c)  

\[ S(24) = 68 - 20\ln(24 + 1) \]  
Substituting 24 for \( t \)

\[ = 68 - 20\ln 25 \]

\[ = 68 - 20(1.397846) \]  
Using a calculator

\[ = 68 - 64.37752 \]

\[ = 3.62248 \]

\[ \approx 3.6\% \]

d) First we rewrite the question:

3.6 (the average score after 24 months) is what percent of 68 (the average score when \( t = 0 \)).

Then we translate and solve:

\[ 3.6 \cdot x = 68 \]

\[ x = \frac{68}{3.6} \]

\[ 0.0528 \cdot x \]

\[ 3.6 \% \approx x \]

c)  

\[ S(t) = 68 - 20\ln(t + 1), \quad t \geq 0 \]

\[ S'(t) = -20 \cdot \frac{1}{t + 1} \]

\[ -20 \]

\[ t + 1 \]

\[ S'(t) < 0 \] for all \( t \geq 0 \). Thus \( S(t) \) is a decreasing function and has a maximum value of 68%, when \( t = 0 \).

g)  

\[ \lim_{t \to \infty} S'(t) \]

\[ \lim_{t \to \infty} 68 - 20 \ln(t + 1) = -\infty. \]

Clearly, the score cannot be less than 0, but this limit indicates that eventually everything will be forgotten.

99)  

\[ e(p) = 0.37 \ln p + 0.05 \quad p \text{ in thousands}, \quad v \text{ in ft per sec} \]

a)  

\[ e(531) = 0.37 \ln 531 + 0.05 \]  
Substituting 531 for \( p \)

\[ = 0.37(6.274702) + 0.05 \]

\[ = 2.231562 + 0.05 \]

\[ = 2.231662 \]

\[ \approx 2.23 \text{ ft/sec} \]

b)  

\[ e(7900) = 0.37 \ln 7900 + 0.05 \]

\[ = 0.37(8.974618) + 0.05 \]

\[ = 3.320099 + 0.05 \]

\[ = 3.370099 \]

\[ \approx 3.37 \text{ ft/sec} \]

c)  

\[ e'(p) = 0.37 \cdot \frac{1}{p} \]

\[ = \frac{0.37}{p} \]

d)  

\[ e'(p) \] is the acceleration of the walker.

101)  

\[ f(x) = \ln \ln x^3 \]

\[ f'(x) = \frac{1}{\ln x^3} \cdot 3 \ln x \cdot \frac{1}{x} \]

\[ = \frac{3}{x \ln x} \]

103) Using L'Hospital's Rule (taking the derivative of the numerator and denominator then applying the limit)

\[ \lim_{h \to 0} \frac{\ln(1 + h)}{h} = \lim_{h \to 0} \frac{\frac{1}{1 + h}}{1} \]

\[ = \frac{\ln 1}{h(1 + h)} \]

\[ = \frac{1}{1 + 0} \]

\[ = 1 \]

105)  

\[ \frac{P}{P_0} = e^{-kt} \]

\[ \frac{P}{P_0} = \frac{\ln \left( \frac{P}{P_0} \right)}{kt \ln(e)} \]
\[ \ln \left( \frac{P}{P_0} \right) = kt \]
\[ \frac{\ln \left( \frac{P}{P_0} \right)}{k} = t \]

107. Left to the student.

109. a) \[ r(T) = 0.124 \left( e^{0.129(T-1.5)} - e^{1.1298 - 0.144(1.5-T)} \right) \]
\[ r(T) = 0.124 \left( e^{0.129(T-1.2255)} - e^{0.144(T-1.5898)} \right) \]

b) \[ \frac{dr}{dT} = 0.124 \left( e^{0.129(T-1.2255)} - 0.144(e^{0.144(T-1.5898)} \right) \]
\[ \frac{dr}{dT} = 0.015996e^{0.125(T-1.2255)} - 0.017856e^{0.144(T-1.5898)} \]

110. c) \[ \frac{dr}{dT} = 0 \]
\[ 0.015996e^{0.125(T-1.2255)} = 0.017856e^{0.144(T-1.5898)} \]
\[ \ln(0.890841286) = 0.151T - 0.6225 \]
\[ T = \frac{\ln(0.890841286) + 0.6225}{0.151} = 34.17 \]

111.
\[ A \left( b(T_1 - T_c) - c(T_c - T_i) - (T_i - T) \right) = 0 \]
\[ b(T_1 - T_c) - c(T_c - T) = b \]
\[ c(T_c - T_i) - (T_i - T) = b \]
\[ b(T_i) - c(T_1) - cT - bT = \ln \left( \frac{b}{c} \right) \]
\[ T_i(b - c) - T(b - c) = \ln \left( \frac{b}{c} \right) \]
\[ \ln \left( \frac{b}{c} \right) - T_i(b - c) = -T(b - c) \]
\[ T_i - \frac{1}{b - c} \ln \left( \frac{b}{c} \right) = T \]

Plugging the values of the constants left to the student.

113.
\[ a - b \]
Exercise Set 4.3

1. The solution of \( \frac{dQ}{dt} = kQ \) is \( Q(t) = ce^{kt} \), where \( t \) is the time. At \( t = 0 \), we have some "initial" population \( Q(0) = Q_0 \). Thus \( Q(0) = ce^{k\cdot0} = ce^{0} \cdot 1 = c \).
Thus, \( Q_0 = c \), so we can express \( Q(t) = Q(0) \cdot e^{kt} \).

3. The solution of \( \frac{dy}{dt} = 2g \) is
\[ y = ce^{2t}, \quad y = 5 \]
when \( t = 0 \) gives
\[ 5 = c \cdot e^{2(0)} \]
\[ 5 = c \cdot 1 \]
\[ 5 = c \]

Therefore, \( y = 5e^{2t} \)

5. a) \( P = 1000 e^{0.003t} \)
b) when \( t = 30 \)
\[ P = 1000 e^{0.003(30)} \]
\[ = 1000 e^{0.09} \]
\[ = 2691 \]
c) when \( t = 60 \)
\[ P = 1000 e^{0.003(60)} \]
\[ = 1000 e^{0.18} \]
\[ = 7233 \]
d) when \( t = 1440 \)
\[ P = 1000 e^{0.003(1440)} \]
\[ = 1000 e^{4.34} \]
\[ = 1.34 \cdot 10^{23} \]
e) The generation time is
\[ T = \frac{\ln(2)}{k} = \frac{\ln(2)}{0.003} \]
\[ = \frac{2.3026}{0.003} \]
\[ = 767.52 \]

7. a) \( k = \frac{\ln(2)}{T} = \frac{\ln(2)}{0.003} \)
\[ = \frac{0.6931}{0.003} \]
\[ = 1.4748 \]

b) \( P = 200 \ e^{0.37481t} \) for \( t \geq 3 \)
\[ P = 200 \ e^{0.37481 \cdot 3} \]
\[ = 200 \ e^{1.12443} \]
\[ = 200 \cdot 3.106 \]
\[ = 621.2 \]

9. Find \( P_0 \) for \( P = 20000 \).

a) \( P = ce^{0.0601t} \), when \( t = 0, \ P = 281 \) gives
\[ 281 = c \cdot e^{0.0601(0)} \]
\[ 281 = c \cdot 1 \]
\[ c = 281 \]

Thus, \( P = 281 \cdot e^{0.0601t} \)

b) Find \( P \) when \( t = 15 \)
\[ P = 281 \cdot e^{0.0601(15)} \]
\[ = 281 \cdot 1.91 \]
\[ = 536.1 \]

13. The balance grows at the rate given by
\[ \frac{dP}{dt} = 0.067P \]
a) \( P(t) = P_0 \cdot e^{0.067t} \)
b) \( P(1) = 1000 \cdot e^{0.067 \cdot 1} \) Substituting 1000 for \( P_0 \) and 1 for \( t \)
\[ = 1000 \cdot e^{0.067} \]
\[ = 1000(1.067150) \]
\[ \approx 1067.16 \]

The balance after 1 year is $1067.16.
27. \((0, 47432)\) and \((40, 432976)\)

a) From the point \((0, 47432)\) we get

\[ y = a \cdot b^t \]

\[ 47432 = a \cdot b^0 \]

\[ 47432 = a \]

From the point \((40, 432976)\) we get

\[ \frac{432976}{47432} = b^{40} \]

\[ \ln \left( \frac{432976}{47432} \right) = 40 \ln(b) \]

\[ \frac{432976}{47432} = \frac{\ln(432976)}{\ln(b)} = \frac{40 \ln(b)}{\ln(b)} \]

\[ 40 \approx 0.0552846 \ln(b) \]

So, \( y = 47432e^{0.0552846t} \)

b) When \( t = 50 \)

\[ y = 47432e^{0.0552846(50)} \]

\[ = 752595 \]

c) When \( y = 1000000 \)

\[ \frac{1000000}{47432} = e^{0.0552846t} \]

\[ \ln \left( \frac{1000000}{47432} \right) = 0.0552846t \]

\[ \frac{1000000}{47432} = \frac{\ln(1000000)}{\ln(47432)} = t \]

\[ 55.14 = t \]

Which corresponds to the year 2015

d) \( T = \ln(2) \)

\[ 0.0552846 = 12.54 \text{ years} \]

e) As expected the answers in this exercise are close to those of Exercise 26.

29. \((0.85)\) and \((4, 109)\)

a) From the point \((0, 85)\) we get

\[ y = a \cdot b^t \]

\[ 85 = a \cdot b^0 \]

\[ 85 = a \]

From the point \((4, 109)\) we get

\[ \frac{109}{85} = b^4 \]

\[ \ln \left( \frac{109}{85} \right) = 4 \ln(b) \]

\[ \frac{109}{85} = \frac{\ln(109)}{\ln(85)} = 4 \ln(b) \]

\[ 0.062174 = \ln(b) \]

So, \( y = 85e^{0.062174t} \)
b) When \( t = 5 \)
\[
\begin{align*}
  y &= 85 e^{0.062174 \cdot 5} \\
  &= 116
\end{align*}
\]

c) When \( y = 1000000 \)
\[
\begin{align*}
  1000 & = 85 e^{0.062174x} \\
  185 & = e^{0.062174x} \\
  \ln \left( \frac{1000}{185} \right) & = 0.062174x \\
  \ln \left( \frac{1000}{185} \right) & = 0.062174 \\
  \ln (5.43) & = 0.062174 \\
  39.65 & = t
\end{align*}
\]

d) \( \frac{\ln(2)}{0.062174} = 11.15 \) years

e) As expected the answers in this exercise are close to those of Exercise 28.

31. a) Using a graphing calculator, \( P(t) = 11.88 \cdot 0.9996^t \)
b) \( \ln(0.9996) = -0.0004 \). So, \( P(t) = 11.88 e^{-0.0004t} \)

c) \[
\begin{align*}
  y = 0.4025 & = 11.88 e^{-0.0004t} \\
  11.88 & = \frac{0.4025}{e^{-0.0004t}} \\
  \ln(11.88) & = \ln \left( \frac{0.4025}{e^{-0.0004t}} \right) \\
  \ln(11.88) & = \ln(0.4025) - 0.0004t \\
  2.47 & = -0.0004t \\
  t & = -\frac{2.47}{0.0004} \\
  t & = 6175
\end{align*}
\]

d) \( P(t) = 11.88 - 0.0004 \cdot 0.9996^t = \frac{11.88 - 0.0004}{0.9996^t} \)

e) When \( x = 0 \), closer to sea level, the slope of the tangent line is \( -0.0005 \).

f) \( P(\text{Atmosphere}) = x = A(\text{Atmosphere}) \) since the exponential function and the logarithmic functions are inverses of each other.

33. \( G(x) = \frac{\ln(100)}{1 + 43.3 e^{-0.0425x}} \)

a) When \( x = 100 \)
\[
\begin{align*}
  G(100) &= \frac{\ln(100)}{1 + 43.3 e^{-0.0425(100)}} \\
  &= \frac{100}{1 + 43.3 e^{-4.25}} \\
  &= \frac{100}{1 + 43.3 e^{-4.25}} \\
  &= 0.0125
\end{align*}
\]

When \( x = 150 \)
\[
\begin{align*}
  G(150) &= \frac{\ln(100)}{1 + 43.3 e^{-0.0425(150)}} \\
  &= \frac{100}{1 + 43.3 e^{-6.375}} \\
  &= \frac{100}{1 + 43.3 e^{-6.375}} \\
  &= 0.0113
\end{align*}
\]

b) When \( x = 300 \)
\[
\begin{align*}
  G(300) &= \frac{\ln(100)}{1 + 43.3 e^{-0.0425(300)}} \\
  &= \frac{100}{1 + 43.3 e^{-12.75}} \\
  &= \frac{100}{1 + 43.3 e^{-12.75}} \\
  &= 0.0099
\end{align*}
\]

c) \( G''(x) = \frac{(-181.025 e^{-0.0425x}) \left[ 0.0425 - 1.84025 e^{-0.0425x} \right]}{(1 + 43.3 e^{-0.0425x})^3} \)

The possible inflection point occurs when \( G''(x) = 0 \) or undefined, which means the only possible inflection points occurs at
\[
\begin{align*}
  0.0425 - 1.84025 e^{-0.0425x} &= 0 \\
  e^{-0.0425x} &= \frac{0.0425}{1.84025} \\
  -0.0425x &= \ln \left( \frac{0.0425}{1.84025} \right) \\
  x &= \ln \left( \frac{0.0425}{1.84025} \right) \\
  x &\approx 5.86
\end{align*}
\]

35. \( A(t) = \frac{105}{1 + 329000 e^{-0.04t}} \)

a) When \( t = 100 \)
\[
\begin{align*}
  A(100) &= \frac{105}{1 + 329000 e^{-0.04(100)}} \\
  &= \frac{105}{1 + 329000 e^{-0.04(100)}} \\
  &= \frac{105}{1 + 329000 e^{-0.04(100)}} \\
  &= 0.171
\end{align*}
\]

When \( t = 150 \)
\[
\begin{align*}
  A(150) &= \frac{105}{1 + 329000 e^{-0.04(150)}} \\
  &= \frac{105}{1 + 329000 e^{-0.04(150)}} \\
  &= \frac{105}{1 + 329000 e^{-0.04(150)}} \\
  &= 0.272
\end{align*}
\]
When $t = 200$

$$A(200) = \frac{105}{1 + 32900 e^{-0.04(200)}} = \frac{105}{1 + 32900 e^{-8}} = 8.723$$

b)

$$A'(x) = 105 \times \frac{(-0.04) (1 + 32900 e^{-0.04t}) - 2 \times 32900 \times (-0.04) e^{-0.04t}}{(1 + 32900 e^{-0.04t})^2}$$

$$= \frac{138180 e^{-0.04t}}{(1 + 32900 e^{-0.04t})^2}$$

c)

$$A''(t) = \frac{e^{-0.04t} [1 + 32900 e^{-0.04t}] + 363689760}{(1 + 32900 e^{-0.04t})^3} \times [-5527.2 (1 + 32900 e^{-0.04t}) + 363689760]$$

The possible inflection point occurs when $A''(t) = 0$ or undefined, which means the possible inflection points occurs at

$$0 = -5527.2 (1 + 32900 e^{-0.04t}) + 363689760$$

$$1 + 32900 e^{-0.04t} = \frac{363689760}{5527.2}$$

$$t = -17.33$$

not an acceptable answer

or

$$1 + 32900 e^{-0.04t} = 0$$

$$e^{-0.04t} = \frac{1}{32900}$$

$$-0.04t = \ln \left( \frac{1}{32900} \right)$$

$$t = \frac{\ln (32900)}{-0.04} = 260.031$$

We find $A(260.031)$

$$A(260.031) = \frac{105}{1 + 32900 e^{-0.04(260.031)}} = \frac{105}{1 + 32900 e^{-10.40124}} = 52.5$$

37. $P(t) = 100\%(1 - e^{-0.4t})$

a) $P(0) = 100\%(1 - e^{-0.4(0)}) = 100\%(1 - 1) = 100\%(0) = 0\%$

$P(1) = 100\%(1 - e^{-0.4(1)}) = 100\%(1 - e^{-0.4}) \approx 33.0\%$

$P(2) = 100\%(1 - e^{-0.4(2)}) = 100\%(1 - e^{-0.8}) \approx 55.1\%$

$P(3) = 100\%(1 - e^{-0.4(3)}) = 100\%(1 - e^{-1.2}) \approx 61.3\%$

$P(5) = 100\%(1 - e^{-0.4(5)}) = 100\%(1 - e^{-2}) \approx 86.5\%$

$P(12) = 100\%(1 - e^{-0.4(12)}) = 100\%(1 - e^{-4.8}) \approx 99.2\%$

$P(18) = 100\%(1 - e^{-0.4(18)}) = 100\%(1 - e^{-6.4}) \approx 99.8\%$

b) $P'(t) = 100\%(-(-0.4) e^{-0.4t})$

$$P'(t) = 100\%(0.4) e^{-0.4t} - 0.4 e^{-0.4t} \quad (100\% = 1)$$

c) The derivative $P'(t)$ exists for all real numbers.

The equation $P'(t) = 0$ has no solution. Thus, the function has no critical points and hence no relative extrema. $P'(t) > 0$ for all real numbers, so $P(t)$ is increasing on $[0, \infty)$. $P''(t) < 0$, so $P''(t)$ is decreasing on $[0, \infty)$.
Exercise Set 4.4

39. \( i = 0.0754 - t = 0.0757 - 7.57\% \)

41. \( k = \ln(0.0924 + 1) - 0.0884 = 0.88\% \)

43. \[
\begin{align*}
3P_0 &= \frac{P_0e^{kt}}{3 - e^{kt}} \\
\ln(3) &= t
\end{align*}
\]

45. Answers vary

47. \( 2 = e^{2t} \)

\[
k = \frac{\ln(2)}{T} \approx 0.0289
\]

49. \( k = \frac{\ln(2)}{T} \) and \( T = \frac{\ln(2)}{k} \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>59.315</th>
<th>15</th>
<th>10</th>
<th>4.951</th>
</tr>
</thead>
</table>

The graph of \( T = \ln(2)/k \) is not linear since the independent variable appears in the denominator.

51. The rule of 69 is used to approximate the doubling time (generation time) of growing things. The name comes from the approximation \( \ln(2) \approx 0.6931 \) which is used to find the doubling time.

Exercise Set 4.4

1. (a)

3. (a)

5. (c)

7. \( k = \frac{\ln(2)}{T} = \frac{\ln(2)}{22} \approx 0.032 \approx 3.2\% \) per yr

9. \( k = \frac{\ln(2)}{T} = \frac{\ln(2)}{25} \approx 0.028 \approx 2.8\% \) per year

11. \( k = \frac{\ln(2)}{T} = \frac{\ln(2)}{23,106} \approx 0.00003 \approx 0.003\% \) per yr

13. \( P(t) - P_0 e^{-kt} \)

\[
P(20) = 1000 e^{-2.01(20)} \]

Substituting 100 for \( P_0 \)

\[
0.231 \text{ for } k \text{ and } 20 \text{ for } t
\]

\[
\approx 9.9
\]

Thus 9.9 grams of polonium will remain after 20 minutes.

15. \( N(t) - N_0 e^{-0.0001205t} \) See Example 3(b).

If a piece of wood has lost 90% of its carbon-14 from an initial amount \( P_0 \), then 10% \( P_0 \) is the amount present. To find the age of the wood, we solve the following equation for \( t \):

\[
10% \text{ } P_0 = P_0 e^{-0.0001205t}
\]

Substituting 10% \( P_0 \) for \( F(t) \)

\[
0.1 = e^{-0.0001205t}
\]

\[
\ln(0.1) = \ln e^{-0.0001205t}
\]

\[
-0.916291 = -0.0001205t
\]

\[
-0.916291 = -0.0001205t
\]

Thus, the piece of wood is about 19.109 years old.

17. If an artifact has lost 60% of its carbon-14 from an initial amount \( P_0 \), then 40% \( P_0 \) is the amount present. To find the age of the wood, we solve the following equation for \( t \):

\[
10% \text{ } P_0 = P_0 e^{-0.0001205t}
\]

Substituting 40% \( P_0 \) for \( P(t) \) (See Example 3(b))

\[
0.4 = e^{-0.0001205t}
\]

\[
\ln(0.4) = \ln e^{-0.0001205t}
\]

\[
-0.916291 = -0.0001205t
\]

\[
-0.916291 = -0.0001205t
\]

Thus, the piece of wood is about 7604 years old.

19. For carbon-14 the decay rate is 0.0001205.

The amount of carbon-14 that remains is

\[
N = N_0 e^{-0.0001205t} \]

\[
= 0.2997 N_0
\]

Amount lost is \( 100 - 29.97 = 70.03\% \)

21. \( k = \frac{\ln(1/2)}{1.3 \times 10^9} \)

\[
-5.332 \times 10^{-16}
\]

\[
0.0980N_0 = 0.0980 \times 5.332 \times 10^{-16}
\]

\[
0.0980 = e^{-5.332 \times 10^{-16}}
\]

\[
\ln(0.0980) = -5.332 \times 10^{-16}
\]

\[
t = \frac{\ln(0.0980)}{-5.332 \times 10^{-16}}
\]

\[
= 4.55 \times 10^9 \text{ years}
\]
23. 

\[ 0.653N_0 = N_0 e^{-0.533 \times 10^{-10} t} \]

\[ 0.653 = e^{-0.533 \times 10^{-10} t} \]

\[ \ln(0.653) = -5.332 \times 10^{-10} t \]

\[ t = \frac{\ln(0.653)}{-5.332 \times 10^{-10}} \approx 7.99 \times 10^8 \text{ years} \]

25. a) When A decomposes at a rate proportional to the amount of A present, we know that

\[ \frac{dA}{dt} = -kA. \]

The solution of this equation is \( A = A_0 e^{-kt} \).

b) We first find \( k \). The half-life of \( A \) is 3 hr.

\[ k = \frac{\ln 2}{3} \]

\[ k = \frac{0.693147}{3} \quad \text{Substituting 0.693147 for } \ln 2 \text{ and 3 for } T \]

\[ \approx 0.23 \text{, or } 23\% \]

We now substitute 8 for \( A_0 \), 1 for \( A \), and 0.23 for \( k \) and solve for \( t \).

\[ A = A_0 e^{-kt} \]

\[ 1 = 8 e^{-0.23t} \]

\[ \frac{1}{8} = e^{-0.23t} \]

\[ 0.125 = e^{-0.23t} \]

\[ \ln 0.125 = \ln e^{-0.23t} \]

\[ -2.079442 = -0.23t \]

\[ t = -\frac{2.079442}{-0.23} \approx 9 \]

After 9 hr there will be 1 gram left.

31. a) \( T(t) = ae^{-kt} + C \) Newton's law of Cooling

At \( t = 0 \), \( T = 100^\circ \). We solve the following equation for \( a \).

\[ 100 = ae^{-k \times 0} + 75 \]

\[ 25 = a \]

\[ a = 25 \]

The value of the constant is 25.

Thus, \( T(t) = 25 e^{-kt} + 75 \).

b) Now we find \( k \) using the fact that at \( t = 10 \), \( T = 90^\circ \).

\[ T(t) = 25 e^{-kt} + 75 \]

\[ 90 = 25 e^{-10k} + 75 \]

\[ 15 = 25 e^{-10k} \]

\[ \frac{15}{25} = e^{-10k} \]

\[ 0.6 = e^{-10k} \]

\[ \ln 0.6 = \ln e^{-10k} \]

\[ \ln 0.6 = -10k \]

\[ -10 = k \]

\[ -0.510826 = k \]

\[ 0.05 \approx k \]

Thus, \( T(t) = 25 e^{-0.05t} + 75 \).

c) \( T(20) = 25 e^{-0.05 \times 20} + 75 \)

\[ T(20) = 25 e^{-0.05 \times 20} + 75 \]

\[ = 25(0.367879) + 75 \]

\[ \approx 84.2^\circ \]

The temperature after 20 minutes is 84.2\(^\circ\).
35. a) Let 1995 correspond to \( t = 0 \)

\[
\begin{align*}
\exp(48.8) &= 51.9 e^{6k} \\
\ln \left( \frac{48.8}{51.9} \right) &= 6k \\
\ln \left( \frac{51.9}{6} \right) &= k \\
\frac{0.010265}{k} &= k \\
\text{Thus, } N &= 51.9 e^{-0.010265} \\
\end{align*}
\]

b) In 2015, \( t = 20 \)

\[
\begin{align*}
N &= 51.9 e^{-0.010265 \cdot 20} \\
&= 51.9 \cdot 0.8144 \approx 42.27
\end{align*}
\]

c) We have \( t = 38.6 - 0.010265 t \)

\[
\begin{align*}
\frac{1}{38.6} &= e^{-0.010265 t} \\
\ln \left( \frac{1}{38.6} \right) &= -0.010265 t \\
\frac{-0.010265}{t} &= t
\end{align*}
\]

The population of Ukraine will be 1 in the year 2380.

37. a) \( P(t) = 50 e^{-0.010265 t} \)

\[
\begin{align*}
P(375) &= 50 e^{-0.010265 \cdot 375} \\
&= 50 \cdot 0.2233 \approx 11
\end{align*}
\]

After 375 days, 11 watts will be available.

b) \( t = \frac{\ln 2}{k} \)

\[
\begin{align*}
\ln \left( \frac{0.693147}{2} \right) &= 0.004 \\
\ln 2 \text{ and 0.004 for } k \\
\approx 173
\end{align*}
\]

The half-life of the power supply is 173 days.

c) \( P(t) = 50 e^{-0.010265 t} \)

\[
\begin{align*}
P(t) &= 50 e^{-0.010265 t} \\
10 &= 50 e^{-0.010265 t} \\
\ln 0.2 &= \ln e^{-0.010265 t} \\
\ln 0.2 &= -0.010265 t \\
\ln 0.2 &= -0.004 t
\end{align*}
\]

\[
\begin{align*}
\frac{-1.609438}{-0.004} &= t \\
102 &\approx t
\end{align*}
\]

The satellite can stay in operation 102 days.
Chapter 4: Exponential and Logarithmic Functions

Exercise Set 4.5

1. \[5^\frac{1}{3} \cdot \cdot 4 \ln 5\]  
   Theorem 13: \[a^r = e^{r \ln a}\]  
   \[\approx e^{(1.0954189)}\] Using a calculator  
   \[\approx e^{12.229}\]

2. \[3.4^{10} = e^{10 \ln 3.4}\]  
   Theorem 13: \[a^r = e^{r \ln a}\]  
   \[\approx e^{(12.23777)}\] Using a calculator  
   \[\approx e^{12.28}\]

3. \[4^4 = e^{4 \ln 4}\]  
   Theorem 13: \[a^r = e^{r \ln a}\]  
   \[\approx e^{(12.23777)}\] Using a calculator  
   \[\approx e^{12.28}\]

4. \[8^{0.5} = e^{0.5 \ln 8}\]  
   Theorem 13: \[a^r = e^{r \ln a}\]  
   \[\approx e^{(3.32193)}\] Using a calculator  
   \[\approx e^{3.32}\]

5. \[f(x) = e^x\]  
   Theorem 14: \[\frac{dy}{dx} = (\ln a) a^r\]  
   \[\frac{dy}{dx} = e^x\]  
   Product Rule  
   \[= e^x (\ln 6.2)(6.2)^x + 1 \cdot (6.2)^x\]  
   \[= (6.2)^x (\ln 6.2 + 1)\]  
   \[\approx e^{(3.32193)}\] Using a calculator  
   \[\approx e^{3.32}\]

6. \[y = \ln x\]  
   Theorem 16: \[\frac{d}{dx} \ln a^x = \frac{1}{a} \cdot \frac{1}{x}\]  
   \[\frac{dy}{dx} = \ln \frac{10}{4} \cdot \frac{1}{x} + \frac{1}{x}\]  
   Product Rule  
   \[= \cdot \frac{1}{x}\]  
   \[= \ln 10 + 3\]

7. \[y = \log x\]  
   Theorem 16: \[\frac{d}{dx} \log a x = \frac{1}{\ln a} \cdot \frac{1}{x}\]  
   \[\frac{dy}{dx} = \frac{1}{\ln 3} \cdot \frac{1}{x}\]  
   \[= \ln 10 + 3\]

8. \[f(x) = 2 \log x\]  
   Theorem 16 (log x = log_{10} x)  
   \[f'(x) = 2 \cdot \frac{d}{dx} \log x\]  
   \[= 2 \cdot \frac{1}{\ln 10} \cdot \frac{1}{x}\]  
   \[= \ln 10 + 3\]

9. \[f(x) = \frac{x}{3}\]  
   f'(x) = \log x - \log 3  
   \(P2\)  
   \[f'(x) = \frac{1}{\ln 10} \cdot \frac{1}{x}\]  
   \[= \ln 10 + 3\]

10. \[f(x) = \frac{x}{3}\]  
    f'(x) = \log x - \log 3  
    \(P2\)  
    \[f'(x) = \frac{1}{\ln 10} \cdot \frac{1}{x}\]  
    \[= \ln 10 + 3\]
23. \( y = x^3 \log_8 x \)
\[
\frac{dy}{dx} = x^3 \left( \frac{1}{\ln 8} - \frac{1}{x} \right) + 3x^2 \cdot \log_8 x
\]
\[\text{Product Rule}\]
\[\text{Theorem 16}\]
\[-x^2 \cdot \frac{1}{\ln 8} + 3x^2 \cdot \log_8 x\]
\[x^2 \left( \frac{1}{\ln 8} - 3 \log_8 x \right)\]

25. \( y = \csc x \log_2 x \)
\[
\frac{dy}{dx} = \csc x \cdot \left( \frac{1}{\ln 2} - \frac{1}{x} \right) - \csc x \cdot \cot x \cdot \frac{\ln x}{\ln 2}
\]
\[\csc x \frac{1}{x} \ln x\]

27. \( y = \frac{\ln 10}{\sin x}\)
\[
\frac{dy}{dx} = \frac{1}{\sin x} - \cot x \cdot \frac{1}{x}
\]
\[\frac{1}{\ln 10} \cdot \cot x\]

29. \( y = \log_8 3 \)
\[
\frac{dy}{dx} = -x \ln 3 \cdot \frac{1}{x \ln 2}
\]
\[-x \ln 3 \cdot \frac{1}{\ln 2} \cdot \frac{1}{x}\]

31. \( g(x) = (\log_{10} 10)(\log_{10} x) \)
\[
g'(x) = \frac{x}{\ln 10} - \frac{1}{x \ln 10} + \log_{10} x \cdot \left( \frac{1}{\ln 10} - \frac{1}{x \ln 2} \cdot \frac{1}{x} \right)
\]
\[\frac{x}{\ln 10} \cdot \frac{1}{x \ln 10} - \frac{1}{x \ln 10} + \log_{10} x \cdot \left( \frac{1}{\ln 10} - \frac{1}{x \ln 2} \cdot \frac{1}{x} \right)\]

33. a) \( N(1) = 250,000 \left( \frac{1}{4} \right)^t \)
\[N'(1) = 250,000 \left( \frac{1}{4} \right)^t \cdot \ln \left( \frac{1}{4} \right) \cdot \left( \frac{1}{4} \right)^t
\]
\[250,000 \ln 4 \cdot \ln \left( \frac{1}{4} \right) \left( \frac{1}{4} \right)^t \] (P2)
\[-250,000 \ln 4 \left( \frac{1}{4} \right)^t \] (P1)

b) The rate of change of the number of cans in use, when 250,000 cans are initially distributed, is
\[250,000 \ln 4 \left( \frac{1}{4} \right)^t \] cans per year.

35. \( R = \log \frac{I}{I_0} \)
\[R = \log \frac{10^5 \cdot I_0}{I_0} \] Substituting \( 10^5 \cdot I_0 \) for \( I \)
\[\log 10^5 \]
\[5 \] (P15)

The magnitude on the Richter scale is 5.

37. a) \( I = I_0 \cdot 10^8 \)
\[I = I_0 \cdot 10^5 \] Substituting 7 for \( R \)
\[10^5 \cdot I_0 \]

b) \( I = I_0 \cdot 10^6 \)
\[I = I_0 \cdot 10^5 \] Substituting 8 for \( R \)
\[= 10^8 \cdot I_0 \]

c) The intensity in (b) is 8 times that in (a).
\[10^8 \cdot I_0 - 10 \cdot 10^7 \cdot I_0 \]

d) \[I = I_0 10^R \]
\[\frac{dI}{dR} = I_0 \cdot \frac{d}{dR} 10^R \]
\[I_0 \cdot \ln 10 \cdot 10^R \]
\[= (I_0 \cdot \ln 10) 10^R \] (P2)

The intensity is changing at a rate of \((I_0 \cdot \ln 10) 10^R\).

39. a) \( R = \frac{1}{I_0} \)
\[R = \log \frac{1}{I_0} \] (P2)
\[\frac{dR}{dI} = -\frac{1}{I_0} \cdot \frac{1}{I} = 0 \] (P16): \( I_0 \) is a constant.
\[\frac{1}{I_0} \cdot \frac{1}{I} \]

b) The magnitude is changing at a rate of \(\frac{1}{I_0} \cdot \frac{1}{I} \).

41. a) \( y = m \log x + b \)
\[\frac{dy}{dx} = m \cdot \frac{d}{dx} \log x = 0 \] \( m \) and \( b \) are constants
\[-m \left( \frac{1}{\ln 10} \cdot \frac{1}{x} \right) \] (P16)
\[-m \left( \frac{1}{\ln 10} \cdot \frac{1}{x} \right) \]

b) The response is changing at a rate of \(-m \left( \frac{1}{\ln 10} \cdot \frac{1}{x} \right) \).

43. a) When \( t \to 0 \) \([\text{OH}^-] = x \cdot 10^{-7} \) moles/liter

b) When \( t \to 0 \) \([H^+] = 10^{-11} \) moles/liter

c) When \( t \to 0 \) \(pH \) \(-\log (10^{-7}) = 7 \)

d) \[\text{log}[H^+] = \text{log}[OH^-] = 10^{-11} \]
\[\text{log}[H^+][OH^-] = \text{log} (10^{-11}) \]
\[\text{log}[H^+] \cdot \text{log}[OH^-] = 14 \]
\[\text{log}[H^+] + \text{log}(0.002 \cdot 10^{-7}) = 14 \]
\[\text{log}[H^+] + \text{log}(0.002 \cdot 10^{-7}) = 14 \]

Thus, \( pH = 14 + \log (0.002 \cdot 10^{-7}) \)

e) \[\frac{d}{dt} [\text{OH}^-] = -\frac{1}{10} (0.002 \cdot 10^{-7}) \]

f) The pH is changing most rapidly at \( t = 0 \) which corresponds to a pH of 7.
45. \[ y = e^{x^y}, \quad x > 0 \]

\[ \frac{dy}{dx} = e^{x^y} \ln x \]

Theorem 13: \( a^x = e^{x \ln a} \)

\[ = (1 + \ln x) x^y \quad \text{Substituting } x^y \text{ for } e^{x \ln a} \]

47. \( f(x) = x^{x^y}, \quad x > 0 \)

\[ f(x) = e^{x^{x^y}} \ln x \quad \text{Theorem 13: } a^x = e^{x \ln a} \]

\[ f'(x) = \left( e^{x^{x^y}} \ln x + e^{x^{x^y}} \right) e^{x^{x^y}} \ln x \]

\[ = e^{x^{x^y}} \left( \ln x \cdot \frac{1}{x} \right) x^{x^y} \quad \text{Substituting } x^{x^y} \text{ for } e^{x \ln a} \]

\[ = e^{x^{x^y}} \ln x \cdot \frac{1}{x} \]

49. \( y = \log_a f(x), \quad f(x) > 0 \)

\[ a^y = f(x) \quad \text{Exponential equation} \]

\[ e^{y \ln a} = f(x) \quad \text{Theorem 13} \]

Differential implicitly to find \( dy/dx \).

\[ \frac{d}{dx} e^{y \ln a} = \frac{d}{dx} f(x) \]

\[ \frac{dy}{dx} \cdot \ln a \cdot e^{y \ln a} = f'(x) \]

\[ \frac{dy}{dx} \cdot \ln a \cdot f(x) = f'(x) \quad \text{Substituting } f(x) \text{ for } e^{y \ln a} \]

\[ \frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{f'(x)}{f(x)} \]

51. Since \( a^x \) can be written as \( e^{x \ln a} \), we can find the derivative of \( f(x) = a^x \) using the rule for differentiating an exponential function, base \( e \).
Chapter 5
Integration

Exercise Set 5.1

1. \[ \int x^5 \, dx = \frac{x^{5+1}}{6+1} + C = \frac{x^6}{7} + C \]

2. \[ \int 2x \, dx = 2x + C \quad \text{(For } k \text{ a constant, } \int k \, dx = kx + C) \]

3. \[ \int x^{1/2} \, dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C \]

4. \[ \int x^{3/2} \, dx = \frac{x^{3/2+1}}{3/2+1} + C = \frac{x^{5/2}}{5/2} + C \]

5. \[ \int (x^2 + x - 1) \, dx = \int x^2 \, dx + \int x \, dx - \int 1 \, dx = \frac{x^3}{3} + \frac{x^2}{2} - x + C \]

6. \[ \int (t^2 - 2t + 3) \, dt = \int t^2 \, dt - 2 \int t \, dt + \int 3 \, dt = \frac{t^3}{3} - t^2 + 3t + C \]

7. \[ \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C \]

8. \[ \int 5e^{2x} \, dx = \frac{5}{2} e^{2x} + C \]

9. \[ \int \frac{\sqrt{x^2 - 1}}{x} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C \]

10. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

11. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

12. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

13. \[ \int (a^x - e^{x/2}) \, dx = \int a^x \, dx - \int e^{x/2} \, dx \]

14. \[ \int (\sqrt{x} - x) \, dx = \int \sqrt{x} \, dx - \int x \, dx = \frac{2}{3} x^{3/2} - \frac{x^2}{2} + C \]

15. \[ \int \frac{1000}{x} \, dx = 1000 \ln |x| + C \]

16. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

17. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

18. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]

19. \[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]
21. \[ \int -\frac{6}{\sqrt{x^4 + x^2}} \, dx = \int -\frac{6}{x^{2/3} + 1} \, dx = -6 \int x^{-2/3} \, dx = -6 \cdot \frac{x^{1/3}}{1} + C = -6 \cdot \frac{3}{3} \cdot x^{1/3} + C = -6 \cdot 3x^{1/3} + C \]

22. \[ \int 8e^{-2x} \, dx = \frac{8}{-2} e^{-2x} + C = -4 e^{-2x} + C \]

25. \[ \int \left( x^2 - \frac{3}{2} \sqrt{x + x^{-1/3}} \right) \, dx = \int x^2 \, dx - \frac{3}{2} \int x^{1/2} \, dx + \frac{3}{2} \int x^{-1/3} \, dx = \frac{x^{3+1}}{3+1} - \frac{3}{2} \cdot \frac{x^{3/2}}{3/2} + \frac{3}{2} \cdot \frac{x^{-1/3+1}}{1 - 1} + C = \frac{x^3}{3} - \frac{3}{2} \cdot \frac{3}{3} x^{3/2} + \frac{3}{3} \cdot \frac{3}{2} x^{-1/3} + C = \frac{x^3}{3} - 3x^{1/2} + C \]

27. \[ \int 5 \sin 20 \, d\theta = \frac{-5}{2} \cos 20 + C \]

29. \[ \int \left( 5 \sin 5x - 4 \cos 2x \right) \, dx = -5 \cos 5x - 2 \sin 2x + C \]

31. \[ \int 3 \sec^2 3x \, dx = \tan 3x + C \]

33. \[ \int \frac{1}{3} \sec \frac{x}{9} \tan \frac{x}{9} \, dx = \frac{1}{3} \sec \frac{x}{9} + C \]

35. \[ \int \left( \sec x + \tan x \right) \sec x \, dx = \int \left( \sec^2 x + \sec x \tan x \right) \, dx = \tan x + \sec x + C \]

37. \[ \int \left( \frac{1}{t} + \frac{1}{t^2} - \frac{1}{e^t} \right) \, dt = \ln t - \frac{1}{t} + \frac{1}{t} + C \]

39. Find the function \( f \) such that \( f'(x) = x - 3 \) and \( f(2) = 0 \).

We first find \( f(x) \) by integrating,

\[ f(x) = \int (x - 3) \, dx = \frac{x^2}{2} - 3x + C \]

The condition \( f(2) = 9 \) allows us to find \( C \).

\[ f(x) = \frac{x^2}{2} - 3x + C \quad \text{and} \quad f(2) = \frac{2^2}{2} - 3 \cdot 2 + C = 9 \]

Substituting 2 for \( x \) and 9 for \( f(2) \)

\[ 2 - 6 + C = 9 \]

\[ C = 13 \]

Thus, \( f(x) = \frac{x^2}{2} - 3x + 13 \).

41. Find the function \( f \) such that \( f'(x) = x^3 - 4 \) and \( f(0) = -7 \).

We first find \( f(x) \) by integrating,

\[ f(x) = \int (x^3 - 4) \, dx = \frac{x^4}{4} - 4x + C \]

The condition \( f(0) = 7 \) allows us to find \( C \).

\[ f(0) = 0 - 4 \cdot 0 + C = 7 \]

Solving for \( C \) we get \( C = 7 \).

Thus, \( f(x) = \frac{x^4}{4} - 4x + 7 \).

43. \[ f(x) = \int 2 \cos 3x \, dx = \frac{2}{3} \sin 3x + C \]

\[ f(0) = \frac{2}{3} \sin 0 + C = 1 \]

\[ C = 1 \]

Thus, \( f(x) = \frac{2}{3} \sin 3x + 1 \).

45. \[ f(x) = \int 5 e^{2x} \, dx = \frac{5}{2} e^{2x} + C \]

\[ f(0) = 5 \cdot 2^0 + C = 10 \]

\[ C = -25 \]

Thus, \( f(x) = \frac{5}{2} e^{2x} - \frac{25}{2} \).

47. \[ f(t) = \int 157t + 1000 \, dt = \frac{157}{2} t^2 + 1000t + C \]

\[ f(0) = 0 + 0 + C \]

\[ 156239 = C \]

\[ f(t) = \frac{157}{2} t^2 + 1000t + 156239 \]

\[ f(2) = \frac{157}{2} (2)^2 + 1000(2) + 156239 \]

\[ = 158553 \]
49. \( v(t) = 3t^2 \), \( s(0) = 4 \)

We find \( s(t) \) by integrating \( v(t) \).

\[
s(t) = \int v(t) \, dt = \int 3t^2 \, dt = 3 \frac{t^3}{3} + C = t^3 + C
\]

The condition \( s(0) = 4 \) allows us to find \( C \).

\[
s(0) = 0^3 + C = 4 \quad \Rightarrow \quad C = 4
\]

Solving for \( C \), we get \( C = 4 \).

Thus, \( s(t) = t^3 + 4 \).

51. \( a(t) = 4t \), \( v(0) = 20 \)

We find \( v(t) \) by integrating \( a(t) \).

\[
v(t) = \int a(t) \, dt = \int 4t \, dt = \frac{4t^2}{2} + C = 2t^2 + C
\]

The condition \( v(0) = 20 \) allows us to find \( C \).

\[
v(0) = 0^2 + C = 20 \quad \Rightarrow \quad C = 20
\]

Solving for \( C \), we get \( C = 20 \).

Thus, \( v(t) = 2t^2 + 20 \).

53. \( a(t) = 6t \), \( v(0) = 6 \) and \( s(0) = 10 \)

We find \( v(t) \) by integrating \( a(t) \).

\[
v(t) = \int a(t) \, dt = \int 6t \, dt = \frac{6t^2}{2} + C = 3t^2 + C
\]

The condition \( v(0) = 6 \) allows us to find \( C_1 \).

\[
v(0) = 0^2 + C_1 = 6 \quad \Rightarrow \quad C_1 = 6
\]

Solving for \( C_1 \), we get \( C_1 = 6 \).

Thus, \( v(t) = 3t^2 + 6t + 6 \).

We find \( s(t) \) by integrating \( v(t) \).

\[
s(t) = \int v(t) \, dt = \int (3t^2 + 6t + 6) \, dt = \frac{3t^3}{3} + 6 \frac{t^2}{2} + 6t + C_2
\]

The condition \( s(0) = 10 \) allows us to find \( C_2 \).

\[
s(0) = \frac{3(0)^3}{3} + 6 \frac{(0)^2}{2} + 6(0) + C_2 = 10 \quad \Rightarrow \quad C_2 = 10
\]

Solving for \( C_2 \), we get \( C_2 = 10 \).

Thus, \( s(t) = \frac{3t^3}{3} + 3t^2 + 6t + 10 \).

55. \( a(t) = -32 \text{ ft/sec}^2 \), \( s(t) = \) initial velocity \( - v_0 \), \( s(0) = \) initial height \( = s_0 \)

We find \( v(t) \) by integrating \( a(t) \).

\[
v(t) = \int a(t) \, dt = \int (-32) \, dt = -32t + C_1
\]

The condition \( v(0) = v_0 \) allows us to find \( C_1 \).

\[
v(0) = -32 \cdot 0 + C_1 = v_0 \quad \Rightarrow \quad C_1 = v_0
\]

Thus, \( v(t) = -32t + v_0 \).

We find \( s(t) \) by integrating \( v(t) \).

\[
s(t) = \int v(t) \, dt = \int (-32t + v_0) \, dt = -16t^2 + v_0t + C_2
\]

The condition \( s(0) = s_0 \) allows us to find \( C_2 \).

\[
s(0) = -16 \cdot 0^2 + v_0 \cdot 0 + C_2 = s_0 \quad \Rightarrow \quad C_2 = s_0
\]

Thus, \( s(t) = -16t^2 + v_0t + s_0 \).

57. \( a(t) = k \), \( s(0) = 0 \)

Constant acceleration

\[
v(t) = \int a(t) \, dt = \int k \, dt = kt \quad \Rightarrow \quad v(0) = 0; \text{ thus } C = 0
\]

\[
s(t) = \int v(t) \, dt = \int kt \, dt = \frac{k^2 t^2}{2} + \frac{1}{2} kt^2
\]

We know that

\[
a(t) = k \quad \Rightarrow \quad \frac{60 \text{ mph}}{2 \text{ min}}
\]

and that

\[
\frac{t}{2} \text{ min.}
\]

Thus

\[
s(t) = \frac{k^2 t^2}{2}
\]

\[
s\left(\frac{1}{2} \text{ min}\right) = \frac{k^2}{2} \cdot \left(\frac{60 \text{ mph}}{2 \text{ min}}\right) \cdot \left(\frac{1}{2} \text{ min}\right)^2
\]

\[
= \frac{k^2}{2} \cdot \frac{60 \text{ mph}}{2 \text{ min}} \cdot \frac{1}{120} \text{ hr}
\]

\[
= \frac{k^2}{2} \cdot \frac{60 \text{ mi}}{2 \text{ min}} \cdot \frac{1}{4} \text{ mi}
\]

The car travels \( \frac{1}{4} \text{ mi during that time.} \)
59. \( a(t) = -68.5 \)

\[ v(t) = \int a(t) \, dt = \int -68.5 \, dt = -68.5t + C \]

\( v(0) = 132 \)

\[ 132 = 0 + C \Rightarrow C = 132 \]

\[ s(t) = \int v(t) \, dt = \int (-68.5t + 132) \, dt = -34.25t^2 + 132t + C \]

\( s(0) = 0; \text{thus} \ C = 0 \)

\[ s(t) = -34.25t^2 + 132t \]

The time it takes to go from 132 ft/sec to 0 is

\[ -68.5t + 132 = 0 \]

\[ 68.5t = 132 \]

\[ t = \frac{132}{68.5} \approx 1.927 \]

Thus

\[ s(1.927) = -34.25(1.927)^2 + 132(1.927) \]

\[ = 127.182 \]

The car travels almost 127.2 feet during that time.

61. \( a(t) = 0.2t - 0.003t^2 \)

a) We integrate to find \( M(t) \).

\[ M(t) = \int (0.2t - 0.003t^2) \, dt = 0.1t^2 - 0.001t^3 + C \]

We use \( M(0) = 0 \) to find \( C \).

\[ M(0) = 0.1(0)^2 - 0.001(0)^3 + C = 0 \Rightarrow C = 0 \]

\[ M(t) = 0.1t^2 - 0.001t^3 \]

b) \( M(8) = 0.1(8)^2 - 0.001(8)^3 \)

\[ = 64 - 512 \]

\[ = 5.888 \]

\[ \approx 6 \text{ words} \]

63. \( N(t) = 38.2e^{0.0076t} \)

a)

\[ N(t) = \int 38.2e^{0.0076t} \, dt \]

\[ = 1015.957e^{0.0076t} + C \]

\[ 1622 = 1015.957 + C \]

\[ 606.043 \]

\[ N(t) = 1015.957e^{0.0076t} + 606.043 \]

b)

\[ N(30) = 1015.957e^{0.0076(30)} + 606.043 \]

\[ = 3742 \]

65. \( f'(t) = t^{2/3} \)

\[ f(t) = \int t^{2/3} \, dt \]

\[ = \frac{3}{5}t^{5/3} + C \]

\[ s = 0 + C \]

\[ s = C \]

\[ f(t) = \frac{3}{5}t^{5/3} + 8 \]

67.

\[ \int (x - 1)^2 x^3 \, dx = \int (x^5 - 2x^4 + x^3) \, dx \]

\[ = \frac{x^6}{6} - \frac{2}{5}x^5 + \frac{x^4}{4} + C \]

69.

\[ \int \left( \frac{1}{3} \right)^2 \frac{1}{\sqrt{t}} \, dt = \int \left( \frac{1}{3} \right)^2 + 6\left( \frac{1}{2} \right)^2 + 9\left( \frac{1}{2} \right)^2 \, dt \]

\[ = \frac{2}{5}\left( \frac{1}{3} \right)^2 + 4\left( \frac{1}{2} \right)^2 + 18\left( \frac{1}{2} \right)^2 + C \]

71.

\[ \int (t + 1)^3 \, dt = \int (t^3 + 3t^2 + 3t + 1) \, dt \]

\[ = \frac{t^4}{4} + t^3 + \frac{3}{2}t^2 + t + C \]

73.

\[ \int be^{at} \, dx = b \int e^{at} \, dx \]

\[ = \frac{b}{a} e^{at} + C \]

75.

\[ \int \sqrt[4]{64x^3} \, dx = \int 4x^{3/4} \, dx \]

\[ = \frac{4}{7}x^{7/4} + C \]

\[ = \frac{12}{7}x^{7/4} + C \]

77.

\[ \int \frac{t^3 + 8}{t + 2} \, dt = \int \frac{(t^2 - 2t + 4)}{(t + 2)} \, dt \]

\[ = \int \frac{(t^2 - 2t + 4)}{(t + 2)} \, dt \]

\[ = \frac{t^3}{3} - t^2 + 4t + C \]
79. 
\[ \int (\cos^2 x + \cos x \sin^2 x) \, dx = \int [\cos x (\cos^2 x + \sin^2 x)] \, dx = \int \cos x \, dx = \sin x + C \]

81. 
\[ \int \csc^2 x \, dx = -\int (\csc^2 x - 1) \, dx = -\csc 2x \cdot \frac{1}{2} - x + C \]

83. Answers could vary. The antiderivative of a function represents the area under the curve of that function.

---

**Exercise Set 5.2**

1. a) \( f(x) = \frac{1}{x^2} \)

   In the drawing in the text the interval \([1, 7]\) has been divided into 6 subintervals, each having width \(1 \left( \frac{7 - 1}{6} = 1 \right) \).

   The heights of the rectangles shown are

   - \( f(1) = \frac{1}{1^2} = 1 \)
   - \( f(2) = \frac{1}{2^2} = \frac{1}{4} \approx 0.2500 \)
   - \( f(3) = \frac{1}{3^2} = \frac{1}{9} \approx 0.1111 \)
   - \( f(4) = \frac{1}{4^2} = \frac{1}{16} \approx 0.0625 \)
   - \( f(5) = \frac{1}{5^2} = \frac{1}{25} \approx 0.0400 \)
   - \( f(6) = \frac{1}{6^2} = \frac{1}{36} \approx 0.0278 \)

   The area of the region under the curve over \([1, 7]\) is approximately the sum of the areas of the 6 rectangles.

   Area of each rectangle:

   - 1st rectangle: \( 1 \cdot 1 \cdot 1 = 1 \)
   - 2nd rectangle: \( 0.2500 \cdot 1 = 0.2500 \)
   - 3rd rectangle: \( 0.1111 \cdot 1 = 0.1111 \)
   - 4th rectangle: \( 0.0625 \cdot 1 = 0.0625 \)
   - 5th rectangle: \( 0.0400 \cdot 1 = 0.0400 \)
   - 6th rectangle: \( 0.0278 \cdot 1 = 0.0278 \)

   The total area is \( 1 + 0.2500 + 0.1111 + 0.0625 + 0.0400 + 0.0278 = 1.4714 \).

   \( f(x) = \frac{1}{x^2} \)

   The interval \([1, 7]\) has been divided into 12 subintervals, each having width \(0.5 \left( \frac{7 - 1}{12} = \frac{6}{12} = 0.5 \right) \).

   The heights of six of the rectangles were computed in part (a). The others are computed below:

   - \( f(1.5) = \frac{1}{(1.5)^2} \approx 0.4444 \)
   - \( f(2.5) = \frac{1}{(2.5)^2} \approx 0.1600 \)
   - \( f(3.5) = \frac{1}{(3.5)^2} \approx 0.0588 \)
   - \( f(4.5) = \frac{1}{(4.5)^2} \approx 0.0294 \)
   - \( f(5.5) = \frac{1}{(5.5)^2} \approx 0.0183 \)
   - \( f(6.5) = \frac{1}{(6.5)^2} \approx 0.0103 \)

   The area of the region under the curve over \([1, 7]\) is approximately the sum of the areas of the 12 rectangles.

   Area of each rectangle:

   - 1st rectangle: \( 1 \cdot 0.5 \cdot 0.5 \)
   - 2nd rectangle: \( 0.4444 \cdot 0.5 \cdot 0.5 \approx 0.2222 \)
   - 3rd rectangle: \( f(1.5) \approx 0.4444 \) and \( \Delta x \cdot 0.5 \)
   - 4th rectangle: \( 0.1600 \cdot 0.5 \cdot 0.5 \approx 0.0800 \)
   - 5th rectangle: \( 0.0588 \cdot 0.5 \cdot 0.5 \approx 0.0294 \)
   - 6th rectangle: \( 0.0294 \cdot 0.5 \cdot 0.5 \approx 0.0147 \)
   - 7th rectangle: \( 0.0183 \cdot 0.5 \cdot 0.5 \approx 0.0092 \)
   - 8th rectangle: \( 0.0183 \cdot 0.5 \cdot 0.5 \approx 0.0092 \)
   - 9th rectangle: \( 0.0147 \cdot 0.5 \cdot 0.5 \approx 0.0066 \)
   - 10th rectangle: \( 0.0066 \cdot 0.5 \cdot 0.5 \approx 0.0033 \)
   - 11th rectangle: \( 0.0033 \cdot 0.5 \cdot 0.5 \approx 0.0016 \)
   - 12th rectangle: \( 0.0016 \cdot 0.5 \cdot 0.5 \approx 0.0008 \)

   The total area is \( 0.5 + 0.2222 + 0.1250 + 0.0800 + 0.0588 + 0.0408 + 0.0313 + 0.0217 + 0.0200 + 0.0166 + 0.0130 + 0.0119 + 0.1429 \). (Answers may vary slightly depending on when rounding was done.)

3. The shaded region represents an antiderivative. It also represents velocity, the antiderivative of acceleration.

5. The shaded region represents an antiderivative. It also represents total energy used in time \( t \).

7. The shaded region represents an antiderivative. It also represents the amount of the drug in the blood.

9. The shaded region represents an antiderivative. It also represents the number of words memorized in time \( t \).
11. \( \Delta x = \frac{2 - 0}{4} = \frac{1}{2} \)

\[
\int_0^3 x^2 \, dx = f(1/2)(1/2) + f(1)(1/2) + f(3/2)(1/2) + \\
f(2)(1/2) = (1/4)(1/2) + (1)(1/2) + (9/4)(1/2) + (4)(1/2) = 1/8 + 1/2 + 9/8 + 2 = 3.75
\]

13. \( \Delta x = \frac{2 - 0}{6} = \frac{1}{3} \)

\[
\int_0^2 x \, dx = f(25/6)(1/6) + f(25/6)(1/6) + f(27/6)(1/6) + \\
f(28/6)(1/6) + f(29/6)(1/6) + f(5)(1/6) = (25/6)(1/6) + (26/6)(1/6) + (27/6)(1/6) + \\
\]

15. \( \Delta x = \frac{2 - 0}{4} = \frac{1}{2} \)

\[
\int_0^\pi \sin x \, dx = \sin(\pi/4)(\pi/4) + \sin(2\pi/4)(\pi/4) + \\
\sin(3\pi/4)(\pi/4) + \sin(\pi)(\pi/4) = (1)(\pi/4) + (1)(\pi/4) + (1)(\pi/4) = 0 = 1.89612
\]

17. a) \( \int_0^1 f(x) \, dx = 0 \), because there is the same area above the \( x \)-axis as below. That is, the area is \( A - A \), or 0.

b) \( \int_0^1 f(x) \, dx < 0 \), because there is more area below the \( x \)-axis than above. The area is \( A - 2A \), or \( -A \).

19. \( P^\prime(t) = 200e^{-t} \)

\[
\Delta t = \frac{2 - 0}{6} - \frac{1}{3} = \frac{1}{3} \]

\[
\int_0^2 P^\prime(t) \, dt = P^\prime(1/3)(1/3) + P^\prime(2/3)(1/3) + P^\prime(1/3)(1/3) + \\
P^\prime(3/3)(1/3) + P^\prime(4/3)(1/3) + P^\prime(2/3)(1/3) = 143.31(1/3) + 102.68(1/3) + 73.58(1/3) + \\
52.72(1/3) + 37.78(1/3) + 27.07(1/3) = 145.71333 \approx 146
\]

21. \( P^\prime(t) = -500(20 - t) \)

\[
\Delta t = \frac{20 - 0}{3} = \frac{20}{3} \]

\[
\int_0^2 P^\prime(t) \, dt = P^\prime(4)(4) + P^\prime(8)(4) + P^\prime(12)(4) + \\
P^\prime(16)(4) + P^\prime(20)(4) = (-8000)(4) + (-6000)(4) + \\
(-4000)(4) + (-2000)(4) = -80000 \]

23. \( v(t) = 3t^2 + 2t \)

\[
\Delta t = \frac{5 - 1}{4} = 1 \]

\[
\int_1^5 v(t) \, dt = v(1)(1) + v(2)(1) + v(3)(1) + v(4)(1) = 5 + 16 + 33 + 56 = 119
\]

25. \( f(x) = x \). Since we are integrating over \([0, 2]\) the length of the subintervals is given by

\[
\Delta x = \frac{2 - 0}{n} = \frac{2}{n}
\]

Now, we need an expression for \( x_i \):

\[
x_0 = 0 \\
x_1 = 0 + 2/n(1) = 2/n \\
x_2 = 2/n + 2/n - 2/n = 2(2/n) \\
x_3 = 2(2/n) + 2/n = 3(2/n)
\]

In general, we can write \( x_i = i \cdot \left( \frac{2}{n} \right) \)

\[
\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f(i \cdot \frac{2}{n}) \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2}{n} \]

\[
= \sum_{i=1}^{n} \frac{2i}{n^2} = \frac{2n(n+1)}{2n} = \frac{2n^2}{n} = \frac{2n}{n} = 2
\]

27. \( f(x) = 3x^2 \). Since we are integrating over \([0, 1]\) the length of the subintervals is given by

\[
\Delta x = \frac{1 - 0}{n} = \frac{1}{n}
\]

Now, we need an expression for \( x_i \):

\[
x_0 = 0 \\
x_1 = 0 + 1/n(1) = 1/n \\
x_2 = 1/n + 1/n - 1/n = 2(1/n) \\
x_3 = 2(1/n) + 1/n = 3(1/n)
\]

In general, we can write \( x_i = i \cdot \left( \frac{1}{n} \right) \)

\[
\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f(i \cdot \frac{1}{n}) \cdot \frac{1}{n} \]
\[ f(x) = x^3. \text{ Since we are integrating over } [0, 4] \text{ the length of the subintervals is given by} \]
\[ \Delta x = \frac{4 - 0}{n} = \frac{4}{n} \]

Now, we need an expression for \( x_i \):
\[ x_0 = 0 \]
\[ x_1 = 0 + \frac{4}{n} = \frac{4}{n} \]
\[ x_2 = 4/n + 4/n = 8/n \]
\[ x_3 = 2(4/n) + 4/n = 3(4/n) \]

In general, we can write \( x_i = i(4/n) \)

\[ \sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f(i \cdot \frac{4}{n}) \Delta x \]
\[ = \sum_{i=1}^{n} i^3 \cdot \frac{4}{n} \]
\[ = \frac{256}{n^3} \sum_{i=1}^{n} i^3 \]
\[ = \frac{256}{n^3} \frac{n^2(n+1)^2}{4} \]
\[ = \frac{64(n^2 + 2n^2 + n^2)}{n^3} \]

\[ \int_{0}^{4} x^3 dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]
\[ = \lim_{n \to \infty} \frac{64(n^2 + 2n^2 + n^2)}{n^3} \]
\[ = \lim_{n \to \infty} 64 \cdot \frac{n^2}{n^3} + \frac{64}{n^2} \]
\[ = 64 \]

31. \( f(x) = x^2. \) Since we are integrating over \([1, 3]\) the length of the subintervals is given by
\[ \Delta x = \frac{3 - 1}{n} = \frac{2}{n} \]

Now, we need an expression for \( x_i \):
\[ x_0 = 1 \]
\[ x_1 = 1 + \frac{2}{n} \]
\[ x_2 = 1 + \frac{2}{n} + \frac{2}{n} = 1 + \frac{4}{n} \]
\[ x_3 = 1 + \frac{4}{n} + \frac{2}{n} = 1 + \frac{6}{n} \]

In general, we can write \( x_i = 1 + \frac{2i}{n} \)

\[ \sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f(1 + \frac{2i}{n}) \frac{2}{n} \]
\[ = \sum_{i=1}^{n} (1 + \frac{2i}{n})^2 \frac{2}{n} \]
\[ = \sum_{i=1}^{n} \left[ \frac{2i}{n} + \frac{i^2}{n^2} \right] \]
\[ \frac{2}{n} \left[ \sum_{i=1}^{n} i + \sum_{i=1}^{n} \frac{i^2}{n^2} \right] \]
\[ \frac{2}{n} \left[ \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} \right] \]
\[ = 2 + \frac{8}{n^2} \left[ \frac{4}{n^3} \right] \]
\[ = \frac{2}{n^2} + \frac{8}{n^3} \]

33. \( \int_{0}^{2} (x + x^2) dx = \int_{0}^{2} x dx + \int_{0}^{2} x^2 dx \)

The first integral was evaluated in Exercise 25, and yielded a value of 2. The second integral is now computed: \( \Delta x = \frac{2 - 0}{n} = \frac{2}{n} \)
\[ x_0 = 0 \]
\[ x_1 = 0 + \frac{2}{n} = \frac{2}{n} \]
\[ x_2 = \frac{2}{n} + \frac{2}{n} = \frac{4}{n} \]
\[ x_3 = \frac{4}{n} + \frac{2}{n} = \frac{6}{n} \]

In general, we can write \( x_i = \frac{2i}{n} \)

\[ \sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f(i \cdot \frac{2}{n}) \frac{2}{n} \]
\[ = \sum_{i=1}^{n} \left[ \frac{2i^2}{n^2} \right] \frac{2}{n} \]
\[ = \sum_{i=1}^{n} i^2 \cdot \frac{8}{n^5} \]
\[ = \frac{8}{n^3} \sum_{i=1}^{n} i^2 \]
\[ = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \]
\[ = \frac{4}{3} \left( \frac{n^3}{n^3} + \frac{3n^2}{n^3} + \frac{n}{n^3} \right) \]
\[ = \frac{4}{3} \left( \frac{2}{n^2} + \frac{3}{n} + \frac{1}{n^2} \right) \]
\[ = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \]
\[ \int_{0}^{1} x^2 \, dx = \lim_{n \to \infty} \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \]
\[ = \frac{8}{3} \]

Thus,
\[ \int_{0}^{1} (x + x^2) \, dx = 2 + \frac{8}{3} = \frac{14}{3} \]

35. - 41. Left to the student

43. \[ \int_{0}^{\pi} \sin x \, dx = 2 \]

45. \[ \int_{0}^{1} \sqrt{x} \, dx = 5.33333 \]

47. \[ \int_{0}^{1} \ln(x) \, dx = 2.15888 \]

**Exercise Set 5.3**

1. Find the area under the curve \( y = 4 \) on the interval \([1, 3]\).
   \[ A(x) = \int A \, dx \]
   \[ = 4x + C \]
   Since we know that \( A(1) = 0 \) (there is no area above the number 1), we can substitute for \( x \) and \( A(x) \) to determine \( C \).
   \[ A(1) = 4 \cdot 1 + C = 0 \quad \text{Substituting 1 for } x \text{ and 0 for } A(1) \]
   Solving for \( C \) we get:
   \[ 4 + C = 0 \]
   \[ C = -4 \]
   Thus, \( A(x) = 4x - 4 \).
   Then the area on the interval \([1, 3]\) is \( A(3) \).
   \[ A(3) = 4 \cdot 3 - 4 \quad \text{Substituting 3 for } x \]
   \[ = 12 - 4 \]
   \[ = 8 \]

3. Find the area under the curve \( y = 2x \) on the interval \([1, 3]\).
   \[ A(x) = \int 2x \, dx \]
   \[ = x^2 + C \]
   Since we know that \( A(1) = 0 \) (there is no area above the number 1), we can substitute for \( x \) and \( A(x) \) to determine \( C \).
   \[ A(1) = 1^2 + C = 0 \quad \text{Substituting 1 for } x \text{ and 0 for } A(1) \]
   Solving for \( C \) we get:
   \[ 1 + C = 0 \]
   \[ C = -1 \]
   Thus, \( A(x) = x^2 - 1 \).
   Then the area on the interval \([1, 3]\) is \( A(3) \).
   \[ A(3) = 3^2 - 1 \quad \text{Substituting 3 for } x \]
   \[ = 9 - 1 \]
   \[ = 8 \]

5. Find the area under the curve \( y = x^2 \) on the interval \([0, 5]\).
   \[ A(x) = \int x^2 \, dx \]
\[ = \frac{x^3}{3} + C \]
Since we know that \( A(0) = 0 \) (there is no area above the number 0), we can substitute for \( x \) and \( A(x) \) to determine \( C \).
   \[ A(0) = \frac{0^3}{3} + C = 0 \quad \text{Substituting 0 for } x \text{ and 0 for } A(0) \]
   Solving for \( C \), we get \( C = 0 \).
   Thus, \( A(x) = \frac{x^3}{3} \).
   Then the area on the interval \([0, 5]\) is \( A(5) \).
   \[ A(5) = \frac{5^3}{3} \quad \text{Substituting 5 for } x \]
\[ = 83.3333 \]

7. Find the area under the curve \( y = x^3 \) on the interval \([0, 1]\).
   \[ A(x) = \int x^3 \, dx \]
\[ = \frac{x^4}{4} + C \]
Since we know that \( A(0) = 0 \) (there is no area above the number 0), we can substitute for \( x \) and \( A(x) \) to determine \( C \).
   \[ A(0) = \frac{0^4}{4} + C = 0 \quad \text{Substituting 0 for } x \text{ and 0 for } A(0) \]
   Solving for \( C \), we get \( C = 0 \).
   Thus, \( A(x) = \frac{x^4}{4} \).
   Then the area on the interval \([0, 1]\) is \( A(1) \).
   \[ A(1) = \frac{1^4}{4} \quad \text{Substituting 1 for } x \]
\[ = \frac{1}{4} \]

9. Find the area under the curve \( y = 4 - x^2 \) on the interval \([-2, 2]\).
   \[ A(x) = \int (4 - x^2) \, dx \]
\[ = 4x - \frac{x^3}{3} + C \]
Exercise Set 5.3

Since we know that \( A(-2) = 0 \), (there is no area above the number \(-2\)), we can substitute for \( x \) and \( A(x) \) to determine \( C \).
\[
A(-2) = A(-2) = \frac{-2^3}{3} + C = 0
\]
Substituting \(-2\) for \( x \) and \( 0 \) for \( A(-2) \)

Solving for \( C \), we get:
\[
\begin{align*}
-2^4 & + \frac{8}{3} + C = 0 \\
- \frac{2^4}{3} & + \frac{8}{3} + C = 0 \\
-16 & + C = 0 \\
C & = \frac{16}{3}
\end{align*}
\]
Thus, \( A(x) = 4x - \frac{x^3}{3} + \frac{16}{3} \).

The area on the interval \([-2, 2]\) is \( A(2) \).
\[
A(2) = 4 \cdot 2 - \frac{2^3}{3} + \frac{16}{3} \
\text{Substituting } 2 \text{ for } x
\]
\[
\begin{align*}
&= 8 - \frac{8}{3} + \frac{16}{3} \\
&= \frac{24}{3} + \frac{8}{3} + \frac{16}{3} \\
&= \frac{40}{3}
\end{align*}
\]

11. Find the area under the curve \( y = e^x \) on the interval \([0, 3]\).
\[
A(x) = \int_0^3 e^x \, dx
\]
Since we know that \( A(0) = 0 \), (there is no area above the number \(0\)), we can substitute for \( x \) and \( A(x) \) to determine \( C \).

\[
A(0) = e^0 + C = 0 \\
\text{Substituting } 0 \text{ for } x \text{ and } 0 \text{ for } A(0)
\]
\[
\begin{align*}
0 & + C = 0 \\
C & = -1
\end{align*}
\]
Thus, \( A(x) = e^x - 1 \).

The area on the interval \([0, 3]\) is \( A(3) \).
\[
A(3) = e^3 - 1
\]
\[
\approx 20.1183
\]
Using a calculator

13. Find the area under the curve \( y = \frac{3}{x} \) on the interval \([1, 6]\).
\[
A(x) = \int_1^6 \frac{3}{x} \, dx
\]
Since we know that \( A(1) = 0 \), (there is no area above the number \(1\)), we can substitute for \( x \) and \( A(x) \) to determine \( C \).

\[
A(1) = 3 \ln 1 + 1 \cdot C = 0 \\
\text{Substituting } 1 \text{ for } x \text{ and } 0 \text{ for } A(1)
\]
\[
\begin{align*}
3 \cdot 0 + C & = 0 \\
C & = 0
\end{align*}
\]
Thus, \( A(x) = 3 \ln x \).

The area on the interval \([1, 6]\) is \( A(6) \).
\[
A(6) = 3 \ln 6 \\
\approx 5.375
\]
Using a calculator

15. \( \int_{1}^{5} (x - x^2) \, dx \)
\[
\begin{align*}
&= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{1}^{5} \\
&= \left( \frac{25}{2} - \frac{125}{3} \right) - \left( \frac{1}{2} - \frac{1}{3} \right) \\
\text{Substituting } 0 \text{ for } x \\
&= \left( \frac{25}{2} - \frac{375}{3} \right) - (0 - 0) \\
&= 1.125 - 1.125 \\
&= 0
\end{align*}
\]
The area above the \( x \)-axis is equal to the area below the \( x \)-axis.

17. \( \int_{0}^{\pi/2} \cos x \, dx \)
\[
\begin{align*}
&= \left[ \sin x \right]_{0}^{\pi/2} \\
&= \sin(\pi/2) - \sin(0) \\
&= 1 - 0 \\
&= 1
\end{align*}
\]
This means that the area between \([0, \pi/2]\) (area above the \( x \)-axis) is less than that area between \([\pi/2, 3\pi/2]\) (area below the \( x \)-axis) by 1 unit of area.

19. Left to the student

37. \( \int_{b}^{a} t^4 \, dt \)
\[
\begin{align*}
&= \left[ \frac{t^5}{5} \right]_{a}^{b} \\
&= \frac{b^5}{5} - \frac{a^5}{5}
\end{align*}
\]

39. \( \int_{a}^{b} 3t^2 \, dt \)
\[
\begin{align*}
&= \left[ \frac{3t^3}{3} \right]_{a}^{b} \\
&= \frac{b^3}{3} - \frac{a^3}{3}
\end{align*}
\]

41. \( \int_{1}^{e} \left( x + \frac{1}{x} \right) \, dx \)
\[
\begin{align*}
&= \left[ \frac{x^2}{2} + \ln x \right]_{1}^{e} \\
&= \frac{e^2}{2} + \ln e - \left( \frac{1^2}{2} + \ln 1 \right) \\
&= \frac{e^2}{2} + 1 - \frac{1}{2} \\
&= \frac{e^2}{2} + 1 - \frac{1}{2} \\
&= \ln e - \ln 1 - 0
\end{align*}
\]
43.  \[
\int_0^{\pi/4} \frac{5}{2} \sin 2x \, dx \\
= \left[ -5 \frac{\cos 2x}{4} \right]_0^{\pi/4} \\
= -5 \cos(\pi/6) - (-5 \cos(0)) \\
= -\frac{5}{8} - (-\frac{5}{4}) \\
= \frac{5}{8} \\
45.  \int_{-1}^{1} \frac{10}{17} t^4 \, dt \\
= \frac{10}{17} \left[ \frac{t^5}{5} \right]_{-1}^{1} \\
= \frac{10}{17} \left( \frac{1}{5} - \frac{(-1)^5}{5} \right) \\
= \frac{10}{17} \left( \frac{1}{5} - \frac{1}{5} \right) \\
= \frac{10}{17} \left( \frac{1}{5} \right) \\
= \frac{10}{85} \\
= \frac{2}{17} \\
= \frac{34}{17} \\
51. \text{Find the area under } y = 5 - x^2 \text{ on } [-1, 2]. \\
\int_{-1}^{2} (5 - x^2) \, dx \\
= \left[ 5x - \frac{x^3}{3} \right]_{-1}^{2} \\
= \left( 5 \cdot 2 - \frac{2^3}{3} \right) - \left( 5(-1) - \frac{(-1)^3}{3} \right) \\
= \left( 10 - \frac{8}{3} \right) - \left( -5 + \frac{1}{3} \right) \\
= \frac{30}{3} - \frac{8}{3} - \frac{15}{3} + \frac{1}{3} \\
= \frac{22}{3} - \frac{14}{3} \\
= \frac{36}{3} - \frac{14}{3} \\
= \frac{22}{3} \\
= \frac{14}{3} \\
= \frac{36}{3} \\
= 12 \\
53. \text{Find the area under } y = e^x \text{ on } [-1, 5]. \\
\int_{-1}^{5} e^x \, dx \\
= \left[ e^x \right]_{-1}^{5} \\
= e^5 - e^{-1} \\
= e^5 - \frac{1}{e} \\
55.  \int_{2}^{3} \frac{x^2 - 1}{x - 1} \, dx \\
= \int_{2}^{3} \frac{(x+1)(x-1)}{x-1} \, dx \\
= \int_{2}^{3} (x+1) \, dx \\
= \left[ \frac{1}{2} x^2 + x \right]_{2}^{3} \\
= \frac{1}{2} (9 + 3) - \frac{1}{2} (4 + 2) \\
= \frac{1}{2} (12) - \frac{1}{2} (6) \\
= 6 - 3 \\
= 3 \\
57.  \int_{4}^{16} \sqrt{x} \, dx \\
= \int_{4}^{16} x^{1/2} \, dx \\
= \left[ \frac{2}{3} x^{3/2} - \frac{2}{3} x^{1/2} \right]_{4}^{16} \\
= \frac{2}{3} (16^{3/2}) - \frac{2}{3} (16^{1/2}) \\
= \frac{2}{3} (64) - \frac{2}{3} (4) \\
= \frac{128}{3} - \frac{8}{3} \\
= \frac{120}{3} \\
= \frac{40}{1} \\
= \frac{5392}{15}
\[ \int_1^3 \frac{\sqrt{2} - 1}{\sqrt{x}} dx = \int_1^3 \left[ x^{1/3} - x^{-1/3} \right] dx \]
\[ = \left[ \frac{3}{4} x^{4/3} - \frac{3}{2} x^{2/3} \right]_1^3 \]
\[ = \left[ \frac{3}{4} 3^{4/3} - \frac{3}{2} 3^{2/3} \right] - \left[ \frac{3}{4} 1^{4/3} - \frac{3}{2} 1^{2/3} \right] \]
\[ = \frac{12 - 6}{3} \cdot \frac{3}{2} \]
\[ = \frac{27}{1} \]

61.
\[ \int_1^2 (5x + 3)(5x - 2) dx \]
\[ = \int_1^2 (25x^2 + 7x - 6) \ dx \]
\[ = \left[ \frac{25}{3} x^3 + \frac{7}{2} x^2 - 6x \right]_1^2 \]
\[ = \frac{160}{3} + 14 - 12 - \frac{20}{3} = \frac{7}{2} + 6 \]
\[ \Rightarrow \quad \int_1^2 \frac{1}{5x + 3} dx = \frac{15}{3} \]

62.
\[ \int_0^1 (t + 1)^2 \ dt \]
\[ = \int_0^1 (t^2 + 2t + 1) \ dt \]
\[ = \left[ \frac{t^3}{3} + t^2 + t \right]_0^1 \]
\[ = \frac{1}{3} + 1 + \frac{1}{2} - 0 \]
\[ = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \]

63.
\[ \int_0^3 \frac{t^2 - 3}{t^3} \ dt \]
\[ = \int_0^3 (t - 3) \ dt \]
\[ = \left[ \frac{t^2}{2} + t^{-1} \right]_0^3 \]
\[ = 9 + \frac{1}{3} - 3 - 1 \]
\[ = \frac{9}{3} - \frac{3}{3} - 1 \]
\[ = \frac{8}{3} \]

64.
\[ \int_3^4 x^2 - 4 \ dx \]
\[ = \int_3^4 (x - 2)(x + 2) \ dx \]
\[ = \left[ \frac{x^3}{3} \right]_3^4 \]
\[ = \frac{25}{2} \cdot 9 - \frac{9}{2} - 6 \]
\[ = 12 \]

69. The average is given by
\[ \frac{1}{1 - (-1)} \int_{-1}^1 2x^3 dx = \frac{1}{2} \int_{-1}^1 2x^3 dx \]
\[ = \frac{1}{2} \left[ \frac{1}{2} \right] \]
\[ = \frac{1}{2} \left[ (1)^3 - (-1)^3 \right] \]
\[ = 0 \]

71. The average is given by
\[ \frac{1}{1 - 0} \int_0^1 e^x dx = \int_0^1 e^x dx \]
\[ = e - 1 \]

73. The average is given by
\[ \frac{1}{2 - 0} \int_0^{2} (x^2 - x + 1) dx \]
\[ = \frac{1}{2} \int_0^2 (x^2 - x + 1) dx \]
\[ = \frac{1}{2} \left[ \frac{1}{3} x^3 - \frac{1}{2} x^2 + x \right]_0^2 \]
\[ = \frac{1}{2} \left[ \frac{8}{3} - 2 + 2 - 0 \right] \]
\[ = \frac{4}{3} \]

75. The average is given by
\[ \frac{1}{6 - 2} \int_0^3 (3x + 1) dx \]
\[ = \frac{1}{4} \int_0^3 (3x + 1) dx \]
\[ = \frac{1}{4} \left[ \frac{3x^2}{2} + x \right]_0^3 \]
\[ = \frac{1}{4} (51 + 6 - 6 - 2) \]
\[ = 13 \]

77. The average is given by
\[ \frac{1}{1 - 0} \int_0^1 x^n dx = \int_0^1 x^n dx \]
\[ = \int_0^1 \left[ x^{n+1} \right]_0^1 \]
\[ = \frac{1}{n + 1} - 0 \]
\[ = \frac{1}{n + 1} \]

79. The distance is given by
\[ \int_3^4 (8t^2 + 2t) dt = \int_3^4 (8t^2 + 2t) \ dt \]
\[ = \left[ \frac{8t^3}{3} + t^2 \right]_3^4 \]
\[ = 125 + 25 - 1 - 1 \]
\[ = 148 \]
81. The population increase is given by
\[ \int_{0}^{2} 200e^{-t} \, dt = \left[ -200e^{-t} \right]_{0}^{2} = -200e^{-2} + 200 = 200 \left( 1 - \frac{1}{e^2} \right) \]

83. The population decrease is given by
\[ \int_{0}^{10} (-500(20 - t)) \, dt = -500 \left[ 20t - \frac{t^2}{2} \right]_{0}^{10} = -500(200 - 50) = -75000 \]

85. The work done is given by
\[ \int_{2}^{11} 71.3x - 4.15x^2 + 0.134x^3 \, dx = \left[ 35.65x^2 - 1.383x^3 + 0.1085x^4 \right]_{2}^{11} = \left[ 4061 - 133.27 \right] = 3927.73 \text{ N} \cdot \text{cm} \]

87. a) The initial dosage is 42.03 \text{ mg/mL}

b) The average amount is given by
\[ \frac{1}{120 - 10} \int_{10}^{120} 42.03e^{-0.01000t} \, dt = \frac{1}{110} \left[ \frac{42.03}{0.01000} - 0.10000e^{-0.01000t} \right]_{10}^{120} = \frac{1135 - (-3601)}{110} = 22.45 \text{ mg/mL} \]

89. a) 
\[ \int_{0}^{b} \pi r^2 \, dv = \pi \left[ \frac{r^2}{2} \right]_{0}^{b} = \pi \frac{b^2}{2} \]

b) The circle could be thought of as a rectangle with length \( \pi b \) and width \( b \)

91. The lower limit of the integral, \( x = 1 \), was not evaluated

Exercise Set 5.4

1. First graph the system of equations and shade the region bounded by the graphs.

Here the boundaries are easily determined by looking at the graph. Note which is the upper graph. Here it is \( x \geq x^3 \) over the interval \([0,1]\).

Compute the area as follows:
\[ \int_{0}^{1} (x - x^3) \, dx = \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_{0}^{1} = \left( \frac{1}{2} - \frac{1}{4} \right) - \left( 0 - 0 \right) = \frac{1}{4} - \frac{1}{4} = \frac{1}{4} \]

3. First graph the system of equations and shade the region bounded by the graphs.

Here the boundaries are easily determined by the graph. Note which is the upper graph. Here it is \( (x+2) \geq x^3 \) over the interval \([-1,2]\).
Compute the area as follows:
\[
\frac{\pi}{4} \left( (x + 1)^2 - x^2 \right) dx
\]
\[
= \left[ -\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^{1}
\]
\[
= \left( -\frac{2}{3} + \frac{2}{2} + 2 \right) - \left( -\frac{1}{3} + \frac{1}{2} + 2 \right)
\]
\[
= \frac{8}{3} + \frac{4}{4} - \frac{1}{3} - \frac{1}{2} + 2
\]
\[
= \frac{9}{3} - \frac{1}{2} + 1 = \frac{9}{2} - \frac{1}{2} + 2
\]
\[
= \frac{9}{2}
\]

5. First graph the system of equations and shade the region bounded by the graphs.

Here the boundaries are easily determined by the graph. Note which is the upper graph. Here it is \((6x - x^2) \geq x\) over the interval \([0, 3]\).

Compute the area as follows:
\[
\int_0^3 \left( (6x - x^2) - x \right) dx
\]
\[
= \left[ -\frac{x^3}{3} + \frac{5x^2}{2} \right]_{0}^{3}
\]
\[
= \left( -\frac{27}{3} + \frac{5 \cdot 9}{2} \right) - \left( 0 + 5 \cdot 0 \right)
\]
\[
= \left( -\frac{27}{3} + \frac{45}{2} \right) - \left( 0 + 5 \cdot 0 \right)
\]
\[
= \frac{\pi}{2} - \frac{27}{3}
\]
\[
= \frac{27}{3} - \frac{27}{3}
\]
\[
= \frac{9}{2}
\]

7. First graph the system of equations and shade the region bounded by the graphs.

The boundaries are easily determined by looking at the graph. Note which is the upper graph. Here it is \((2x - x^2) \geq x\) over the interval \([0, 3]\).

Compute the area as follows:
\[
\int_0^3 \left( (2x - x^2) - x \right) dx
\]
\[
= \left[ -\frac{x^3}{3} + \frac{x^2}{2} \right]_{0}^{3}
\]
\[
= \left( \frac{27}{3} - \frac{9}{2} \right) - \left( 0 + 0 \right)
\]
\[
= \frac{27}{3} - \frac{9}{2}
\]
\[
= \frac{27}{6} - \frac{9}{2}
\]
\[
= \frac{9}{2}
\]

9. First graph the system of equations and shade the region bounded by the graphs.

The boundaries are easily determined by looking at the graph. Note which is the upper graph. Here it is \(\sqrt{x} \geq x\) over the interval \([0, 1]\).

Compute the area as follows:
\[ \int_0^1 (\sqrt{x} - x) \, dx \]
\[ = \int_0^1 (x^{1/4} - x) \, dx \]
\[ = \left[ \frac{4}{5} x^{5/4} - \frac{1}{2} x^2 \right]_0^1 \]
\[ = \left( \frac{4}{5} \cdot 1^{5/4} - \frac{1}{2} \cdot 1^2 \right) - \left( \frac{4}{5} \cdot 0^{5/4} - \frac{1}{2} \cdot 0^2 \right) \]
\[ = \frac{4}{5} - \frac{1}{2} = 0 \]
\[ = \frac{8}{10} - \frac{5}{10} = \frac{3}{10} \]

11. Graph the system of equations and shade the region bounded by the graphs.

The boundaries are easily determined by looking at the graph. Here \( 5 \geq \sqrt{x} \) over the interval \([0, 25]\).

Compute the area as follows:
\[ \int_0^{25} (5 - \sqrt{x}) \, dx \]
\[ = \int_0^{25} (5 - x^{1/2}) \, dx \]
\[ = \left[ 5x - \frac{x^{3/2}}{3/2} \right]_0^{25} \]
\[ = \left[ 5 \cdot 25 - \frac{2 \cdot 25^{3/2}}{3} \right] - \left[ 5 \cdot 0 - \frac{2 \cdot 0^{3/2}}{3} \right] \]
\[ = 125 - \frac{250}{3} - 0 \quad [25^{3/2} = (5^2)^{3/2} = 5^3 = 125] \]
\[ = \frac{375}{3} - \frac{250}{3} = \frac{125}{3}, \text{ or } 41 \frac{2}{3} \]

13. First graph the system of equations and shade the region bounded by the graphs.

Thus determine the first coordinates of possible points of intersection by solving a system of equations as follows.

At the points of intersection, \( y = 4 - x^2 \) and \( y = 4 - 4x \), so
\[ 4 - x^2 = 4 - 4x \]
\[ 0 = x^2 - 4x \]
\[ 0 = x(x - 4) \]
\[ x = 0 \text{ or } x = 4 \]

Thus the interval with which we are concerned is \([0, 4]\).

Note that \( 1 - x^2 \geq 1 - 4x \) over the interval \([0, 4]\).

Compute the area as follows:
\[ \int_0^4 [(4 - x^2) - (4 - 4x)] \, dx \]
\[ = \int_0^4 (-x^2 + 4x) \, dx \]
\[ = \left[ -\frac{x^3}{3} + 2x^2 \right]_0^4 \]
\[ = \left( -\frac{4^3}{3} + 2 \cdot 4^2 \right) - \left( -\frac{0^3}{3} + 2 \cdot 0^2 \right) \]
\[ = \frac{96}{3} = 32 \]
\[ = \frac{54}{3} + \frac{96}{3} \]
\[ = \frac{32}{3} \]

15. First graph the system of equations and shade the region bounded by the graphs.

From the graph we can easily determine the interval with which we are concerned. Here \( x^2 - 3 \geq x^2 \) over the interval \([1, 2]\).
17. First graph the system of equations and shade the region bounded by the graphs.

From the graph we can easily determine the interval with which we are concerned. Here $x + 3 \geq \sin x + \cos x$ over the interval $[0, \frac{\pi}{2}]$.

Compute the area as follows:

$$\int_0^{\frac{\pi}{2}} [(x + 3) - \sin x - \cos x] \, dx$$

$$= \int_0^{\frac{\pi}{2}} (x + 3) \, dx - \left[ \frac{1}{3}x^3 - 3x + 1 \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8} + \frac{3\pi}{2} - 2$$

19. First graph the system of equations and shade the region bounded by the graphs.

From the graph we can easily determine the interval with which we are concerned. Here $x + 3 \geq \sin x + \cos x$ over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

Compute the area as follows:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [\cot x \cdot \sin x - \sin x] \, dx$$

$$= \left[ -\cot x + \sin x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 0 - \left[ -\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} \right]$$

$$= -1 + \frac{\sqrt{2}}{2}$$

21. First graph the system of equations and shade the region bounded by the graphs.

From the graph we can easily determine the interval with which we are concerned. Here $x + 3 \geq \sin x + \cos x$ over the interval $[2, 10]$.

Compute the area as follows:

$$\int_2^{10} [(x^2 + 6x - 15 - 2x^2 + 6x - 5) \, dx$$

$$= \int_2^{10} [-x^2 + 12x - 20] \, dx$$

$$= \left[ -\frac{1}{3}x^3 + 6x^2 - 20x \right]_2^{10}$$

$$= \left[ -\frac{100}{3} + 600 - 200 \right] - \left[ -\frac{8}{3} + 12 - 40 \right]$$

$$= \frac{200}{3}$$

23. $f(x) \geq g(x)$ on $[-5, -1]$, and $g(x) \geq f(x)$ on $[-1, 3]$. We use two integrals to find the total area.

$$\int_{-5}^{-1} [(x^3 + 3x^2 - 9x + 12) - (4x + 3)] \, dx + \int_{-1}^{3} [(4x + 3) - (x^3 + 3x^2 - 9x + 12)] \, dx$$

$$= \left[ \frac{-1}{4}x^4 + \frac{3}{2}x^3 - 15x^2 \right]_{-1}^{1} + \left[ \frac{-1}{4}x^4 + \frac{13}{2}x^2 + 15x \right]_{-1}^{3}$$

$$= \left[ \frac{-1}{4} \cdot 4 + \frac{3}{2} \cdot 3 - 15 \cdot 1 \right] - \left[ \frac{-1}{4} \cdot 4 + \frac{13}{2} \cdot 3 + 15 \cdot 3 \right]$$

$$= \left[ -\frac{1}{4} \cdot 4 - \frac{3}{2} \cdot 3 - \frac{13}{2} \cdot 3 + 15 \cdot 1 \right]$$

$$= -\frac{31}{4} - \frac{9}{2} - \frac{39}{2} + 15$$

$$= -\frac{31}{4} - \frac{9}{2} - \frac{39}{2} + 15$$

$$= -\frac{31}{4} - \frac{9}{2} - \frac{39}{2} + 15$$

$$= -\frac{31}{4} - \frac{9}{2} - \frac{39}{2} + 15$$

$$= -\frac{31}{4} - \frac{9}{2} - \frac{39}{2} + 15$$
25. \( f(x) \geq g(x) \) on \([1, 4]\). We find the area.

\[
\int_1^4 \left[ (4x - x^2) - (x^2 - 6x + 8) \right] \, dx
\]

\[
= \int_1^4 (-2x^2 + 10x - 8) \, dx
\]

\[
= \left[ -\frac{2x^3}{3} + 5x^2 - 8x \right]_1^4
\]

\[
= \left( \frac{-2 \cdot 4^3}{3} + 5 \cdot 4^2 - 8 \cdot 4 \right) - \left( \frac{-2 \cdot 1^3}{3} + 5 \cdot 1^2 - 8 \cdot 1 \right)
\]

\[
= (\frac{-128}{3} + 80 - 32) - \left( \frac{-2}{3} + 5 - 8 \right)
\]

\[
= \frac{16}{3} \cdot \frac{11 + 27}{3} = 9
\]

27. Find the area under:

\[
f(x) = \begin{cases} 
4 - x^2, & \text{if } x \leq 0 \\
4, & \text{if } x \geq 0
\end{cases}
\text{ on } [-2, 3]
\]

We have to break the integral into two parts in order to complete this problem.

\[
\int_{-2}^3 f(x) \, dx
\]

\[
= \int_{-2}^0 f(x) \, dx + \int_0^3 f(x) \, dx
\]

\[
= \int_{-2}^0 (4 - x^2) \, dx + \int_0^3 4 \, dx
\]

\[
= \left[ 4x - \frac{x^3}{3} \right]_0^1 + [4x]_0^3
\]

\[
= \left[ 4 - \frac{0^3}{3} \right] + \left[ 4(-2) - \frac{(-2)^3}{3} \right] + (12 - 0)
\]

\[
= (0 - 0) - \left( -8 + \frac{8}{3} \right) + (12 - 0)
\]

\[
= \left( -\frac{24}{3} + \frac{8}{3} \right) + 12
\]

\[
= \left( -\frac{16}{3} \right) + 12
\]

\[
= \frac{16}{3} + \frac{36}{3}
\]

\[
= \frac{52}{3}, \text{ or } 17\frac{1}{3}
\]

29. First graph the system of equations and shade the region bounded by the graph.

From the graph we can easily determine the interval with which we are concerned. Here \( x^2 \geq x^3 \) over the interval \([-1, 1]\).

\[
\int_{-1}^1 (x^2 - x^3) \, dx
\]

\[
= \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_{-1}^1
\]

\[
= \left( \frac{1}{3} - \frac{1}{4} \right) - \left( \frac{-1}{3} - \frac{-1}{4} \right)
\]

\[
= \left( \frac{1}{3} - \frac{1}{4} \right) - \left( \frac{1}{3} - \frac{1}{4} \right)
\]

\[
= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} + \frac{1}{4} = \frac{2}{3}
\]

31. First graph the system of equations and shade the region bounded by the graph.

From the graph we can easily determine the interval with which we are concerned. Here \( e^x \geq e^{-x} \) over the interval \([0, 1]\).

\[
\int_0^1 (e^x - e^{-x}) \, dx
\]

\[
= \left[ e^x + e^{-x} \right]_0^1
\]

\[
= (e - 1) - (e^0 + e^{-0})
\]

\[
= (e + \frac{1}{e}) - (1 + 1)
\]

\[
= e + \frac{1}{e} - 2
\]

\[
= \frac{e^2 - 2e + 1}{e}
\]

\[
\approx \frac{(e - 1)^2}{e} \approx 1.086
\]

33. On \([0, 1]\), \( f(x) \geq g(x) \).

\[
\int_0^1 (f(x) - g(x)) \, dx
\]

\[
= \int_0^1 0.3x + 6.15x^2 - 0.552x^3 \, dx
\]

\[
= \left[ 0.15x^2 + \frac{1.45}{3}x^3 - 0.138x^4 \right]_{0}^{11}
\]

\[
= [726.24 - 0]
\]

\[
= 726.24 \text{ N} \cdot \text{mm}
\]
35. \( T(t) = 25 + 3e^{-t} - 20(t - 0.5)^2 \), \( f(t) = 8 \)
\[
\int_0^t [T(t) - f(t)] \, dt
\]
\[
\int_0^t [17 + 3e^{-t} - 20(t - 0.5)^2] \, dt
\]
\[
[17 - \frac{3}{3} - \frac{20}{3}(t - 0.5)]^t_0
\]
\[
17 - \frac{3}{3} - \frac{20}{3} - [-3 + \frac{3}{3}]
\]
17.23 degree days

37.
\[
\begin{array}{c|c|c|c|c}
-4 & -2 & 0 & 2 & 4 \\
\hline
-100 & 0 & 100 & 0 & -100
\end{array}
\]
From the graph we see that the first coordinate of the relative maximum is \( x = 2 \) and the first coordinate of the relative minimum is \( x = -2 \).

\[
\int_{-2}^2 [3x^2 - 20x^3 - 0] \, dx = \int_0^2 [0 - 3x^2 + 20x^3] \, dx
\]
\[
\left[ \frac{1}{2}x^5 - 5x^2 \right]_{-2}^2 + \left[ -\frac{1}{2}x^6 + 5x^2 \right]_0^2
\]
\[
= [32 + 80] - [32 - 80]
\]
96

39. \( V = \frac{p}{4Lv^2}(R^2 - r^2) \)
\[
Q = \frac{p}{2Lv} \int_0^R [2\pi \cdot \frac{p}{4Lv}(R^2 - r^2)] \, dr
\]
\[
= \frac{\pi p}{2Lv} \int_0^R [R^2r - r^3] \, dr
\]
\[
= \frac{\pi p}{2Lv} \left[ \frac{R^3}{2} - \frac{1}{4}R^4 \right]_0^R
\]
\[
= \frac{\pi p R^4}{8Lv}
\]

41. The area bounded in the graph is 21.961
43. The area bounded in the graph is 416.708

45. a) The area is 64.5239
b) \( x = -1.8623 \), \( x = 0 \), and \( x = 1.1594 \)
c) The area is 17.683

d) The area is 64.5239

Exercise Set 5.5

1. \( \int \frac{3x^2 \, dx}{1 + x^3} \)
Let \( u = 7 + x^3 \), then \( du = 3x^2 \, dx \).
\[
= \int \frac{du}{u} \quad \text{Substituting } u \text{ for } 7 + x^3 \text{ and } du \text{ for } 3x^2 \, dx
\]
\[
= \int \frac{1}{u} \, du \quad \text{Using Formula C}
\]
\[
= \ln |u + C|
\]
\[
= \ln (7 + x^3) + C
\]

3. \( \int e^{x^2} \, dx \)
Let \( u = 4x \), then \( du = 4 \, dx \).
We do not have \( 4 \, dx \). We only have \( dx \) and need to supply \( \frac{1}{4} \cdot 4 \) as follows.
\[
\frac{1}{4} \cdot 4 \int e^{u^2} \, du \quad \text{Multiplying by } \frac{1}{4}
\]
\[
= \frac{1}{4} \int 4e^{u^2} \, du
\]
\[
= \frac{1}{4} \int e^{u^2} \, du
\]
\[
= \frac{1}{4} \int c \, du \quad \text{Substituting } u \text{ for } 4x \text{ and } du \text{ for } 4 \, dx
\]
\[
= \frac{1}{4}c^u + C \quad \text{Using Formula B}
\]
\[
= \frac{1}{4}e^{x^2} + C
\]

5. \( \int e^{x^2} \, dx \)
Let \( u = \frac{1}{2}x \), then \( du = \frac{1}{2} \, dx \).
We do not have \( \frac{1}{2} \, dx \). We only have \( dx \) and need to supply \( \frac{1}{2} \cdot \frac{1}{2} \) as follows.
\[
\frac{1}{2} \cdot \frac{1}{2} \int e^{u^2} \, du
\]
\[
= \frac{1}{2} \int e^{u^2} \, du
\]
\[
= \frac{1}{2} \int e^{u^2} \, du
\]
7. \[ \int e^{x^2} e^{x^4} \, dx \]
Let \( u = x^2 \), then \( du = 2x \, dx \).
We do not have \( 4x^3 \, dx \). We only have \( x^3 \, dx \) and need to supply a 4. We do this by multiplying by \( \frac{1}{4} \cdot 4 \) as follows:
\[ \frac{1}{4} \cdot \int \frac{1}{2} e^{x^2} e^{x^4} \, dx \]
Multiplying by 1
\[ \cdot \frac{1}{4} \int 4x^3 \, dx \]
\[ = \frac{1}{4} \int e^{x^2} (4x^3 \, dx) \]
\[ = \frac{1}{4} \int e^{x^2} \, du \]
Substituting \( u = x^2 \) and \( du = 2x \, dx \) for \( e^{x^2} \, dx \)
\[ \cdot \frac{1}{3} e^{x^2} + C \]
Using Formula B
\[ \cdot \frac{1}{4} e^{x^2} + C \]

9. \[ \int \frac{1}{t} e^{-t^2} \, dt \]
Let \( u = -t^2 \), then \( du = -2t \, dt \).
We do not have \(-3t^2 \, dt \). We only have \( t^2 \, dt \). We need to supply a \(-3\) by multiplying by \( -\frac{1}{3} \cdot (-3) \) as follows:
\[ -\frac{1}{3} \cdot (-3) \int \frac{1}{t} e^{-t^2} \, dt \]
Multiplying by 1
\[ = -\frac{1}{3} \int -3t e^{-t^2} \, dt \]
\[ = -\frac{1}{3} \int e^{-t^2} (-3t^2 \, dt) \]
\[ = -\frac{1}{3} \int e^{u} \, du \]
Substituting \( u = -t^2 \) and \( du = -2t \, dt \) for \( e^{-t^2} \, dt \)
\[ = -\frac{1}{3} e^{-t^2} + C \]
Using Formula B
\[ = -\frac{1}{3} e^{-t^2} + C \]
21. \( \int \frac{t^2(t^3 - 1)^2}{dt} \)

Let \( u = t^3 - 1 \), then \( du = 3t^2 \, dt \).

We do not have \( 3t^2 \, dt \). We only have \( t^2 \, dt \). We need to supply a 3 by multiplying by \( \frac{1}{3} \) as follows.

\[- \frac{1}{3} \int \left( \frac{t^3}{3} - 1 \right)^2 \, dt \]

Multiplying by \( \frac{1}{3} \)

\[- \frac{1}{3} \int 2t^2 \left( t^3 - 1 \right)^2 \, dt \]

\[- \frac{1}{3} \int t^3 \, du \]

Substituting \( a \) for \( t^3 - 1 \) and \( du \) for \( 3t^2 \, dt \)

\[- \frac{1}{3} \left( \frac{u^3}{3} - u \right) + C \]

Using Formula A

\[- \frac{1}{24}(t^3 - 1)^4 + C \]

23. \( \int x \sin(x^2) \, dx \)

Let \( u = x^2 \), then \( du = 2x \, dx \).

\[- \frac{1}{2} \int \sin u \, du \]

\[- \frac{1}{2} \cos x + C \]

\[- \frac{1}{2} \cos x^2 + C \]

25. \( \int (x + 1) \csc(x^2 + 2x + 3) \, dx \)

Let \( u = x^2 + 2x + 3 \), then \( du = 2x + 2 \, dx \).

\[- \frac{1}{2} \int u \csc u \, du \]

\[- \frac{1}{2} \ln(u) + C \]

\[- \frac{1}{2} \ln(x^2 + 2x + 3) + C \]

27. \( \int \frac{e^u}{4 + e^u} \, dx \)

Let \( u = 1 + e^x \), then \( du = e^x \, dx \).

\[- \int \frac{du}{u} \]

Substituting \( u \) for \( 1 + e^x \) and \( du \) for \( e^x \, dx \)

\[- \frac{1}{u} \ln u + C \]

Using Formula C

\[- \ln (1 + e^x) + C \]

29. \( \int \frac{\ln x^2}{x} \, dx \)

Let \( u = \ln x^2 \), then \( du = \frac{2x}{x^2} \, dx = \frac{2}{x} \, dx \).

We do not have \( \frac{2}{x} \, dx \). We only have \( \frac{1}{x} \, dx \) and need to supply a 2 by multiplying by \( \frac{1}{2} \) as follows.

\[- \frac{1}{2} \int 2 \cdot \frac{\ln x^2}{x} \, dx \]

Multiplying by \( \frac{1}{2} \)

\[- \frac{1}{2} \int 2 \cdot \frac{\ln x^2}{x} \, dx \]

\[- \frac{1}{2} \int \ln x^2 \cdot \frac{2}{x} \, dx \]

\[- \frac{1}{2} \int u \, du \]

Substituting \( a \) for \( x^2 \) and \( du \) for \( \frac{2}{x} \, dx \)

\[- \frac{1}{2} \cdot \frac{u^2}{2} + C \]

Using Formula A

\[- \frac{u^2}{4} + C \]

\[- \frac{1}{4} \ln (\ln x^2) + C \]

or \( \frac{1}{4} \ln (\ln x)^2 + C \)

\[- \frac{1}{4} \cdot 4(\ln x)^2 + C \]

\[- \ln (\ln x)^2 + C \]

31. \( \int \frac{dx}{x \ln x} \)

Let \( u = \ln x \), then \( du = \frac{1}{x} \, dx \).

\[- \int \frac{1}{u} \, du \]

\[- \ln u + C \]

\[- \ln (\ln x) + C \]

33. \( \int \sqrt{ax+b} \, dx \), or \( f(ax+b)^{1/2} \, dx \)

Let \( u = ax + b \), then \( du = a \, dx \).

We do not have \( a \, dx \). We only have \( dx \) and need to supply an \( a \) by multiplying by \( \frac{1}{a} \) as follows.

\[- \ln (ax+b) + C \]

\[- \ln (ax+b)^{1/2} + C \]
\[ \frac{1}{a} \cdot a \int \sqrt{ax + b} \, dx \quad \text{Multiplying by 1} \]
\[ \frac{1}{a} \int a \sqrt{ax + b} \, dx \]
\[ = \frac{1}{a} \int \sqrt{ax + b} \, (a \, dx) \]
\[ = \frac{1}{a} \int \sqrt{u} \, du \quad \text{Substituting} \ u = ax + b \text{and} \ du \text{for} \ a \, dx \]
\[ = -\frac{1}{a} \int u^{3/2} \, du \]
\[ = -\frac{1}{a} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + C \quad \text{Using Formula A} \]
\[ = -\frac{2}{3a} \cdot u^{\frac{3}{2}} + C \]
\[ = -\frac{2}{3a} \cdot (ax + b)^{\frac{3}{2}} + C \]

35. \[ \int b e^{ax} \, dx \]
\[ = b \int e^{ax} \, dx \]
\[ = b \cdot \frac{1}{a} \cdot a \int e^{ux} \, dx \]
\[ = b \int e^{ux} \, (e \, dx) \]
\[ = b \int e^{ux} \, du \quad \text{Substituting} \ u = ax \text{and} \ du \text{for} \ a \, dx \]
\[ = \frac{b}{u} \cdot e^{ux} + C \quad \text{Using Formula A} \]
\[ = \frac{b}{a} \cdot e^{ax} + C \]

37. \[ \int a \sin(bx + c) \, dx \]
\[ \text{Let} \ a = bx + c, \text{then} \ du = b \, dx. \]
\[ \frac{1}{b} \int \sin(u) \, du \]
\[ = \frac{1}{b} \cos(u) + D \]
\[ = \frac{1}{b} \cos(bx + c) + D \]

39. \[ \int \frac{3x^2 \, dx}{(1 + x^3)^2} \]
\[ \text{Let} \ u = 1 + x^3, \text{then} \ du = 3x^2 \, dx. \]
\[ = \int \frac{1}{u^2} \, du \quad \text{Substituting} \ u \text{for} \ 1 + x^3 \text{and} \ du \text{for} \ 3x^2 \, dx \]
\[ = u^{-1} + C \quad \text{Using Formula A} \]
\[ = -\frac{1}{4a^4} + C \]
\[ = -\frac{1}{4(1 + x^3)^3} + C \]

41. \[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx \]
\[ \text{Let} \ u = \sin x, \text{then} \ du = \cos x \, dx. \]
\[ = \int \frac{du}{u} \quad \text{in} \ u + C \]
\[ = \ln |u| + C \]
\[ = \ln (\sin x) + C \]

43. \[ \int 5t \sqrt{1 - 4t^2} \, dt \]
\[ \text{Let} \ u = 1 - 4t^2, \text{then} \ du = -8t \, dt. \]
\[ = -\frac{5}{8} \int u^{1/2} \, du \]
\[ = -\frac{5}{12} \cdot u^{3/2} + C \]
\[ = -\frac{5}{12} (1 - 4t^2)^{3/2} + C \]

45. \[ \int e^t \sin t \, dt \]
\[ \text{Let} \ u = \sqrt{w}, \text{then} \ du = \frac{dw}{2\sqrt{w}}. \]
\[ = \frac{1}{2} \int e^u \, du \]
\[ = \frac{1}{2} e^u + C \]
\[ = \frac{1}{2} e^{\sqrt{w}} + C \]

47. \[ \int \sin^2 x \cos x \, dx \]
\[ \text{Let} \ u = \sin x, \text{then} \ du = \cos x \, dx. \]
\[ = \int u^2 \, du \]
\[ = \frac{1}{3} u^3 + C \]
\[ = \frac{1}{3} \sin^3 x + C \]
49. \[ \int e^t \sin e^t \, dt \]
Let \( u = -\sin e^t \), then \( du = -e^t \, dt \)
\[ - \int e^t \, du \]
\[ -e^t + C \]
\( -e^t \sin e^t + C \)

51. \[ \int r^2 \sin(3r^3 + 7) \, dr \]
Let \( u = 3r^3 \), then \( du = 9r^2 \, dr \)
\[ - \frac{1}{9} \int \sin u \, du \]
\[ -\frac{1}{9} \cos u + C \]
\[ -\frac{1}{9} \cos(3r^3 + 7) + C \]

53. \( \int_0^1 e^r e^e \, dx \)
First find the indefinite integral.
\[ \int 2x e^e \, dx \]
Let \( u = e^e \), then \( du = 2x \, dx \)
\[ \ln 2x + e^e + C \]

Then evaluate the definite integral on \([0, 1]\).
\[ \int_0^1 2x e^e \, dx = \left[ 2xe^e \right]_0^1 \]
\[ = e^e - e^e \]
\[ = 0 \]

55. \( \int_0^1 x(x^2 + 1)^5 \, dx \)
First find the indefinite integral.
\[ \int x(x^2 + 1)^5 \, dx \]
Let \( u = x^2 + 1 \), then \( du = 2x \, dx \).

We only have \( x \, dx \) and need to supply a \( 2 \) by multiplying
\[ \frac{1}{2} \cdot 2 = 1 \]
\[ \frac{1}{2} \int x(x^2 + 1)^5 \, dx = \int x(x^2 + 1)^5 \, dx \quad \text{Multiplying by 1} \]
\[ \frac{1}{2} \int 2x(x^2 + 1)^5 \, dx \]
\[ \frac{1}{2} \int (x^2 + 1)^5 \cdot 2x \, dx \]
\[ \frac{1}{2} \int 2x(x^2 + 1)^5 \, dx \quad \text{Substituting} \ a \ \text{for} \ x^2 + 1 \ \text{and} \ \text{for} \ 2x \, dx \]
\[ \frac{1}{2} \int u^5 \, du \quad \text{Using Formula A} \]
\[ \frac{1}{2} \frac{u^6}{6} + C \]
\[ \frac{x^2 + 1)^6}{12} + C \]

Then evaluate the definite integral on \([0, 1]\).
\[ \int_0^1 x(x^2 + 1)^5 \, dx = \left[ \frac{(x^2 + 1)^6}{12} \right]_0^1 \]
\[ = \frac{61}{12} - \frac{1}{12} \]
\[ = \frac{60}{12} \]
\[ = 5 \]
\[ = \ln 5 + C \]

57. \( \int_0^1 \frac{dt}{1 + t} \)
First find the indefinite integral.
\[ \int \frac{dt}{1 + t} \]
Let \( u = 1 + t \), then \( du = dt \).
\[ = \int \frac{du}{u} \quad \text{Substituting} \ u \ \text{for} \ 1 + t \ \text{and} \ \text{for} \ dt \]
\[ = \int \frac{1}{u} \, du \]
\[ = \ln u + C \quad \text{Using Formula C} \]
\[ = \ln (1 + t) + C \]

Then evaluate the definite integral on \([0, 4]\).
\[ \int_0^1 \frac{dt}{1 + t} = \left[ \ln (1 + t) \right]_0^1 \]
\[ = \ln (1 + 1) - \ln (1 + 0) \]
\[ = \ln 5 - \ln 1 \]
\[ = \ln 5 \]
\[ = 0 \]

59. \( \int_0^1 \frac{2x + 1}{x^2 + x - 1} \, dx \)
First find the indefinite integral.
\[ \int \frac{2x + 1}{x^2 + x - 1} \, dx \]
Let \( u = x^2 + x = 1 \), then \( du = (2x + 1) \, dx \).
\[ = \int \frac{1}{x^2 + x - 1} \, dx \]
\[ = \ln (x^2 + x - 1) + C \]
\[ = \ln (x^2 + x - 1) + C \]
Then evaluate the definite integral on $[1,4]$.

\[ \int_{1}^{4} \frac{2x + 1}{x^2 + x - 1} \, dx = \left. [\ln (x^2 + x - 1)] \right|_{1}^{4} \]

\[ = \ln (4^2 + 4 - 1) - \ln (1^2 + 1 - 1) \]

\[ = \ln 19 - \ln 1 \]

\[ = \ln 19 \] (since $\ln 1 = 0$)

61. $\int_{a}^{b} e^{-x} \, dx$

First find the indefinite integral.

\[ \int e^{-x} \, dx \]

Let $u = -x$, then $du = -dx$.

We only have $dx$ and need to supply a $-1$ by multiplying by $-1 \cdot (-1)$.

\[ -1 \cdot (-1) \int e^{-x} \, dx \]

\[ = - \int e^{-x} \, dx \]

\[ = - e^{-x} + C ]

Then evaluate the definite integral on $[0, b]$.

\[ \int_{0}^{b} e^{-x} \, dx \]

\[ = \left. [-e^{-x}] \right|_{0}^{b} \]

\[ = (-e^{-b}) - (-e^{-0}) \]

\[ = -e^{-b} + e^{0} \]

\[ = -e^{-b} + 1 \]

\[ = 1 - \frac{1}{e^{b}} \]

63. $\int_{a}^{b} m e^{-mx} \, dx$

First find the indefinite integral.

\[ \int m e^{-mx} \, dx \]  

$m$ is a constant

Let $u = -mx$, then $du = -m \, dx$.

We only have $m \, dx$ and need to supply a $-1$ by multiplying by $-1 \cdot (-1)$.

\[ -1 \cdot (-1) \int m e^{-mx} \, dx \]

\[ = - \int -m e^{-mx} \, dx \]

\[ = - \int e^{-mx} \, (-m \, dx) \]

\[ = - \int e^{u} \, du \] (Substituting $u$ for $-mx$ and $du$ for $-m \, dx$)

\[ = -e^{u} + C \] (Using Formula B)

\[ = -e^{-mx} + C \]

Then evaluate the definite integral on $[0, b]$.

\[ \int_{0}^{b} m e^{-mx} \, dx \]

\[ = \left. \left[ \frac{-1}{m} \, e^{-mx} \right] \right|_{0}^{b} \]

\[ = \left\{ \frac{-1}{m} \, e^{-mb} - \frac{-1}{m} \, e^{-m0} \right\} \]

\[ = \frac{-e^{-mb}}{m} - \frac{-e^{-0}}{m} \]

\[ = \frac{-e^{-mb}}{m} + 1 \] (since $e^{-m0} = e^{0} = 1$)

\[ = 1 - \frac{1}{e^{mb}} \]

65. $\int_{0}^{1} (x - 6)^{3} \, dx$

First find the indefinite integral.

\[ \int (x - 6)^{3} \, dx \]

Let $u = x - 6$, then $du = dx$.

\[ = \int u^{3} \, du \] (Substituting $u$ for $x - 6$ and $du$ for $dx$)

\[ = \frac{u^{4}}{4} + C \]

\[ = \frac{(x - 6)^{4}}{4} + C \]

Then evaluate the definite integral on $[0,1]$.

\[ \int_{0}^{1} (x - 6)^{3} \, dx \]

\[ = \left[ \frac{(x - 6)^{4}}{4} \right]_{0}^{1} \]

\[ = \left\{ \frac{(1 - 6)^{4}}{4} - \frac{(0 - 6)^{4}}{4} \right\} \]

\[ = \frac{8}{3} - \left( \frac{216}{3} \right) \]

\[ = \frac{8}{3} - 72 \]

\[ = \frac{8}{3} + 216 \]

\[ = \frac{208}{3} \]

67. $\int_{-1/3}^{0} \cos(\pi x + \pi/3) \, dx$

First find the indefinite integral.

\[ \int \cos(\pi x + \pi/3) \, dx \]

Let $u = \pi x + \pi/3$, then $du = \pi \, dx$.

\[ = \frac{1}{\pi} \cos u \, du \] (Substituting $u$ for $\pi x + \pi/3$ and $\frac{1}{\pi} \, du$ for $\pi \, dx$)

\[ = \frac{1}{\pi} \sin u + C \]

\[ = \frac{1}{\pi} \sin(\pi x + \pi/3) + C \]

Then evaluate the definite integral on $[-1/3,0]$.

\[ \int_{-1/3}^{0} \cos(\pi x + \pi/3) \, dx \]

\[ = \left[ \frac{1}{\pi} \sin(\pi x + \pi/3) \right]_{-1/3}^{0} \]

\[ = \frac{\sqrt{3}}{2\pi} - 0 \]

\[ = \frac{\sqrt{3}}{2\pi} \]
69. \[ \int_0^2 \frac{3x^2}{(1 + x^2)^2} \, dx \]

From Exercise 39 we know that the indefinite integral is

\[ \int \frac{3x^2}{(1 + x^2)^2} \, dx = -\frac{1}{3(1 + x^2)^{\frac{3}{2}}} + C. \]

Now we evaluate the definite integral on [0, 2].

\[ \int_0^2 \frac{3x^2}{(1 + x^2)^2} \, dx = \left. \right|_0^2 \left[ -\frac{1}{4(1 + x^2)^{\frac{3}{2}}} \right] \]

\[ = \left[ -\frac{1}{4(1 + 2^2)^{\frac{3}{2}}} \right] - \left[ -\frac{1}{4(1 + 0^2)^{\frac{3}{2}}} \right] \]

\[ = -\frac{1}{20} \frac{1}{2} + \frac{1}{4} \]

\[ = \frac{53}{50} \]

71. \[ \int_0^{\sqrt{7}} x \sqrt{\frac{1}{x^2} + 3} \, dx = 7 \int_0^{\sqrt{7}} x \sqrt{1 + x^2} \, dx \]

First find the indefinite integral.

\[ 7 \int x \sqrt{1 + x^2} \, dx \]

Let \( u = x + 1 \), then \( du = dx \).

\[ = \frac{7}{2} \int (2x + 2) \sqrt{1 + x^2} \, dx \]

\[ = \frac{7}{2} \left[ 2x \sqrt{1 + x^2} + \frac{7}{2} \frac{x^{1/2}}{1/3} + C \right] \]

\[ = \frac{7}{2} \left[ 2x \sqrt{1 + x^2} + \frac{21}{8} x^{1/2} + C \right] \]

\[ = \frac{7}{2} \left[ 2x \sqrt{1 + x^2} + \frac{21}{8} \sqrt{1 + x^2} \right] \]

Then evaluate the definite integral on [0, \( \sqrt{7} \)].

\[ \left. \right|_0^{\sqrt{7}} \left[ 2x \sqrt{1 + x^2} + \frac{21}{8} \sqrt{1 + x^2} \right] \]

\[ = \frac{21}{8} \frac{8}{8} + \frac{21}{8} \frac{8}{8} = \frac{21}{8} + \frac{21}{8} \cdot 8 \cdot \sqrt{2} \]

\[ = \frac{21}{8} \cdot 16 \frac{21}{8} \cdot 1 \]

\[ = 2 - \frac{21}{8} \]

\[ = \frac{315}{8} \]

73. a) \[ K \int_0^2 (H - x)^{5/2} \, dx \]

\[ \left. \right|_0^2 \left[ -\frac{2K}{5} (H - x)^{7/2} \right] \]

\[ = 0 - \frac{2K}{5} \]

\[ = \frac{2}{5} K H^{5/2} \]

b) \[ K \int_0^{H/2} (H - x)^{5/2} \, dx \]

\[ \left. \right|_0^{H/2} \left[ -\frac{2K}{5} (H - x)^{7/2} \right] \]

\[ = \frac{2K}{5} \frac{H^2}{2} - \frac{2K}{5} \frac{H^2}{2} \]

\[ = \frac{\sqrt{2}KH^{5/2} + 2KH^{5/2}}{20} \]

\[ = \frac{1}{20} \left( 8 - \sqrt{2} \right) K H^{5/2} \]

c) We divide the answers from the previous two parts

\[ \frac{\frac{1}{20} \left( 8 - \sqrt{2} \right) K H^{5/2}}{\frac{2}{5} K H^{5/2}} \]

\[ = \frac{\frac{8 - \sqrt{2}}{8}}{2} \]

\[ = 0.8232 \]

74. Upper half proportion is given by

\[ 1 - \frac{(8 - \sqrt{2})}{8} \sqrt{2} \]

\[ = 0.1768 \]

75. We have to break the integral into two parts

\[ \int_0^2 \left[ -x \sqrt{4 - x^2} - 0 \right] \, dx + \int_0^2 \left[ 0 - (-x \sqrt{4 - x^2}) \right] \, dx \]

\[ \left. \right|_0^2 \left[ \frac{1}{3} (1 - x^2)^{3/2} \right] \]

\[ = \frac{8}{3} - 0 - \frac{8}{3} \]

\[ = \frac{16}{3} \]

77. Using the hint.

\[ \int_1^2 \frac{t^2 + 2t}{(t + 1)^3} \, dt \]

\[ = \int_1^2 \frac{1}{(t + 1)^2} \, dt \]

\[ = \int_1^2 \frac{1}{(t + 1)^2} \, dt \]

\[ = \int_1^2 \frac{1}{(t + 1)^2} \, dt \]

\[ = C \]

\[ = C \]
79. Using the hint
\[
\int \frac{x + 3}{x + 1} \, dx = \int \left( 1 + \frac{2}{x + 1} \right) \, dx = x + 2\ln |x + 1| + C.
\]

81. Let \( u = \ln x \) then \( du = \frac{dx}{x} \)
\[
\int \frac{dx}{x} = \int u^{-n} \, du = \frac{-n + 1}{n-1} u + C = \ln |\ln x|^{-n+1} + C.
\]

83. Let \( u = \ln(\ln x) \) then \( du = \frac{dx}{x \ln x} \)
\[
\int \frac{dx}{\ln x} = \int du = \ln u + C = \ln |\ln(\ln x)| + C.
\]

85. Using the hint
\[
\int \sec z \, dz = \int \frac{\sec z \tan z + \sec^2 z}{\tan z + \sec z} \, dz
\]
Let \( u = \tan z + \sec z \) then \( du = \sec^2 z + \sec z \tan z \, dz \)
\[
\int \frac{du}{u} = \ln u + C = \ln |\sec z + \tan z| + C.
\]

87. Left to the student

Exercise Set 5.6

1. \( \int 5x e^{3x} \, dx = \int x(5e^{3x} \, dx) \)
Let
\[
u = x \quad \text{and} \quad dv = 5e^{3x} \, dx.
\]
Then \( du = dx \) and \( v = \frac{e^{3x}}{3} \).
\[
\int 5x e^{3x} \, dx = x \cdot e^{3x} - \int \frac{e^{3x}}{3} \, dx
\]
Using Theorem 7:
\[
\int u \, dv = uv - \int v \, du
\]
\[
x e^{3x} - \frac{1}{3} e^{3x} + C
\]

3. \( \int x \sin x \, dx \)
Let
\[
u = x \quad \text{and} \quad dv = \sin x \, dx.
\]
Then \( du = dx \) and \( v = -\cos x \)
\[
\int x \sin x \, dx = x \cdot -\cos x + \int \cos x \, dx
\]
\[
= -x \cos x + \sin x + C
\]

5. \( \int xe^{2x} \, dx \)
Let
\[
u = x \quad \text{and} \quad dv = e^{2x} \, dx.
\]
Then \( du = dx \) and \( v = \frac{1}{2} e^{2x} \)
\[
\int xe^{2x} \, dx = x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx
\]
\[
= \frac{1}{2} e^{2x} - \frac{1}{4} e^{2x} + C
\]

7. \( \int xe^{-2x} \, dx \)
Let
\[
u = x \quad \text{and} \quad dv = e^{-2x} \, dx.
\]
Then \( du = dx \) and \( v = -\frac{1}{2} e^{-2x} \).
\[
\int xe^{-2x} \, dx = x \cdot \left(-\frac{1}{2} e^{-2x}\right) - \int \left(-\frac{1}{2} e^{-2x}\right) \, dx
\]
\[
= \frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} + C
\]
\[
= -\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} + C
\]

9. \( \int e^{2x} \ln x \, dx = \int (\ln x) e^{2x} \, dx \)
Let
\[
u = \ln x \quad \text{and} \quad dv = e^{2x} \, dx.
\]
Then \( du = \frac{dx}{x} \) and \( v = \frac{e^{2x}}{2} \).
\[
\int (\ln x) e^{2x} \, dx = \ln x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot \frac{1}{x} \, dx
\]
Integration by Parts
\[
= \frac{e^{2x}}{3} \ln x - \frac{1}{3} \int e^{2x} \, dx
\]
\[
= \frac{e^{2x}}{3} \ln x - \frac{1}{3} \cdot \frac{e^{2x}}{2} + C
\]
\[
= \frac{e^{2x}}{3} \ln x - \frac{e^{2x}}{6} + C
\]

11. \( \int x \ln x \, dx \)
Let
\[
u = x \quad \text{and} \quad dv = \ln x \, dx.
\]
Then \( du = \frac{dx}{x} \) and \( v = \ln x \).
\[
\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx
\]
\[
= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C
\]
\[
= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C
\]
13. \( \int \ln(x + 3) \, dx \)

Let 
\( u = \ln(x + 3) \) and \( dv = dx \).

Then \( du = \frac{1}{x + 3} \, dx \) and \( v = x + 3 \).

\[
\int \ln(x + 3) \, dx = \ln(x + 3)(x + 3) - \int (x + 3) \cdot \frac{dx}{x + 3}
\]
\[
= (x + 3) \ln(x + 3) - x + C.
\]

15. \( \int (x + 2) \ln x \, dx = \int \{\ln x\}(x + 2) \, dx \)

Let 
\( u = \ln x \) and \( dv = (x + 2) \, dx \).

Then 
\( du = \frac{1}{x} \, dx \) and \( v = \frac{(x + 2)^2}{2} \).

\[
\int (x + 2) \ln x \, dx = \left( \ln x \right) \frac{(x + 2)^2}{2} - \int \frac{(x + 2)^2}{2} \cdot \frac{1}{x} \, dx
\]
\[
= \frac{x^2 + 4x + 4}{2} \ln x - \int \frac{x^2 + 4x + 4}{2x} \, dx
\]
\[
= \frac{x^2 + 4x + 4}{2} \ln x - \int \frac{x^2}{2x} \ln x + 2 \int \frac{1}{x} \, dx
\]
\[
= \frac{x^2 + 4x + 4}{2} \ln x - \frac{x^2}{2} - 2 \ln x + 2 - \frac{1}{x} + C.
\]

19. \( \int x\sqrt{x^2 + 2} \, dx \)

Let 
\( u = x \) and \( dv = \sqrt{x^2 + 2} \, dx \).

Then 
\( du = dx \) and 
\( v = \frac{1}{3} \ln(x^2 + 2) \).

\[
\int x\sqrt{x^2 + 2} \, dx = \frac{x}{3} \ln(x^2 + 2) - \int \frac{1}{3} \ln(x^2 + 2) \, dx
\]
\[
= \frac{x}{3} \ln(x^2 + 2) - \frac{1}{15} (x^2 + 2)^{3/2} + C.
\]

21. \( \int x^3 \ln 2x \, dx = \int \{\ln 2x\}(x^3) \, dx \)

Let 
\( u = \ln 2x \) and \( dv = x^3 \, dx \).

Then 
\( du = \frac{1}{x} \, dx \) and \( v = \frac{x^4}{4} \).

\[
\int \{\ln 2x\}(x^3) \, dx = \left( \ln 2x \right) \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} \, dx
\]
\[
= \frac{x^4}{4} \ln 2x - \frac{1}{4} \int x^3 \, dx
\]
\[
= \frac{x^4}{4} \ln 2x - \frac{1}{15} x^4 + C.
\]

22. \( \int x^2 e^x \, dx \)

Let 
\( u = x^2 \) and \( dv = e^x \, dx \).

Then 
\( du = 2x \, dx \) and \( v = e^x \).

\[
\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx
\]
\[
= x^2 e^x - 2xe^x + C.
\]

We evaluate \( \int 2xe^x \, dx \) using the Integration by Parts formula.

\[
\int 2xe^x \, dx
\]

Let 
\( u = 2x \) and \( dv = e^x \, dx \).

Then 
\( du = 2 \, dx \) and \( v = e^x \).

\[
\int 2xe^x \, dx = 2x e^x - \int 2e^x \, dx
\]
\[
= 2xe^x - 2e^x + K.
\]
Thus,
\[ \int x^3 e^x \, dx = x^2 e^x - (2x e^x - 2e^x + K) \]
\[ = x^2 e^x - 2x e^x + 2e^x + C \quad (C = -K) \]

Since we have an integral \( \int f(x)g(x) \, dx \) where \( f(x) \), or \( x^3 \), can be differentiated repeatedly to a derivative that is eventually 0 and \( g(x) \), or \( e^x \), can be integrated repeatedly easily, we can use tabular integration.

<table>
<thead>
<tr>
<th>( f(x) ) and repeated derivatives</th>
<th>( g(x) ) and repeated integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 )</td>
<td>( + )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( - )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( + )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( e^x )</td>
</tr>
</tbody>
</table>

We add the products along the arrows, making the alternate sign changes.

\[ \int x^3 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C \]

25. \( \int x^2 \sin 2x \, dx \)

Let
\[ u = x^2 \quad \text{and} \quad dv = \sin 2x \, dx. \]
Then \( du = 2x \, dx \) and \( v = -\frac{1}{2} \cos 2x. \)

\[
\int x^2 \sin 2x \, dx = x^2 \left( -\frac{\cos 2x}{2} \right) \left( -\frac{1}{2} \right) \left( \cos 2x \right) \, dx
\]

27. \( \int x e^{-2x} \, dx \)

We will use tabular integration.

<table>
<thead>
<tr>
<th>( f(x) ) and repeated derivatives</th>
<th>( g(x) ) and repeated integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( -\frac{1}{2} e^{-2x} )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( -\frac{1}{4} e^{-2x} )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( -\frac{1}{8} e^{-2x} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \frac{1}{16} e^{-2x} )</td>
</tr>
</tbody>
</table>

We add the products along the arrows, making the alternate sign changes.

\[ \int x^2 \sin 2x \, dx = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{2} \cos 2x + C \]

29. \( \int x \sec^2 x \, dx \)

Let
\[ u = x \quad \text{and} \quad dv = \sec^2 x \, dx. \]
Then
\[ du = dx \quad \text{and} \quad v = \tan x. \]

\[
\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx
\]
\[ = x \tan x + \ln |\cos x| + C \]
31. \( \int_1^2 x^2 \ln x \, dx \)

In Exercise 9 above we found the indefinite integral.

\[
\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C
\]

Evaluate the definite integral.

\[
\int_1^2 x^2 \ln x \, dx = \left[ \frac{x^3}{3} \ln x - \frac{x^3}{9} \right]_1^2
\]

\[- \left( \frac{2^3}{3} \ln 2 - \frac{2^3}{9} \right) - \left( \frac{1^3}{3} \ln 1 - \frac{1^3}{9} \right)\]

\[- \left( \frac{8}{3} \ln 2 - \frac{8}{9} \right) - \left( \frac{1}{3} \ln 1 - \frac{1}{9} \right)\]

\[- \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9} \quad (\ln 1 = 0)\]

\[- \frac{8}{3} \ln 2 - \frac{7}{9}\]

33. \( \int_1^3 \ln(x + 1) \, dx \)

In Exercise 13 above we found the indefinite integral.

\[
\int \ln(x + 1) \, dx = (x + 1) \ln (x + 1) - x + C
\]

Evaluate the definite integral.

\[
\int_1^3 \ln(x + 1) \, dx = \left[ (x + 1) \ln (x + 1) - x \right]_1^3
\]

\[- \left( (3 + 1) \ln (3 + 1) - 3 \right) - \left( (2 + 1) \ln (2 + 1) - 2 \right)\]

\[- \left( 4 \ln 4 - 4 \right) - \left( 3 \ln 3 - 3 \right)\]

\[- 4 \ln 4 + 4 + 3 \ln 3 - 3\]

\[- 4 \ln 4 + 3 \ln 3 + 1\]

35. a) We first find the indefinite integral.

\[
\int x e^x \, dx
\]

Let

\[ v = x\, \text{and} \, dv = e^x \, dx\]

Then

\[ du = dx \, \text{and} \, u = e^x\]

\[
\int x e^x \, dx = x e^x - \int e^x \, dx
\]

\[ - e^x + C\]

b) Evaluate the definite integral.

\[
\int_0^1 x e^x \, dx = \left[ x e^x - e^x \right]_0^1
\]

\[- (1 - 1) - (0 - 0)\]

\[- 0 - (0)\]

\[- 0\]

Thus,

\[
\int_0^1 3x \cos x \, dx = 3x \sin x - \int 3 \sin x \, dx
\]

\[- 3 \cos x + C\]

39. \( M(t) = 10t \sqrt{t + 15} \)

Let \( u = t \) and \( dv = \sqrt{t + 15} \, dt \)

Then \( du = dt \) and \( v = \frac{2}{3}(t + 15)^{3/2} \)

\[
M(t) = \int 10t \sqrt{t + 15} \, dt
\]

\[- 20 \int \frac{1}{3} (t + 15)^{3/2} - 20 \int (t + 15)^{3/2} \, dt\]

\[- \frac{20}{3} (t + 15)^{3/2} - \frac{8}{3} (t + 15)^{3/2} + C\]

\[- 150000 = 0 - \frac{8}{3} (15)^{3/2} + C\]

\[- C = 150000 + \frac{8(15)^{3/2}}{3}\]

\[ M(10) = \frac{20}{3} (10 + 15)^{3/2} - \frac{8}{3} (15)^{3/2} = 150000 + \frac{8(15)^{3/2}}{3}\]

\[- \frac{250000}{3} \quad \frac{25000}{3} \quad \frac{150000}{3} \quad \frac{8(15)^{3/2}}{3}\]

\[- 152524\]

41. a) We first find the indefinite integral.

\[
\int 10 e^{-t} \, dt = 10 \int e^{-t} \, dt
\]

Let

\[ u = -t \, \text{and} \, du = -e^{-t} \, dt\]

Then

\[ du = dt \, \text{and} \, u = -t\]

\[ 10 \int e^{-t} \, dt = 10 \left[ e^{-t} - e^{-t} \right] + K\]

\[ = 10(-e^{-t}) + C \quad (C = 10K)\]

Then evaluate the definite integral.

\[
\int_0^T e^{-t} \, dt = [1 - 10e^{-t} - 10e^{-t} T + 10e^{-t} T]_0^T\]

\[ = (10K) - 10e^{-T} + 10e^{-T} T - 10e^{-T} T + 10e^{-T} + 10e^{-T} T + 10e^{-T} T \]

\[ = 10(-e^{-T} - e^{-T} + 1) \text{ or } 10(e^{-T} - e^{-T} + 1)\]
b) Substitute 4 for \( T \),

\[ \int_0^1 te^{-t} \, dt = -10[e^{-t}(1+t) - 1] = -50e^{-1} + 10 \approx -50(0.0018316) + 10 \approx -0.915800 + 10 \approx 9.084 \]

43. Left to the student.

45. Let \( u = \ln x \) and \( dv = x^{1/2} \)

Then \( du = \frac{dx}{x} \) and \( v = \frac{2}{3} x^{3/2} \)

\[ \int \sqrt{x} \ln x \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C \]

47. Let \( u = t^2 \) and \( dv = \frac{1}{(t+1)^2} \, dt \)

Then \( du = (2t) \, dt \) and \( v = -\frac{1}{t+1} \)

\[ \int \frac{t^2}{(t+1)^2} \, dt = -\frac{t^2}{t+1} + \int \frac{2t}{t+1} \, dt + \int \frac{1}{t+1} \, dt \]

For the first integral on the right hand side

Let \( u = \frac{t}{t+1} \) and \( dv = t^2 \, dt \)

Then \( du = \frac{1}{(t+1)^2} \, dt \) and \( v = \frac{1}{3} t \)

\[ \int \frac{t^2}{(t+1)^2} \, dt = -\frac{t^2}{t+1} + \int \frac{2t}{t+1} \, dt + \frac{1}{3} t \]

For the second integral on the right hand side

Let \( u = \frac{1}{t+1} \) and \( dv = t^2 \, dt \)

Then \( du = -\frac{1}{(t+1)^2} \, dt \) and \( v = \frac{1}{3} t \)

\[ \int \frac{t^2}{(t+1)^2} \, dt = -\frac{t^2}{t+1} + \int \frac{2t}{t+1} \, dt + \frac{1}{3} t \]

49. Let \( u = \ln x \) and \( dv = x^{-1/2} \)

Then \( du = \frac{dx}{x} \) and \( v = 2x^{1/2} \)

\[ \int \frac{\ln x}{x^{1/2}} \, dx = 2 \sqrt{x} \ln x - 2 \int x^{-1/2} \, dx = 2 \sqrt{x} \ln x - 4 \sqrt{x} + C \]

51. Let \( u = 13t^2 - 48 \) and \( dv = (4t + 7)^{-1/5} \)

Then \( du = 26t \, dt \) and \( v = \frac{5}{16} (4t + 7)^{4/5} \)

\[ \int \frac{13t^2 - 48}{\sqrt{4t + 7}} \, dt = \frac{5}{16} (13t^2 - 48)(4t + 7)^{4/5} - \frac{65}{8} t(4t + 7)^{4/5} \]

Let \( u = \frac{65}{8} t \) and \( dv = (4t + 7)^{4/5} \, dt \)

Then \( du = 65 \, dt \) and \( v = \frac{5}{36} (4t + 7)^{9/5} \int \frac{13t^2 - 48}{\sqrt{4t + 7}} \, dt = \frac{5}{16} (13t^2 - 48)(4t + 7)^{4/5} - \frac{65}{8} t(4t + 7)^{4/5} \]

53. Left to the student.

55. Left to the student.

57. Left to the student.

59. \[ \int_{10}^{15} x^5 \ln x \, dx = 355086.43 \]

Exercise Set 5.7

1. \[ \int x e^{-3x} \, dx \]

This integral fits Formula 6 in Table 1.

\[ \int x e^{ax} \, dx = \frac{a x e^{ax}}{a^2 + 1} + C \]

In our integral \( a = -3 \), so we have, by the formula,

\[ \int e^{-3x} \, dx = \frac{\frac{1}{3} e^{-3x}}{-3} + C \]

\[ = -\frac{1}{9} e^{-3x}(-3x + 1) + C \]

\[ \text{or} \quad -\frac{1}{9} e^{-3x}(3x + 1) + C \]

3. \[ \int 5^x \, dx \]

This integral fits Formula 11 in Table 1.

\[ \int a^x \, dx = \frac{a^x}{\ln a} + C, \quad a > 0, \quad a \neq 1 \]

In our integral \( a = 5 \), so we have, by the formula,

\[ \int 5^x \, dx = \frac{5^x}{\ln 5} + C \]

5. \[ \int \frac{1}{16 - x^2} \, dx \]

This integral fits Formula 27 in Table 1.

\[ \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln \left( \frac{a + x}{a - x} \right) + C \]

In our integral \( a = 4 \), so we have, by the formula,

\[ \int \frac{1}{16 - x^2} \, dx = \frac{1}{8} \ln \left( \frac{4 + x}{4 - x} \right) + C \]

\[ = \frac{1}{2} \ln \left( \frac{4 + x}{4 - x} \right) + C \]

\[ = \frac{1}{2} \ln \left( \frac{4 + x}{4 - x} \right) + C \]
7. \[ \int \frac{x}{5-x} \, dx \]

This integral fits Formula 30 in Table 1.

\[ \int \frac{x}{a+x} \, dx = \frac{b}{a^2} \ln \left( \frac{a+x}{b} \right) + C \]

In our integral, \( a = -1 \) and \( b = 5 \), so we have, by the formula,

\[ \int \frac{x}{5-x} \, dx = -\frac{5}{(-1)^2} \ln\left(\frac{-1}{5}\right) + C = -5 \ln (5-x) + C \]

9. \[ \int \frac{1}{x(5-x)^2} \, dx \]

This integral fits Formula 34 in Table 1.

\[ \int \frac{1}{x(a+x)^2} \, dx = -\frac{1}{b(a+x)} + \frac{1}{b^2} \ln \left( \frac{x}{a} \right) + C \]

In our integral, \( a = -1 \) and \( b = 5 \), so we have, by the formula,

\[ \int \frac{1}{x(5-x)^2} \, dx = -\frac{1}{5(-1)} + \frac{1}{25} \ln\left(\frac{-1}{5}\right) + C = -\frac{1}{5} \ln (5-x) + C \]

11. \[ \int \ln 3 \, dx \]

\[ \int \ln 3 \, dx = \int \ln x \, dx + \int \ln 3 \, dx \]

The integral in the second term fits Formula 8 in Table 1.

\[ \int \ln x \, dx = x \ln x - x + C \]

13. \[ \int x^3 e^{5x} \, dx \]

This integral fits Formula 7 in Table 1.

\[ \int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \]

In our integral, \( n = 3 \) and \( a = 5 \), so we have, by the formula,

\[ \int x^3 e^{5x} \, dx = \frac{x^3 e^{5x}}{5} - \frac{3}{5} \int x^2 e^{5x} \, dx \]

We continue to apply Formula 7.

\[ = \frac{x^3 e^{5x}}{5} - \frac{4}{25} x^2 e^{5x} + \frac{12}{125} x e^{5x} - \frac{2}{5} \int x e^{5x} \, dx \]

\[ = \frac{x^3 e^{5x}}{5} - \frac{4}{25} x^2 e^{5x} + \frac{12}{125} x e^{5x} - \frac{21}{125} \int x e^{5x} \, dx \]

\[ = \frac{x^3 e^{5x}}{5} - \frac{4}{25} x^2 e^{5x} + \frac{12}{125} x e^{5x} - \frac{21}{625} e^{5x} + C \]

15. \[ \int x^3 \sin x \ln x \, dx \]

This integral fits Formula 22 in Table 1.

\[ \int x^3 \sin x \, dx = -\cos x + 3 \int x^2 \cos x \, dx \]

\[ = -x^3 \cos x + 3 \left[ x^2 \sin x - 2 \int x \sin x \, dx \right] \]

\[ = -x^3 \cos x + 3 x^2 \sin x - 6 \left[ -x \cos x + \sin x \right] + C \]

\[ = -x^3 \cos x + 3 x^2 \sin x + 6x \cos x + 6 \sin x + C \]

17. \[ \int \sec 2x \, dx \]

This integral fits Formula 16 in Table 1.

\[ \int \sec 2x \, dx = \frac{1}{2} \ln | \sec 2x + \tan 2x | + C \]

19. \[ \int x \tan(2x + 1) \, dx \]

Let \( u = 2x + 1 \) then \( du = 2 \, dx \)

This integral fits Formula 11 in Table 1.

\[ \int \tan u \, du = \ln | \cos u | + C \]

\[ = -\ln | \cos(2x + 1) | + C \]

21. \[ \int \frac{dx}{\sqrt{x^2 + 7}} \]

This integral fits Formula 24 in Table 1.

\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln (x + \sqrt{x^2 + a^2}) + C \]

In our integral, \( a^2 = 7 \), so we have, by the formula,

\[ \int \frac{dx}{\sqrt{x^2 + 7}} = \ln (x + \sqrt{x^2 + 7}) + C \]
23. \[ \int \frac{10}{x(5-7x)} \, dx = 10 \int \frac{1}{x(-7x + 5)} \, dx \]
This integral fits Formula 35 in Table 1.
\[ \int \frac{1}{x(5-7x + 6)} \, dx = \frac{1}{b^2} \ln \left( \frac{x}{5-7x + 6} \right) + C \]
In our integral \( a = -7 \) and \( b = 5 \), so we have, by the formula,
\[ = 10 \int \frac{1}{x(-7x + 5)} \, dx \]
\[ = 10 \int \frac{1}{5(-7x + 5)} \, dx + \frac{2}{5} \ln \left( \frac{x}{5-7x + 5} \right) + C \]
\[ = \frac{2}{5} \ln \left( \frac{x}{5-7x + 5} \right) + C \]

25. \[ \int \frac{-5}{4x^2 - 1} \, dx = 5 \int \frac{1}{4x^2 - 1} \, dx \]
This integral almost fits Formula 26 in Table 1.
\[ \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left( \frac{x-a}{x+a} \right) + C \]
But the \( x^2 \) coefficient needs to be 1. We factor out 4 as follows. Then we apply Formula 26.
\[ = 5 \int \frac{1}{4x^2 - 1} \, dx \]
\[ = \frac{5}{4} \int \frac{1}{(x^2 - \frac{1}{4})} \, dx \]
\[ = \frac{5}{4} \int \frac{1}{x^2 - \frac{1}{4}} \, dx \]
\[ = \frac{5}{4} \frac{1}{\sqrt{2}} \ln \left( \frac{x + \frac{1}{2}}{x - \frac{1}{2}} \right) + C \]
\[ = \frac{5}{4} \ln \left( \frac{x^{1/2}}{x^{1/2}} \right) + C \]

27. \[ \int \sqrt{3m^2 + 16 \, dm} \]
This integral almost fits Formula 34 in Table 1.
\[ \int \sqrt{x^2 + a^2} \, dx = \frac{1}{2} \left[ x \sqrt{x^2 + a^2} + a^2 \ln (x + \sqrt{x^2 + a^2}) \right] + C \]
But the \( x^2 \) coefficient needs to be 1. We factor out 4 as follows. Then we apply Formula 34.
\[ \int \sqrt{4m^2 + 16 \, dm} \]
\[ = 2 \int \sqrt{m^2 + 4} \, dm \]
\[ = 2 \left[ \frac{m \sqrt{m^2 + 4}}{2} + 4 \ln (m + \sqrt{m^2 + 4}) \right] + C \]
\[ = m \sqrt{m^2 + 4} + 4 \ln (m + \sqrt{m^2 + 4}) + C \]

29. \[ \int \frac{-5 \ln x}{x^3} \, dx = 5 \int \frac{1}{x^2 - 1} \ln x \, dx \]
This integral fits Formula 38 in Table 1.
\[ \int x^n \ln x \, dx = x^{n+1} \left[ \frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right] + C, \, n \neq -1 \]
In our integral \( n = -3 \) so we have, by the formula,
\[ -5 \int \frac{1}{x^3} \ln x \, dx \]
\[ = -5 \left[ \frac{x^{-3+1}}{-3+1} \left( \frac{\ln x}{-3+1} - \frac{1}{(-3+1)^2} \right) \right] + C \]
\[ = -5 \left[ \frac{x^{-2} \ln x}{-2 - 4} \right] + C \]
\[ = \frac{5 \ln x}{2x^2} + \frac{5}{4x^2} + C \]

31. \[ \int \frac{e^x}{x-3} \, dx = \int x^3 \, e^x \, dx \]
This integral fits Formula 7 in Table 1.
\[ \int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \]
In our integral \( n = 3 \) and \( a = 1 \), so we have, by the formula,
\[ \int x^3 e^x \, dx \]
\[ = x^3 e^x - 3 \int x^2 e^x \, dx \]
We continue to apply Formula 7.
\[ = x^3 e^x - 3 \left( x^2 e^x - 2 \int x e^x \, dx \right) \]
\[ = x^3 e^x - 3 x^2 e^x + 6 \int x e^x \, dx \]
\[ = x^3 e^x - 3 x^2 e^x + 6 \left( x e^x - \int e^x \, dx \right) \]
\[ = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 \int e^x \, dx \]
\[ = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x + C \]

33. \[ \int \frac{x \sqrt{1 + 2x}}{x} \, dx \]
This integral fits Formula 35 in Table 1.
\[ \int x \sqrt{a + bx} \, dx = \frac{2}{15a^2} \left( 3ax - 2a \right) (a + bx)^{3/2} + C \]
In our integral \( a = 1 \) and \( b = 2 \).
\[ \int x \sqrt{1 + 2x} \, dx \]
\[ = \frac{2}{15} \left( 3 \cdot 2x - 2 \cdot 1 \right) (1 + 2x)^{3/2} + C \]
\[ = \frac{2}{60} \left( 6x - 2 \right) (1 + 2x)^{3/2} + C \]
\[ = \frac{1}{30} \cdot 2(3x - 1)(1 + 2x)^{3/2} + C \]
\[ = \frac{1}{15} (3x - 1)(1 + 2x)^{3/2} + C \]

35. \[ \int (\ln x)^4 \, dx \]
\[ = -x \ln^3 x - 4x \ln^2 x + 12x \ln x - 24x \ln x + 24x + C \]

37. \[ \int \frac{1}{x(x^2 - 1)} \, dx \]
\[ = \frac{1}{2} \ln \left| x^2 - 1 \right| - \ln | x | + C \]
39. \[ \int x^3 \sin 3x \, dx \]

\[ = -\frac{1}{3} x^4 \cos 3x + \frac{4}{9} x^3 \sin 3x + C \]

\[ + \frac{1}{9} e^x \cos 3x - \frac{8}{9} \cos 3x - \frac{8}{27} e^x \sin 3x + C \]

41. \[ \int e^{2x} \sin 3x \, dx \]

\[ = -\frac{1}{13} e^{2x} \cos 3x + \frac{2}{13} e^{2x} \sin 3x + C \]

43. a) Trapezoid: 0.742984098
b) Simpson: 0.7498538

45. a) Trapezoid: 0.443975605
b) Simpson: 0.447139901

47. a) Trapezoid: 1.503577487
b) Simpson: 1.507173088

49. a) Trapezoid: 0.270958739
b) Simpson: 0.270918581

51. \[ p(t) = \frac{1}{4[(2 + t)^2]} \] The integral follows formula number 33 in Table 1 with \( a = 1 \) and \( b = 2 \)

\[ p(t) = \frac{1}{4} \int_1^2 \frac{1}{(2 + t)^2} dt \]

\[ = \left[ \frac{3}{2} \ln |2 + t| \right]_1^2 \]

\[ = 0.8267 \]

53. Using a spreadsheet and following Example 6, the number of degree days between 9:00 P.M. July 5 and 9:00 P.M. July 7, 2004 are 17.

55. \[ \int \frac{x}{x^2 - 12x + 9} \, dx \]

Using formula number in Table 1 with \( a = 2 \) and \( b = -3 \)

\[ \int \frac{x}{(x^2 - 3x - 9)} \]

\[ = \frac{-3}{4(3 - 2x)} \left| \frac{1}{3} \ln \left| \frac{x}{2x - 3} \right| \right. + C \]

57. \[ \int e^x \sqrt{e^{2x} + 1} \, dx \]

Let \( y = e^x \) then \( dy = e^x \, dx \)

Using formula number in Table 1 with \( a = 1 \)

\[ \int e^x \sqrt{e^{2x} + 1} \, dx = \int \sqrt{y^2 + 1} \, dy \]

\[ = \ln \left| y + \sqrt{y^2 + 1} \right| + C \]

59. Let \( y = \ln x \) then \( dy = \frac{dx}{x} \)

Using formula number in Table 1 with \( a = 7 \)

\[ \int \frac{\sqrt{[\ln(x)]^2 + 49}}{2x} \, dx \]

\[ = \frac{1}{2} \int \sqrt{y^2 + 49} \, dy \]

\[ = \frac{1}{2} \ln \left| y + \sqrt{y^2 + 49} \right| + C \]

61. a) \[ \int_{-h}^{h} 4x^3 + b x^2 + c x \, dx \]

\[ = \left[ \frac{4h^4}{3} + \frac{bh^3}{2} + \frac{cx^2}{2} \right]_{-h}^{h} \]

\[ = \frac{2bh^3}{3} + 2ch \]

b) \[ f(h) = ah^3 + bh^2 + ch + d \]

\[ f(0) = d \]

\[ f(-h) = -ah^3 + bh^2 - ch + d \]

\[ c \int_{-h}^{h} 4x^3 + b x^2 + c x \, dx \]

\[ = \frac{2bh^3}{3} + 2ch \]

\[ = \frac{h}{3} \left[ 2bh^2 + d \right] \]

\[ = \frac{h}{3} \left[ \left| -ah^3 + bh^2 - ch + d + 4d + d + ch + bh^2 + ah \right| \right] \]

\[ = \frac{h}{3} \left| f(-h) - 4f(0) + f(h) \right| \]

d) Left to the student (answers vary)
Exercise Set 5.8

1. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = x$ from $x = 0$ to $x = 1$.

$$V = \int_a^b \pi x^2 \, dx$$  
Volume of a solid of revolution

Substituting 0 for $a$, 1 for $b$, and $x$ for $f(x)$

$$= \pi \left[ \frac{x^3}{3} \right]_0^1$$

$$= \pi \left[ \frac{1}{3} - 0 \right]$$

$$= \frac{\pi}{3}$$

3. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = \sqrt{\sin x}$ from $x = 0$ to $x = \frac{\pi}{2}$.

$$V = \int_0^{\frac{\pi}{2}} \pi (\sin x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{2}} \pi \sin x \, dx$$

$$= \left[ \cos x \right]_0^{\frac{\pi}{2}}$$

$$= \cos \left( \frac{\pi}{2} \right) - \cos 0$$

$$= \frac{2}{3} \pi$$

5. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = e^x$ from $x = -2$ to $x = 5$.

$$V = \int_a^b \pi (e^x)^2 \, dx$$  
Volume of a solid of revolution

Substituting $-2$ for $a$, 5 for $b$, and $e^x$ for $f(x)$

$$= \int_{-2}^5 \pi e^{2x} \, dx$$

$$= \pi \left[ e^{2x} \right]_{-2}^5$$

$$= \pi \left[ e^{10} - e^{-4} \right]$$

7. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = \frac{1}{x}$ from $x = 1$ to $x = 3$.

$$V = \int_1^3 \pi \left( \frac{1}{x} \right)^2 \, dx$$  
Volume of a solid of revolution

Substituting 1 for $a$, 3 for $b$, and $\frac{1}{x}$ for $f(x)$

$$= \int_1^3 \pi \frac{1}{x^2} \, dx$$

$$= \int_1^3 \pi e^{-2x} \, dx$$

$$= \left[ \frac{1}{e^x} \right]_1^3$$

$$= \frac{1}{e^3} - \frac{1}{e}$$

$$= \frac{1}{e} \left( -\frac{2}{3} \right)$$

9. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = \frac{2}{\sqrt{x}}$ from $x = 1$ to $x = 3$.

$$V = \int_1^3 \pi \left( \frac{2}{\sqrt{x}} \right)^2 \, dx$$  
Volume of a solid of revolution

Substituting 1 for $a$, 3 for $b$, and $\frac{2}{\sqrt{x}}$ for $f(x)$

$$= \int_1^3 \pi \frac{4}{x} \, dx$$

$$= 4 \pi \ln x \bigg|_1^3$$

$$= 4 \pi (\ln 3 - \ln 1)$$

$$= 12 \pi$$

11. Find the volume of the solid of revolution generated by rotating about the $x$-axis the region under the graph of $y = 4$ from $x = 1$ to $x = 3$.

$$V = \int_1^3 \pi \left( \frac{4}{x} \right)^2 \, dx$$  
Volume of a solid of revolution

Substituting 1 for $a$, 3 for $b$, and 4 for $f(x)$

$$= \int_1^3 16 \pi \, dx$$

$$= 16 \pi \ln x \bigg|_1^3$$

$$= 16 \pi (3 - 1)$$

$$= 32 \pi$$
13. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = x^2 \]
from \( x = 0 \) to \( x = 2 \).
\[ V = \int_0^2 \pi [f(x)]^2 \, dx \quad \text{Volume of a solid of revolution} \]
\[ V = \int_0^2 \pi \left[ x^2 \right]^2 \, dx \quad \text{Substituting 0 for a,} \]
\[ = \left[ \pi \cdot \frac{5}{3} \right]_0^2 \]
\[ = \frac{32}{5} \pi \]

15. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = \cos x \]
from \( x = 0 \) to \( x = \frac{\pi}{2} \).
\[ V = \int_0^{\pi/2} \pi [f(x)]^2 \, dx \quad \text{Volume of a solid of revolution} \]
\[ V = \int_0^{\pi/2} \pi \cos^2 x \, dx \]
\[ = \left[ \frac{1}{2} x + \frac{1}{2} \sin x \cos x \right]_0^{\pi/2} \]
\[ = \frac{\pi}{2} \left( \frac{\pi}{2} + 0 - 0 \right) \]
\[ = \frac{\pi^2}{4} \]

17. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = \tan x \]
from \( x = 0 \) to \( x = \frac{\pi}{4} \).
\[ V = \int_0^{\pi/4} \pi [f(x)]^2 \, dx \quad \text{Volume of a solid of revolution} \]
\[ V = \int_0^{\pi/4} \pi \tan^2 x \, dx \]
\[ = \int_0^{\pi/4} \pi \left[ \tan x + x \right] \, dx \]
\[ = \left[ \pi \left( \tan x + x \right) \right]_0^{\pi/4} \]
\[ = \pi + \frac{\pi^2}{4} \]

19. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = \sqrt{4 - x^2} \]
from \( x = 2 \) to \( x = 10 \).
\[ V = \int_2^{10} \pi \left( \sqrt{4 - x^2} \right)^2 \, dx \quad \text{Volume of a solid of revolution} \]
\[ V = \int_2^{10} \pi \left( \frac{4 - (x^2)}{2} \right) \, dx \quad \text{Substituting 2 for a,} \]
\[ = \left[ \pi \cdot \frac{(1 + x)^2}{2} \right]_2^{10} \]
\[ = \frac{\pi}{2} [1 + 99] \]
\[ = \frac{\pi}{2} [100 - 10] \]
\[ = \frac{\pi}{2} [90] \]
\[ = \frac{90 \pi}{2} \]

21. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = \sqrt{4 - x^2} \]
from \( x = 2 \) to \( x = 10 \).
\[ V = \int_2^{10} \pi \left( \sqrt{4 - x^2} \right)^2 \, dx \quad \text{Volume of a solid of revolution} \]
\[ V = \int_2^{10} \pi \left( \frac{4 - x^2}{2} \right) \, dx \quad \text{Substituting 2 for a,} \]
\[ = \left[ \pi \cdot \frac{(1 + x)^2}{2} \right]_2^{10} \]
\[ = \frac{\pi}{2} [100 - 10] \]
\[ = \frac{\pi}{2} [90] \]
\[ = \frac{90 \pi}{2} \]

23. Find the volume of the solid with cross-sectional area
\[ A(x) = \frac{1}{2} x^2 \]
from \( x = 0 \) to \( x = 6 \).
25. Find the volume of the solid with cross-sectional area
\[ A(x) = \frac{\sqrt{3}}{2} x^2 \]
from \( x = 0 \) to \( x = 9 \).

\[
V = \int_0^9 A(x) \, dx
\]
\[
= \int_0^9 \left( \frac{\sqrt{3}}{2} x^2 \right) \, dx
\]
\[
= \left[ \frac{\sqrt{3}}{6} x^3 \right]_0^9
\]
\[
= \left( \frac{243 \sqrt{3}}{2} \right) - 0
\]
\[
= \frac{243 \sqrt{3}}{2}
\]

27. \( A(x) = x^4 \)
from \( x = 0 \) to \( x = 4 \).

\[
V = \int_0^4 A(x) \, dx
\]
\[
= \int_0^4 (x^4) \, dx
\]
\[
= \left[ \frac{1}{5} x^5 \right]_0^4
\]
\[
= \frac{1024}{5} - 0
\]
\[
= \frac{1024}{5}
\]

29. \( A(x) = 2x^2 \)
from \( x = 0 \) to \( x = 5 \).

\[
V = \int_0^b A(x) \, dx
\]
\[
= \int_0^5 (2x^2) \, dx
\]
\[
= \left[ \frac{2}{3} x^3 \right]_0^5
\]
\[
= \frac{250}{3} - 0
\]
\[
= \frac{250}{3}
\]

31. \( 10 = K \cdot 8 \rightarrow K = \frac{5}{4} \) (See Example 4 on page 110 as a reference)

\[
A(x) = \frac{25}{16} x^2
\]
from \( x = 0 \) to \( x = 8 \).

\[
V = \int_0^b A(x) \, dx
\]
\[
= \int_0^8 \left( \frac{25}{16} x^2 \right) \, dx
\]
\[
= \left[ \frac{25}{48} x^3 \right]_0^8
\]
\[
= \frac{800}{3} - 0
\]
\[
= \frac{800}{3}
\]

33. For \( r = 1.5 \) when \( H = 75 \) and \( x = 0 \), \( K = 38.2285 \)
Thus, \( r(x) = 0.05886(75 - x)^{3/4} \)

\[
V = \int_0^{75} \pi [r(x)]^2 \, dx
\]
\[
= \int_0^{75} \pi \left( 0.05886(75 - x)^{3/4} \right)^2 \, dx
\]
\[
= \int_0^{75} \pi \left( 0.003464(75 - x)^{3/2} \right) \, dx
\]
\[
= -0.003464 \left[ \frac{2}{5} (75 - x)^{5/2} \right]_0^{75}
\]
\[
= -0.003464 \left( 0 - \frac{-2}{5} \right) \frac{75^{5/2}}{5}
\]
\[= -212.058 \]

35. Find the volume of the solid of revolution generated by rotating about the x-axis the region under the graph of
\[ y = \sqrt{\ln x} \]
from \( x = e \) to \( x = e^3 \).

\[
V = \int_0^b \pi [f(x)]^2 \, dx
\]
\[
= \int_e^{e^3} \pi \left[ \sqrt{\ln x} \right]^2 \, dx
\]
\[
= \int_e^{e^3} \pi \ln x \, dx
\]
\[
= \pi \left[ \ln x \cdot x \right]_e^{e^3}
\]
\[
= \pi (3e - e - x \ln x)
\]
\[
= \pi (3e^3 - e^3 - (e^3 - e))
\]
\[-2ae^3 \]

37. a) The resulting solid of revolution is a cone with a base radius of \( r \) and a height of \( h \)

b) \[ V = \pi \int_0^{b} y^2 \, dx \]
\[
= \pi \int_0^{b} \frac{r^2}{h^2} x^2 \, dx
\]
\[
= \frac{\pi r^2}{3h^2} \left[ x^3 \right]_0^h
\]
Exercise Set 5.9

1. \[
\int_0^\infty \frac{dx}{x^2} = \lim_{b \to \infty} \int_0^b x^{-2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_0^b = -1 - \frac{1}{b} + 0 = \frac{1}{b} - 1
\]

The limit does exist. Thus the improper integral is convergent.

3. \[
\int_1^\infty \frac{dx}{x} = \lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln x \bigg|_1^b = \lim_{b \to \infty} \ln(b - \ln 4)
\]

Note that \(\ln b\) increases indefinitely as \(b\) increases. Therefore, the limit does not exist. If the limit does not exist, we say the improper integral is divergent.

5. \[
\int_{-\infty}^{-1} \frac{dt}{t^2} = \lim_{b \to -\infty} \int_b^{-1} t^{-2} \, dt = \lim_{b \to -\infty} \left[ -\frac{1}{t} \right]_b^{-1} = -1 - \frac{1}{b} + 1 = 1 - \frac{1}{b}
\]

The limit does exist. Thus the improper integral is convergent.

7. \[
\int_0^\infty 4 e^{-3x} \, dx = \lim_{b \to \infty} \int_0^b 4 e^{-3x} \, dx = \lim_{b \to \infty} \left[ \frac{1}{3} e^{-3x} \right]_0^b = \frac{1}{3} e^{-3b} - \frac{1}{3} e^0 = \frac{1}{3} e^{-3b} \to 0
\]

The integral is convergent.

9. \[
\int_0^\infty e^{cx} \, dx = \lim_{b \to \infty} \int_0^b e^{cx} \, dx = \lim_{b \to \infty} \left[ \frac{e^{cx}}{c} \right]_0^b = \frac{e^{cb} - 1}{c}
\]

As \(b \to \infty\), \(e^b \to \infty\). Thus the limit does not exist. The improper integral is divergent.

11. \[
\int_{-\infty}^\infty e^{2x} \, dx = \lim_{b \to \infty} \int_{-b}^b e^{2x} \, dx = \lim_{b \to \infty} \left[ \frac{1}{2} e^{2x} \right]_{-b}^b = \frac{e^{2b} - 1}{2}
\]

As \(b \to \pm \infty\), \(e^b \to \infty\). Thus the limit does not exist. The improper integral is divergent.

13. \[
\int_0^\infty \frac{lt}{(1 + t^2)^3} \, dt = \lim_{b \to \infty} \int_0^b \frac{lt}{(1 + t^2)^3} \, dt = \lim_{b \to \infty} \left[ -\frac{1}{3} t \right]_0^b = 0
\]

The integral is convergent.

15. \[
\int_{-\infty}^\infty 2x e^{-3x^2} \, dx = \lim_{b \to \infty} \int_{-b}^b 2x e^{-3x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{3} e^{-3x^2} \right]_{-b}^b = \frac{1}{3} e^{-3b^2} - \frac{1}{3} e^0 = \frac{1}{3} e^{-3b^2} \to 0
\]

As \(b \to \pm \infty\), \(e^b \to \infty\). Thus the limit does not exist. The improper integral is divergent.

17. \[
\int_0^\infty 2t e^{-2t} \, dt = \lim_{b \to \infty} \int_0^b 2t e^{-2t} \, dt = \lim_{b \to \infty} \left[ -t e^{-2t} - \frac{1}{2} e^{-2t} \right]_0^b = \frac{1}{2} e^{-2b} - \frac{1}{2} e^{-2b} = \frac{1}{2}
\]

The integral is convergent.
19. \[ \int_{-\infty}^{1} 2x e^{3x} \, dx = \lim_{b \to -\infty} \int_{b}^{1} 2x e^{3x} \, dx \]
\[ = \lim_{b \to -\infty} \left[ \frac{2}{3} e^{3x} \right]_{b}^{1} \]
\[ = \lim_{b \to -\infty} \left( \frac{2}{3} e^{3} - \frac{2}{3} e^{3b} - \frac{2}{3} e^{3b} + \frac{2}{3} e^{3b} \right) \]
\[ = \frac{2}{3} e^{3} \]

The integral is convergent.

21. \[ \int_{1}^{\infty} \frac{dx}{x^{1/3}} = \lim_{b \to \infty} \int_{1}^{b} x^{-1/3} \, dx \]
\[ = \lim_{b \to \infty} \left[ \frac{3}{2} x^{2/3} \right]_{1}^{b} \]
\[ = \lim_{b \to \infty} \left( \frac{3}{2} b^{2/3} - \frac{3}{2} \right) \]
\[ = \infty \]

The integral is divergent.

23. \[ \int_{0}^{\infty} \frac{1 - \sin x}{(x + \cos x)^{2}} \, dx \approx \lim_{b \to \infty} \int_{0}^{b} \left( 1 - \sin x \right) (x + \cos x)^{-2} \, dx \]
\[ = \lim_{b \to \infty} \left[ \frac{-1}{x + \cos x} \right]_{0}^{b} \]
\[ = \lim_{b \to \infty} \left( \frac{-1}{b + \cos b + 1} \right) \]

The integral is convergent.

25. \[ \int_{1}^{\infty} \frac{1}{x(x + 1)^{2}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^{3} + 2x^{2} + 3x} \, dx \]
\[ = \lim_{b \to \infty} \left[ \ln \left( \frac{1}{x} \ln(x + 1) \right) \right]_{0}^{b} \]
\[ = \lim_{b \to \infty} \left( \ln 1 \cdot 0 - \ln \frac{1}{2} - \frac{1}{2} \right) \]
\[ = \ln 2 - \frac{1}{2} \]

The integral is convergent.

27. \[ \int_{1}^{\infty} \frac{x}{\sqrt{x^{2} + 2}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{\sqrt{x^{2} + 2}} \, dx \]
\[ = \lim_{b \to \infty} \left[ x - 2 \ln(x + \sqrt{x^{2} + 2}) \right]_{1}^{b} \]
\[ = \lim_{b \to \infty} \left( b - 2 \ln(b + \sqrt{b^{2} + 2}) - 1 - 2 \ln(3) \right) \]

The limit does not exist. The integral is divergent.

29. \[ \int_{e}^{\infty} \frac{\ln x}{x} \, dx = \lim_{b \to \infty} \int_{e}^{b} \frac{\ln x}{x} \, dx \]
\[ = \lim_{b \to \infty} \left[ \ln^{2} x \right]_{e}^{b} \]
\[ = \lim_{b \to \infty} \left[ \ln^{2} b - 1 \right] \]

The limit does not exist. The integral is divergent.

31. \[ \int_{-\infty}^{\infty} 3xe^{-x^{2}/2} \, dx = \lim_{b \to \infty} \int_{-b}^{b} 3xe^{-x^{2}/2} \, dx \]
\[ = \lim_{b \to \infty} \left[ -3e^{-x^{2}/2} \right]_{-b}^{b} \]
\[ = \lim_{b \to \infty} \left[ -3e^{-b^{2}/2} - 3e^{-b^{2}/2} \right] \]
\[ = 0 \]

The integral is convergent.

33. \[ \int_{0}^{\infty} m e^{-mx} \, dx, \ m > 0 \]
\[ \approx \lim_{b \to \infty} \int_{0}^{b} m e^{-mx} \, dx \]
\[ = \lim_{b \to \infty} \left[ -e^{-mx} \right]_{0}^{b} \]
\[ = \lim_{b \to \infty} \left( -e^{-mb} - (-e^{-m0}) \right) \]
\[ = \lim_{b \to \infty} \left( 1 - \frac{1}{e^{mb}} \right) \]
\[ = 1 \]

The limit does exist. Thus the improper integral is convergent.

35. The area is given by
\[ \int_{2}^{\infty} \frac{1}{x^{2}} \, dx = \lim_{b \to \infty} \int_{2}^{b} x^{-2} \, dx \]
\[ = \lim_{b \to \infty} \left[ -x^{-1} \right]_{2}^{b} \]
\[ = \lim_{b \to \infty} \left( -\frac{1}{b} - \left( -\frac{1}{2} \right) \right) \]
\[ = \frac{1}{2} \]

The area of the region is $\frac{1}{2}$.

37. The area is given by
\[ \int_{0}^{\infty} 2x e^{-x^{2}} \, dx \]
\[ = \lim_{b \to \infty} \int_{0}^{b} 2x e^{-x^{2}} \, dx \]

(We use the substitution $u = -x^{2}$ to integrate.)
\[ = \lim_{b \to \infty} \left[ -e^{-x^{2}} \right]_{0}^{b} \]
\[ = \lim_{b \to \infty} \left( -e^{-b^{2}} - (-e^{-0^{2}}) \right) \]
\[ = \lim_{b \to \infty} \left( -\frac{1}{e^{b^{2}}} + 1 \right) \]
\[ = 1 \quad \text{(As } b \to \infty, -\frac{1}{e^{b^{2}}} \to 0 \text{ and } -\frac{1}{e^{b^{2}}} + 1 \to 1) \]

The area of the region is 1.
39. Note that 60.1 days \( \cdot \frac{60.1}{365} \text{ yr} \approx 0.164658.

\[ P_0 = 2.5 \cdot e^{-0.164658 \cdot t} \]
\[ \ln 0.5 = \ln e^{-0.164658 \cdot k} \]
\[ \ln 0.5 = -0.164658 \cdot k \]
\[-0.164658 \cdot k \approx 4.20963 \approx k \]
The decay rate is 420.963% per year.

b) The first month is \( \frac{1}{12} \) yr.
\[ E = \int_0^{1/12} 10e^{-4.20963t} \, dt \]
\[ = \frac{10}{-4.20963} \left[ e^{-4.20963t} \right]_0^{1/12} \]
\[ = \frac{10}{-4.20963} \left( e^{-0.508925} - 1 \right) \]
\[ \approx 0.702858 \text{ rens} \]

c) \[ E = \int_0^\infty 10e^{-4.20963t} \, dt \]
\[ = \left[ \frac{10}{-4.20963} \right]_0^{\infty} \]
\[ \approx 2.37554 \text{ rems} \]

41. \[ \frac{P}{k} = \frac{1}{0.0000286} \approx 31963 \text{ lbs} \]

43. \[ \lim_{b \to \infty} \int_1^b x^r \, dx \]
\[ \lim_{b \to \infty} \left[ \frac{1}{r+1} x^{r+1} \right]_1^b \]
In order for the limit to converge, the exponent \( r+1 \) must be negative, that is \( r+1 < 0 \).
Therefore, the integral is convergent for \( r < -1 \) and divergent otherwise.

45. \[ \lim_{b \to \infty} \int_0^b ke^{-kt} \, dt \]
\[ \lim_{b \to \infty} \left[ \frac{-ke^{-kt}}{k} - \frac{e^{-kt}}{k^2} \right]_0^b \]
\[ \lim_{b \to \infty} \left[ \frac{-ke^{-kb}}{k} - \frac{e^{-bk}}{k^2} + 0 + \frac{1}{k^2} \right] \]
\[ = \frac{1}{k^2} \]
The total amount of the drug dosage that goes through the body is \( \frac{1}{k^2} \).

47. If \( y = \frac{1}{x^2} \) then \( \int_1^\infty y \, dx = \frac{-1}{x} \)
\[ = \lim_{b \to \infty} \left[ \frac{-1}{b} + 1 \right] \]
\[ = 1 \]

If \( y = \frac{1}{x} \) then \( \int_1^\infty y \, dx = \ln x \)
\[ = \lim_{b \to \infty} \left[ \ln b - 0 \right] \]
\[ = \infty \]
The region under \( y = \frac{1}{x^2} \) could be painted. Since the integral of that \( y \) is convergent while the other integral is divergent.

49. \[ \int_1^\infty \frac{1}{1 + x^2} \, dx = \pi \]
Chapter 6
Matrices

Exercise Set 6.1

1.
\[
\begin{bmatrix}
4 + 3 & -1 + 6 \
7 + 2 & -9 + (-2)
\end{bmatrix}
= \begin{bmatrix} 7 & 8 \\
9 & -11
\end{bmatrix}
\]
\[
\begin{bmatrix}
3 + 4 & 9 + (-1) \\
2 + 7 & -2 + (-9)
\end{bmatrix}
= \begin{bmatrix} 7 & 8 \\
9 & -11
\end{bmatrix}
\]

3.
\[
\begin{bmatrix}
3 + 8 & 9 + 3 \\
2 + 0 & -2 + (-3)
\end{bmatrix}
= \begin{bmatrix} 11 & 12 \\
2 & -5
\end{bmatrix}
\]

5. Not possible, dimensions not the same

7.
\[
B^2
= \begin{bmatrix}
3 \cdot 2 & 9 \cdot 2 \\
2 \cdot 2 & -2 \cdot 2
\end{bmatrix}
\begin{bmatrix} 6 & 19 \\
4 & -1
\end{bmatrix}
\]

9.
\[
A - B
= \begin{bmatrix}
1 - 3 & -1 - 0 \\
7 - 2 & -9 - (-2)
\end{bmatrix}
= \begin{bmatrix} 1 & -10 \\
5 & -7
\end{bmatrix}
\]
\[
A - (-1)B
= \begin{bmatrix}
1 + (-1)(3) & -1 + (-1)(9) \\
7 + (-1)(2) & -9 + (-1)(-2)
\end{bmatrix}
= \begin{bmatrix} 1 & -10 \\
5 & -7
\end{bmatrix}
\]
The answers are identical.

11.
\[
AD
= \begin{bmatrix}
30 - 10 & -21 - 3 & 4 + 1 \\
35 + 90 & -42 - 27 & 7 + 9
\end{bmatrix}
\begin{bmatrix} 10 & 27 & 5 \\
-55 & -69 & 16
\end{bmatrix}
\]

13.
\[
A^2e
= \begin{bmatrix}
71 & -51 \\
378 & -631
\end{bmatrix}
\begin{bmatrix} 2 & -3 \\
(71)(2)(-51)(-3) & (378)(2)(-631)(-3)
\end{bmatrix}
= \begin{bmatrix} 301 & 2619 \\
142 & 2619
\end{bmatrix}
\]

15.
\[
B^3e
= \begin{bmatrix}
90 & 225 \\
56 & -26
\end{bmatrix}
\begin{bmatrix} 2 & -3 \\
(90)(2)(225)(-3) & (50)(2)(-26)(-3)
\end{bmatrix}
= \begin{bmatrix} -477 & 178 \\
\end{bmatrix}
\]

17.
\[
DE
= \begin{bmatrix}
-35 + 18 & 12 - 20 & 12 - 1 \\
70 - 0 & 20 - 16 & 11
\end{bmatrix}
= \begin{bmatrix} -15 & 7 \\
7 & 47
\end{bmatrix}
\]
\[
A \cdot DE
= \begin{bmatrix} 1 & 7 \\
-11 & 7
\end{bmatrix}
\begin{bmatrix} -15 & 7 \\
-71 & 47
\end{bmatrix}
= \begin{bmatrix} -105 & 9 \\
-10 & -18
\end{bmatrix}
\]

19.
\[
BD
= \begin{bmatrix}
15 + 90 & -18 + 27 & 3 + 9 \\
10 - 20 & 12 - 6 & 2 + 2
\end{bmatrix}
= \begin{bmatrix} 105 & 9 & 6 \\
10 & -18 & 1
\end{bmatrix}
\]
\[
(BD)^F
= \begin{bmatrix}
105 & 9 & -6 \\
-10 & 18 & 1
\end{bmatrix}
\begin{bmatrix} -1 & 2 & 3 \\
0 & -1 & 2 \\
-7 & -2 & 5
\end{bmatrix}
= \begin{bmatrix} -120 + 10 & 12 - 9 + 12 & 315 + 18 - 30 \\
40 - 0 & 28 - 0 & 18 - 20 + 30
\end{bmatrix}
= \begin{bmatrix} -378 & 213 & 393 \\
12 & -10 & -46
\end{bmatrix}
\]

21. (B + C) = \begin{bmatrix} 11 & 12 \\
2 & -5
\end{bmatrix}

A(B + C)
= \begin{bmatrix}
4 & 1 \\
7 & -9
\end{bmatrix}
\begin{bmatrix} 11 & 12 \\
2 & -5
\end{bmatrix}
= \begin{bmatrix} 44 + 2 & 110 + 5 \\
77 - 18 & 84 - 45
\end{bmatrix}
= \begin{bmatrix} 42 & 53 \\
59 & 32
\end{bmatrix}
\]

AB
= \begin{bmatrix}
12 & 2 \\
21 & 0
\end{bmatrix}
\begin{bmatrix} 36 & 2 \\
63 & 18
\end{bmatrix}
= \begin{bmatrix} 10 & 38 \\
3 & 81
\end{bmatrix}
\]

AC
= \begin{bmatrix}
32 & 0 \\
56 & 0
\end{bmatrix}
\begin{bmatrix} 12 & 3 \\
21 & 27
\end{bmatrix}
= \begin{bmatrix} 32 & 15 \\
56 & 48
\end{bmatrix}
\]

AB + AC
= \begin{bmatrix}
10 & 32 \\
3 & 90
\end{bmatrix}
\begin{bmatrix} 38 & 45 \\
81 & 18
\end{bmatrix}
= \begin{bmatrix} 42 & 53 \\
59 & 129
\end{bmatrix}
\]
23. 

\[ AB = \begin{bmatrix} 18 & 0 & -22 \\ -16 & 14 & -3 \\ -15 & -10 & -11 \end{bmatrix} \]

\[
= \begin{bmatrix} 18 & 0 & -22 \\ -16 & 14 & -3 \\ -15 & -10 & -11 \end{bmatrix} \]

Clearly, \( AB \neq BA \)

25.

\[ A(B-C) = \begin{bmatrix} 1 & 6 & -2 \\ 4 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} -4 & 0 & -2 \\ 7 & -3 & -11 \\ 5 & 5 & -5 \end{bmatrix} \]

\[
= \begin{bmatrix} -1 & 12 & -10 \\ 16 & 14 & 5 \\ 5 & 8 & 36 \end{bmatrix} \]

\[ B = \begin{bmatrix} 18 & 0 & -22 \\ -16 & 14 & -3 \\ -15 & -10 & -11 \end{bmatrix} \]

\[ AC = \begin{bmatrix} 4 & -12 & 2 \\ 10 & 16 & 6 \\ 0 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0 & 18 & 6 \\ 0 & 14 & 8 \end{bmatrix} \]

\[
= \begin{bmatrix} -10 & 28 & 36 \\ 10 & 13 & -22 \\ -11 & 24 & -6 \end{bmatrix} \]

\[ AB - BC = \begin{bmatrix} 18 & 0 & -22 \\ -16 & 14 & -3 \\ -15 & -10 & -11 \end{bmatrix} - \begin{bmatrix} -10 & 28 & 36 \\ 19 & 13 & -22 \\ 11 & 24 & -6 \end{bmatrix} \]

\[
= \begin{bmatrix} 28 & -26 & -58 \\ -35 & 1 & 19 \\ -4 & -34 & -8 \end{bmatrix} \]

27.

a) 

\[ \begin{bmatrix} 0 & 4 & -2 \\ 5 & 0 & -3 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 10 \\ 6 & 8 & 4 \end{bmatrix} \]

\[ = \begin{bmatrix} 16 \\ 0 \\ 4 \end{bmatrix} \]

b) 

\[ \begin{bmatrix} 0 & 4 & -2 \\ 5 & 0 & -3 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 2 \end{bmatrix} \]

29.

\[ B - AB = \begin{bmatrix} 0 & 4 & -2 \\ 5 & 0 & -3 \\ 6 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 10 \\ 6 & 8 & 4 \end{bmatrix} \]

\[
= \begin{bmatrix} 0 & 4 & -2 \\ 5 & 0 & -3 \\ 6 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 0 & 4 \end{bmatrix} \]

31. a) Left to the student

b) 

\[ \begin{bmatrix} 1.1 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} = \begin{bmatrix} 55 \\ 20 \end{bmatrix} \]

\[ = \begin{bmatrix} 114 \\ 36 \end{bmatrix} \]

33. a) 

\[ \begin{bmatrix} 0.8 & 2 \\ 0.5 & 0.4 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.8 & 2 \\ 0.5 & 0.4 \end{bmatrix} \]

\[ = \begin{bmatrix} 1200 & 1520 \\ 4000 & 4298 \end{bmatrix} \]

35. a) Left to the student

b) 

\[ \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.4 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.4 \end{bmatrix} \]

\[ = \begin{bmatrix} 300 \\ 16 \end{bmatrix} \]

\[ = \begin{bmatrix} 4620 \\ 240 \end{bmatrix} \]

37. a) 

\[ A^2 = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \]

\[ = \begin{bmatrix} 16 \\ 5 \end{bmatrix} \]

\[ = \begin{bmatrix} 9 & -8 \\ 12 & 20 \end{bmatrix} \]

\[ = \begin{bmatrix} 32 & -16 \end{bmatrix} \]
b) 
\[
\begin{bmatrix}
2 & 7 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
2 & 7 \\
1 & 3
\end{bmatrix}
= 
\begin{bmatrix}
17 & 14+21 \\
-2.3 & 7+9
\end{bmatrix}
= 
\begin{bmatrix}
11 & -35 \\
-3 & 16
\end{bmatrix}
\]

45. Left to the student

51. \[
\begin{bmatrix}
200502 \\
63757
\end{bmatrix}
\]

53. \[
\begin{bmatrix}
3.14 \times 10^7 \\
1.39 \times 10^7
\end{bmatrix}
\]

55. \[
\begin{bmatrix}
7.64 \times 10^9 \\
1.56 \times 10^8
\end{bmatrix}
\]

c) 
\[
(A + B)^2 = 
\begin{bmatrix}
1 & 5 \\
5 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 5 \\
5 & 2
\end{bmatrix}
- 
\begin{bmatrix}
11 & 25 \\
5 & 10
\end{bmatrix}
\begin{bmatrix}
26 & 15 \\
25 & 14
\end{bmatrix}
\]

60. a) \(A^T\) is \(1 \times n\) and \(B^T\) is \(n \times 1\)

b) \(A \cdot B\) will have dimensions of \(1 \times 1\)

\[
A \cdot B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}
\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1}
\]

\(B^T \cdot A^T\) will have dimensions of \(1 \times 1\)

\[
B^T \cdot A^T = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \end{bmatrix}
\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} + \cdots + b_{1n}a_{n1}
\]

Thus, \(A \cdot B = B^T \cdot A^T\)

41. This is true due to the associative property of addition for real numbers

43. Since each entry in \(A\) is added to its additive inverse each entry in \((A + (-A))\) will be zero

Exercise Set 6.2

1. From the second equation we have \(y = 2x\). Then
\[
x + 2y = 5 \\
x + 4x = 5 \\
x = 5 \\
x = 1
\]
Which means \(y = 2(1) = 2\)

3. From the first equation we have \(z = 5w - 14\). Then
\[
2w + 3z = 26 \\
2w + 3(5w - 14) = 26 \\
2w + 15w - 42 = 26 \\
17w = 68 \\
w = 4
\]
Which means \(z = 5(4) - 14 = 6\)

5. From the first equation we have \(s = t + 7\). Then
\[
-2s + 2t = -5 \\
-2(t + 7) + 2t = -5 \\
-2t + 14 + 2t = -5 \\
14 = -5
\]
This is a contradiction, which means there is no solution.

7. From the first equation we have \(x = y + 7\). Then
\[
-2x + 2y = -14 \\
-2(y + 7) + 2y = -14 \\
-2y - 14 + 2y = -14 \\
-14 = -14
\]
This is an identity, which means there are many solutions.
9. - 15.] Left to the student

17. Write the augmented matrix
\[
\begin{bmatrix}
0 & 1 & 3 & -1 \\
1 & 0 & 6 & 37 \\
0 & 2 & 1 & -2
\end{bmatrix}
\]
Interchange \( R_1 \) and \( R_2 \)
\[
\begin{bmatrix}
1 & 0 & 6 & 37 \\
0 & 1 & 3 & -1 \\
0 & 2 & 1 & -2
\end{bmatrix}
\]
\[-2R_2 + R_3\]
\[
\begin{bmatrix}
1 & 0 & 6 & 37 \\
0 & 1 & 3 & -1 \\
0 & 0 & -5 & 0
\end{bmatrix}
\]
This means the \( z = 0 \)
\( y + 3(0) = -1 \rightarrow y = -1 \)
\( x + 6(0) = 37 \rightarrow x = 37 \)

19. Write the augmented matrix
\[
\begin{bmatrix}
7 & -1 & -9 & -1 \\
2 & 0 & -4 & -4 \\
-4 & 0 & 6 & -3
\end{bmatrix}
\]
Interchange \( R_2 \) and \( R_1 \)
\[
\begin{bmatrix}
2 & 0 & -4 & -4 \\
7 & -1 & -9 & 1 \\
-4 & 0 & 6 & -3
\end{bmatrix}
\]
\( R_1 / 2 \)
\[
\begin{bmatrix}
1 & 0 & -2 & -2 \\
7 & -1 & -9 & 1 \\
-4 & 0 & 6 & -3
\end{bmatrix}
\]
\(-7R_1 + R_2\)
\[
\begin{bmatrix}
1 & 0 & -2 & -2 \\
0 & -1 & 5 & 15 \\
-4 & 0 & 6 & -3
\end{bmatrix}
\]
\( 4R_1 + R_3\)
\[
\begin{bmatrix}
1 & 0 & -2 & -2 \\
0 & -1 & 5 & 15 \\
0 & 0 & -2 & -11
\end{bmatrix}
\]
\(-R_2\)
\[
\begin{bmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & -5 & -15 \\
0 & 0 & -2 & -11
\end{bmatrix}
\]
This means \( z = 11/2 = 5.5 \)
\( y - 5(5.5) = -15 \rightarrow y = 12.5 \)
\( x - 2(5.5) = -3 \rightarrow x = 9 \)

21. Write the augmented matrix
\[
\begin{bmatrix}
2 & -2 & 3 & 3 \\
4 & -3 & 3 & 2 \\
1 & 1 & -1 & 4
\end{bmatrix}
\]
Interchange \( R_1 \) and \(-R_3\)
\[
\begin{bmatrix}
1 & -1 & 1 & 1 \\
4 & -3 & 3 & 2 \\
2 & -2 & 3 & 3
\end{bmatrix}
\]
\(-4R_1 + R_2\)
\[
\begin{bmatrix}
1 & -1 & 1 & -4 \\
0 & 1 & -1 & 18 \\
2 & -2 & 3 & 3
\end{bmatrix}
\]
\(-2R_1 + R_3\)
\[
\begin{bmatrix}
1 & -1 & 1 & -4 \\
0 & 1 & -1 & 18 \\
0 & 0 & 1 & 11
\end{bmatrix}
\]
This means \( z = 11 \)
\( y - 11 = 18 \rightarrow y = 29 \)
\( x - 29 + 11 = -4 \rightarrow x = 14 \)

23. - 27.] Left to the student

29. Write the augmented matrix
\[
\begin{bmatrix}
1 & 1 & 2 & 5 \\
1 & 1 & 1 & -10 \\
2 & 3 & 4 & 2
\end{bmatrix}
\]
\(-R_1 + R_2\)
\[
\begin{bmatrix}
1 & 1 & 2 & 5 \\
0 & 0 & -1 & -15 \\
2 & 3 & 4 & 2
\end{bmatrix}
\]
Interchange \(-R_2 \) and \( R_3 \)
\[
\begin{bmatrix}
1 & 1 & 2 & 5 \\
0 & 1 & 0 & -5 \\
0 & 0 & 1 & 15
\end{bmatrix}
\]
This means \( z = 15 \)
\( y = -5 \)
\( x + (-5) + 2(15) = 5 \rightarrow x = -17 \)

31. Write the augmented matrix
\[
\begin{bmatrix}
1 & 1 & -2 & 4 \\
4 & 7 & 3 & 3 \\
14 & 23 & 5 & 5
\end{bmatrix}
\]
\(-4R_1 + R_2\)
\[
\begin{bmatrix}
1 & 1 & -2 & 4 \\
0 & 3 & 11 & -13 \\
14 & 23 & 5 & 17
\end{bmatrix}
\]
\(-14R_1 + R_3\)
\[
\begin{bmatrix}
1 & 1 & -2 & 4 \\
0 & 3 & 11 & -13 \\
0 & 0 & 33 & -39
\end{bmatrix}
\]
\(-3R_2 + R_3\)
\[
\begin{bmatrix}
1 & 1 & -2 & 4 \\
0 & 3 & 11 & -13 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
This means the system has many solutions.
Let \( z = z \), then \( y = -\frac{11}{3} z - \frac{18}{3} \)
\( x - \frac{11}{3} z - \frac{18}{3} + 2 z + 4 = \frac{17}{2} z + \frac{33}{2} \)
33. Write the augmented matrix
\[
\begin{bmatrix}
1 & -1 & 3 & 2 \\
2 & 3 & -1 & 5 \\
-1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\(-2R_1 + R_2\)
\[
\begin{bmatrix}
1 & -1 & 3 & 2 \\
0 & 5 & -7 & 1 \\
-1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\(-R_2 + R_3\)
\[
\begin{bmatrix}
1 & -1 & 3 & 2 \\
0 & 5 & -7 & 1 \\
0 & 0 & 6 & 3 \\
\end{bmatrix}
\]

This means the system has no solution.

35. Write the augmented matrix
\[
\begin{bmatrix}
1 & -2 & -5 & 0 \\
2 & 3 & 15 & 0 \\
-2 & -1 & 8 & 1 \\
\end{bmatrix}
\]

\(-2R_1 + R_2\)
\[
\begin{bmatrix}
1 & -2 & -5 & 0 \\
0 & 7 & 25 & 0 \\
-2 & -1 & 8 & 1 \\
\end{bmatrix}
\]

\(R_1 + R_3\)
\[
\begin{bmatrix}
1 & -2 & -5 & 0 \\
0 & 7 & 25 & 0 \\
0 & -5 & -18 & 1 \\
\end{bmatrix}
\]

\(\frac{1}{2}R_2\)
\[
\begin{bmatrix}
1 & -2 & -\frac{5}{2} & 0 \\
0 & 1 & \frac{25}{2} & 0 \\
0 & -5 & -18 & 1 \\
\end{bmatrix}
\]

\(5R_2 + R_3\)
\[
\begin{bmatrix}
1 & -2 & -\frac{5}{2} & 0 \\
0 & 1 & \frac{25}{2} & 0 \\
0 & 0 & \frac{-1}{2} & 1 \\
\end{bmatrix}
\]

This means: \(-7\)
\[
g = -\frac{\frac{5}{2}}{5} = -7 - 25 \\
x = 2(\frac{25}{2}) + 5(-7) = 15 \\
\]

37. Write the augmented matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 5 \\
1 & 0 & 1 & 1 & 6 \\
0 & 1 & 1 & 1 & 4 \\
1 & 0 & 1 & 0 & 3 \\
\end{bmatrix}
\]

\(-R_1 + R_2\)
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 5 \\
0 & -1 & 0 & 0 & 4 \\
0 & 1 & 1 & 1 & 4 \\
1 & 0 & 1 & 0 & 3 \\
\end{bmatrix}
\]

39. Write the augmented matrix
\[
\begin{bmatrix}
-2 & -1 & 6 & -1 & 2 \\
-3 & -5 & 6 & 1 & -3 \\
1 & 1 & 2 & 0 & 1 \\
0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\]

Interchange \(R_1\) and \(R_3\)
\[
\begin{bmatrix}
1 & 1 & -2 & 0 & 4 \\
-3 & -5 & 6 & 1 & -3 \\
-2 & -1 & 6 & -1 & 2 \\
0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\]

Interchange \(R_2\) and \(R_4\)
\[
\begin{bmatrix}
1 & 1 & -2 & 0 & 4 \\
0 & 2 & 0 & 1 & 9 \\
0 & 1 & 2 & -1 & 10 \\
0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\]

Interchange \(R_2\) and \(R_4\)
\[
\begin{bmatrix}
1 & 1 & -2 & 0 & 4 \\
0 & 1 & 0 & 0 & 2 \\
0 & 1 & 2 & -1 & 10 \\
0 & 1 & -2 & 0 & 9 \\
\end{bmatrix}
\]
This means that $w = 20$

$y = 20 + 1 = 21$

$x = -21 + 2(20) + 1 = 23$

41. We are looking for the values of $B$ and $F$ for which $B' = 0$ and $F' = 0$. Thus,

$$0 = -0.01B + 0.1 \rightarrow B = 10$$

and

$$0 = 0.01(10) - 0.02F \rightarrow F = 5$$

43. Write the segmented matrix

\[
\begin{bmatrix}
0 & 0.76 & 0.95 & 579 \\
0.2 & 0 & 0 & 80 \\
0 & 0.91 & 0.91 & 619
\end{bmatrix}
\]

Interchange the $R1$ and $R2$

\[
\begin{bmatrix}
0 & 0.76 & 0.95 & 579 \\
0.2 & 0 & 0 & 80 \\
0 & 0.91 & 0.91 & 619
\end{bmatrix}
\]

\[
R1/0.2
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 290 \\
0.76 & 0.95 & 579 \\
0.91 & 0.91 & 619
\end{bmatrix}
\]

\[
R2/0.76
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 480 \\
0 & 1.25 & 761.842 \\
0 & 0.91 & 619
\end{bmatrix}
\]

\[
-0.91R2 + R3
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 400 \\
0 & 1.25 & 761.842 \\
0 & -0.2275 & -84.276
\end{bmatrix}
\]

\[
R3/ -0.2275
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 400 \\
0 & 1.25 & 761.842 \\
0 & 1 & 370.45
\end{bmatrix}
\]

This means that $A = 370, S = 761.842 - 1.25(370) = 299,$

and $H = 400$

45. Left to the student. [Answers may vary]

47. a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 16 & 20 & 24 \\ 7 & 8 & 9 \end{bmatrix}$

b) $AC = \begin{bmatrix} -4 & -2 & 7 \\ 4 & 4 & -12 \\ -4 & 3 & 6 \end{bmatrix}$

c) Matrix $A$ is the identity matrix with second row multiplied by 4. Therefore, the effect of multiplying $A$ on the left with another $3 \times 3$ matrix will give a matrix with the second row multiplied by 4.

49. a) Since $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ are solutions then

$$ax_0 + by_0 + cz_0 = d$$

$$ax_1 + by_1 + cz_1 = d$$

Now consider $(tx_0 + (1 - t)x_1, ty_0 + (1 - t)y_1, tz_0 + (1 - t)z_1) = a(tx_0 + ax_1, ty_0 + by_1 + cz_0 + cz_1, (1-t)z_1)$

Rearranging the terms we get

$$= (ax_0 + by_0 + cz_0) + (ax_1 + by_1 + cz_1)$$

$$- (ax_1 + by_1 + cz_1)$$

$$= \lambda (ax_0 + by_0 + cz_0) + \lambda (ax_1 + by_1 + cz_1)$$

$$- \lambda (ax_1 + by_1 + cz_1)$$

$$= \lambda (ax_0 + by_0 + cz_0) + \lambda (ax_1 + by_1 + cz_1)$$

$$= \lambda (ax_0 + by_0 + cz_0) + \lambda (ax_1 + by_1 + cz_1)$$

$$= \lambda (ax_0 + by_0 + cz_0) + \lambda (ax_1 + by_1 + cz_1)$$

$$= \lambda (ax_0 + by_0 + cz_0) + \lambda (ax_1 + by_1 + cz_1)$$

Therefore, $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1, tz_0 + (1-t)z_1)$ is also a solution to $ax_0 + by_0 + cz_0 = d$ for any value of $t$.

b) As seen in the previous part, since any value of $t$ could be used to find a solution of the form $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1, tz_0 + (1-t)z_1)$, then if a system has two solutions then it has infinitely many solutions.

c) The answers in part b) may be generalized since the technique used in part a) can be easily extended to any number of variables.

51. - 55. Left to the student.

Exercise Set 6.3

1. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Thus, the inverse is $\begin{bmatrix} 0 & -1 \\ i & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Thus, the inverse is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. $\text{det} = (3 \cdot 7) - (5 \cdot 4) = 1$

Thus, the inverse is $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$

7. $\text{det} = (3 \cdot -2) - (8 \cdot 7) = 62$

Thus, the inverse is $\begin{bmatrix} -2 & -7 \\ -8 & -3 \end{bmatrix}$
9. \[
\begin{vmatrix}
-2 & 2 & 1 \\
1 & 2 & 0 \\
0 & -1 & 1
\end{vmatrix}
= \begin{vmatrix}
1 & 2 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{vmatrix}
= \begin{vmatrix}
1 & 2 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{vmatrix}
\begin{vmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{vmatrix}
= \begin{vmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
1 & 0 \\
1 & 2
\end{vmatrix}
\begin{vmatrix}
0 & 1 \\
1 & 2 \\
0 & 6
\end{vmatrix}
\begin{vmatrix}
1 & 0 \\
1 & 0 \\
1 & 0
\end{vmatrix}
= \begin{vmatrix}
1 & 2 \\
0 & 1 \\
0 & 1
\end{vmatrix}
Thus, the inverse is
\begin{vmatrix}
0 & 1 & 2 \\
0 & 0 & -1 \\
1 & 2 & 6
\end{vmatrix}
\]

31. \[\det 1(5)(2) - (-1)(-3) - 8(-2)(2) - (7)(-3) + 3(-2)(-1) - (7)(5) = -2 + 12 - 81 = -21\]
The matrix is not invertible.

33. \[\det 1[2]{(1)} - (1)(3)] - 1[2]{(1)} - (3)(1)] + 1[2]{(3)} - (2)(3)]
= -1 + 1 - 9
= -9
The matrix is not invertible.

35. \[\det - 0(6)(6) - (1)(2)] - 1[-1](6) - (-1)(2) - 2(14) - (-1)(6)]
= -16 - 16
= 0
The matrix is not invertible.

37. \[\det = 2((-5)(5) - (-1)(7)) + 4((-3)(5) - (-1)(7)) + 8((-3)(-1) - (7)(-1))
= -36 + 52 - 136
= -120
The matrix is invertible.

39. \[\det = -96 (technology used) The matrix is invertible.

41. \[\det = 12 (technology used) The matrix is invertible.

43. \[P_1 = \begin{bmatrix}
0.5 & 1.25 \\
0.75 & 0.25
\end{bmatrix}
\begin{bmatrix}
156 \\
48
\end{bmatrix}
= \begin{bmatrix}
26 \\
113
\end{bmatrix}
\]

45. a) \[G^{-1} = \begin{bmatrix}
0 & 5/4 \\
2/15 & -5/2
\end{bmatrix}
\]

b) \[G_2 = \begin{bmatrix}
0 & 5/4 \\
2/15 & -5/2
\end{bmatrix}
\begin{bmatrix}
3825 \\
192
\end{bmatrix}
= \begin{bmatrix}
3825 \\
192
\end{bmatrix}
\]

c) \[G_3 = \begin{bmatrix}
0 & 5/4 \\
2/15 & -5/2
\end{bmatrix}
\begin{bmatrix}
240 \\
30
\end{bmatrix}
= \begin{bmatrix}
240 \\
30
\end{bmatrix}
\]

25. \[\det = (2 \cdot 6) - (3 \cdot 1) = 12 - 3 = 9 (The matrix is invertible)

27. \[\det = (9 \cdot 3) - (-5 \cdot 2) = 27 + 10 = 37 (The matrix is invertible)

29. \[\det = (2 \cdot 3) - (6 \cdot 3) = 18 - 18 = 0 (The matrix is not invertible)\]
47. Left to the student.

49. a) \( AB = \begin{bmatrix} -8 & 25 \\ -10 & 41 \end{bmatrix} \) Thus
\[
\text{det}(AB) = -328 + 250 = -78
\]
\[
= -6 \cdot 13
\]
\[
= \text{det}(A) \cdot \text{det}(B)
\]
b) \( AB = \begin{bmatrix} 8 & -26 & 20 \\ -2 & -7 & 0 \end{bmatrix} \) Thus
\[
\text{det}(AB) = 1032
\]
\[
= 12 \cdot 86
\]
\[
= \text{det}(A) \cdot \text{det}(B)
\]
c) Left to the student (answers vary)

51. - 53. Left to the student.

55. - 57. Left to the student.

Exercise Set 6.4

1. 
\[
\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]
\[
= 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
Thus, the vector is an eigenvector with eigenvalue of 2.

3. 
\[
\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\]
The vector is not an eigenvector.

5. 
\[
\begin{bmatrix} 5 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}
\]
\[
= -2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
Thus, the vector is an eigenvector with eigenvalue -2.

7. 
\[
\begin{bmatrix} -8.5 & -4.5 \\ 21 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 15.5 \\ -35 \end{bmatrix}
\]
The vector is not an eigenvector.

9. 
\[
\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}
\]
\[
= 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]
Thus, the vector is an eigenvector with eigenvalue 3.

11. 
\[
\begin{bmatrix} -25 & 40 & 39 \\ -32 & 47 & 39 \\ 16 & -20 & -1 \end{bmatrix} \begin{bmatrix} -13 \\ -13 \\ 4 \end{bmatrix} = \begin{bmatrix} -39 \\ -39 \\ 48 \end{bmatrix}
\]
The vector is not an eigenvector.

13. \( v = 3w + 4u \)

15. \( v = -2w + u \)

17. \( v = -w + 3u \)

19. 
\[
\text{det}(A - rt) = \begin{bmatrix} 1-r & 0 \\ -1 & 2-r \end{bmatrix} = 0
\]
\[
(1-r)(2-r) - 0 = 0
\]
\[
r = 1
\]
\[
r = 2
\]
For \( r = 1 \)
\[
\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
x = y
\]
\[
x = t
\]
\[
y = t
\]
Thus for the eigenvalue of \( r = 1 \) the eigenvector is \( \begin{bmatrix} t \\ t \end{bmatrix}, \{ t \neq 0 \} \).

For \( r = 2 \)
\[
\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
x = 0
\]
\[
y = t
\]
Thus, for the eigenvalue of \( r = 2 \) the eigenvector is \( \begin{bmatrix} 0 \\ t \end{bmatrix}, \{ t \neq 0 \} \).

21. 
\[
\text{det}(A - rt) = 0
\]
\[
\begin{bmatrix} 5-r & 2 \\ -24 & -9-r \end{bmatrix} = 0
\]
\[
(5-r)(-9-r) - (-24) = 0
\]
\[
r^2 + 4r + 3 = 0
\]
\[
r = -1
\]
\[
r = -3
\]
For $r = -1$

\[
\begin{bmatrix}
6 & 2 \\
-24 & -8
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Thus, for the eigenvalue of $r = -1$ the eigenvector is

\[
\begin{bmatrix}
1 \\
3t
\end{bmatrix}, (t \neq 0)
\]

For $r = 3$

\[
\begin{bmatrix}
8 & 2 \\
-24 & -6
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Thus, for the eigenvalue of $r = 9$ the eigenvector is

\[
\begin{bmatrix}
i \\
-3t
\end{bmatrix}, (t \neq 0)
\]

23.

\[
\begin{vmatrix}
-7.5-r & -15.75 \\
6 & -12
\end{vmatrix}
= 0
\]

\[
(9.5 - r)(-7 - r) - (-67.5) = 0
\]

\[
r^2 - 2.5r + 1 = 0
\]

Thus, for the eigenvalue of $r = 0.5$

\[
\begin{bmatrix}
9 & -15.75 \\
15 & -7
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Thus, for the eigenvalue of $r = 6.5$ the eigenvector is

\[
\begin{bmatrix}
i \\
2t
\end{bmatrix}, (t \neq 0)
\]

For $r = 2$

\[
\begin{bmatrix}
7.5 & -15.75 \\
15 & -9
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Thus, for the eigenvalue of $r = 2$ the eigenvector is

\[
\begin{bmatrix}
3t \\
5t
\end{bmatrix}, (t \neq 0)
\]

27.

The characteristic equation is $r^3 - 3r^2 + 2r - 0$

The eigenvalues are $r = 0$, $r = 1$, and $r = 2$

For $r = 0$

\[
\begin{bmatrix}
10 & -4 & 15 \\
8 & -2 & 15 \\
-4 & 2 & -5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Solving the matrix above using any method discussed earlier this chapter yields that $x = 5t$, $y = 5t$, and $z = 2t$, which are the eigenvectors for $r = 0$.

For $r = 1$

\[
\begin{bmatrix}
0 & -4 & 15 \\
8 & -3 & 15 \\
-4 & 2 & -6
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
9x - 4y + 15z = 0
\]

\[
8x - 3y - 15z = 0
\]

\[
-4x + 2y - 6z = 0
\]
Solving the matrix above using any method discussed earlier this chapter yields that \( x = 3t, y = 3t, \) and \( z = t, \) which are the eigenvectors for \( r = 1. \)

For \( r = 2 \)

\[
\begin{bmatrix}
8 & -4 & 15 \\
8 & -4 & 15 \\
-4 & 2 & -7
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
8x - 4y + 15z = 0 \\
8x - 4y - 15z = 0 \\
-4x + 2y - 7z = 0
\]

Solving the matrix above using any method discussed earlier this chapter yields that \( x = t, y = 2t, \) and \( z = 0, \) which are the eigenvectors for \( r = 2. \)

29. \[
\begin{bmatrix}
-5t & 8 & 2 \\
-15 & 18 - r & 4 \\
45 & -48 & -10 - r
\end{bmatrix}
\]

The characteristic equation is \( r^3 - 3r^2 + 2r = 0. \)
The eigenvalues are \( r = 0, r = 1, \) and \( r = 2. \)

For \( r = 0 \)

\[
\begin{bmatrix}
-5 & 8 & 2 \\
-15 & 18 & 4 \\
45 & -48 & -10
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
-5x + 8y + 2z = 0 \\
-15x + 18y + 4z = 0 \\
45x - 48y - 10z = 0
\]

Solving the matrix above using any method discussed earlier this chapter yields that \( x = 5t, y = -5t, \) and \( z = -2t, \) which are the eigenvectors for \( r = 0. \)

For \( r = 1 \)

\[
\begin{bmatrix}
-6 & 8 & 2 \\
-15 & 17 & 4 \\
45 & -48 & -11
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
-6x + 8y + 2z = 0 \\
-15x + 17y + 4z = 0 \\
45x - 48y - 11z = 0
\]

Solving the matrix above using any method discussed earlier this chapter yields that \( x = -5t, \) \( y = -3t, \) and \( z = 9t, \) which are the eigenvectors for \( r = 1. \)

For \( r = 2 \)

\[
\begin{bmatrix}
-7 & 8 & 2 \\
-15 & 16 & 4 \\
45 & -48 & -12
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
-7x + 8y + 2z = 0 \\
-15x + 16y + 4z = 0 \\
45x - 48y - 12z = 0
\]

Solving the matrix above using any method discussed earlier this chapter yields that \( x = 0, y = -t, \) and \( z = 3t, \) which are the eigenvectors for \( r = 2. \)

31. The characteristic equation is \( r^3 + 2r^2 - r - 2 = 0. \)
The eigenvalues are \( r = -2, r = -1, \) and \( r = 1. \)
The eigenvectors are \([t, t, 2t], [t, -2t, -4t], \) and \([0, t, 0] \) respectively.

32. The characteristic equation is \( r^3 - 2r^2 - 16r + 32 = 0. \)
The eigenvalues are \( r = -4, r = 2, \) and \( r = 4. \)
The eigenvectors are \([t, t, t], [0, 0, t], \) and \([2t, t, 0] \) respectively.

33. The characteristic equation is \( r^3 - 2r^2 - r + 2 = 0. \)
The eigenvalues are \( r = -1, r = 1, \) and \( r = 2. \)
The eigenvectors are \([t, t, -1], [2t, t, 0], \) and \([-2t, 2t, t] \) respectively.

37.

\[
A^{w}_{w} = 2(2)^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3(1)^{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 2018 \\ 0 \\ 3 \end{bmatrix}
\]

39.

\[
A^{w}_{w} = 2(-2)^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3(0)^{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 2018 \\ 0 \\ 0 \end{bmatrix}
\]

41. Long-term growth rate: 1.5
Long-term growth rate percentage: 50% (See page 483 for a reference; Percentage = 100 • Long-term growth rate 100%)

43. Long-term growth rate: 2.1
Long-term growth rate percentage: 110% (See page 483 for a reference; Percentage = 100 • Long-term growth rate 100%)

45. The characteristic equation for the Leslie matrix is

\[
(0.5 - r)^2 - 1 = 0 \\
r^2 - r + 0.25 - 1 = 0 \\
r^2 - r - 0.75 = 0
\]

\[
r = 1.5 \\
r = -0.5
\]

Thus, the long-term growth rate is 1.5

47. The characteristic equation for the Leslie matrix is

\[
(15 - r)(10 - r) - 6 = 0 \\
r^2 - 15r - 6 = 0 \\
r = 15.38986092 \\
r = -0.38986092
\]

Thus, the long-term growth rate is \( \approx 15.39 \)
49. a) 
\[ G^np = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \]
\[ G^np = 3(1.1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4(0.8)^n \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]
\[ \lim_{n \to \infty} \left( \frac{1}{1.1} \right)^n G^np = \lim_{n \to \infty} \left( \frac{1}{1.1} \right)^n 3(1.1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lim_{n \to \infty} \left( \frac{1}{1.1} \right)^n 4(0.8)^n \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]
\[ = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \]

b) If the long term rate is not 1.1 then the
\[ \lim_{n \to \infty} \left( \frac{1}{1.1} \right)^n G^np \text{ may not converge} \]

51. \[ B^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \]

a) \[ HA = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \]
\[ B^{-1}HA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -2 & -3 \end{bmatrix} \]

b) For matrix \( A = (1-r)(2-r) - 0 = r^2 - 3r + 2 \)
For matrix \( B^{-1}AB = (6-r)(-3-r) + 20 = r^2 - 3r + 2 \)
The answers are identical 

c) Since the two matrices have the same characteristic equation, they will have the same eigenvalues

53. Left to the student

55. a) \( (2-r)(-5-r) - 3 = r^2 + 3r - 13 \)
b) \[ A^2 + 3A - 13I = \begin{bmatrix} 7 & -9 \\ -3 & 28 \end{bmatrix} \begin{bmatrix} 8 & 9 \\ 3 & 15 \end{bmatrix} - \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

c) Left to the student
27. \( c = -2c + 8c - 45 \rightarrow c = 9 \\
\text{Homogeneous case:} \\
r^2 + 2r - 8 = 0 \\
r = -4 \text{ and } r = 2 \\
x_n = c_1(-4)^n + c_2(2)^n + 9 \\
\text{For } n = 0, \ c_1 + c_2 = 9 = 11 \\
\text{For } n = 1, -2c_1 + 3c_2 + 9 = -5 \\
\text{Solving the system gives } c_1 = 3 \text{ and } c_2 = -1 \\
\text{Therefore, } x_n = 3(-4)^n - (2)^n + 9 \\
\)

29. \( c = 9c - 18c + 20 \rightarrow c = 2 \\
\text{Homogeneous case:} \\
r^2 - 9r + 18 = 0 \\
r = 3 \text{ and } r = 6 \\
x_n = c_1(3)^n + c_2(6)^n + 2 \\
\text{For } n = 0, \ c_1 + c_2 = 2 \\
\text{For } n = 1, 3c_1 + 6c_2 = 2 + 26 \\
\text{Solving the system gives } c_1 = 2 \text{ and } c_2 = 3 \\
\text{Therefore, } x_n = 3(3)^n + 3(6)^n + 2 \\
\)

31. \( a = 0.35 \) and \( b = 0.45 \\
\text{Since } a + b < 1 \\
The population decreases exponentially to 0 \\

33. \( a = 2.5 \) and \( b = 1.5 \\
\text{Since } a + b > 1 \\
The population grows exponentially to \( \infty \) \\

35. \( a = 0.01 \) and \( b = 1.5 \\
\text{Since } a + b > 1 \\
The population grows exponentially to \( \infty \) \\

37. \( a) \ r^2 - 0.92r - 0.15 = 0 \\
r = -0.14133 \text{ and } r = 1.06133 \\
x_n = c_1(-0.14133)^n + c_2(1.06133)^n \\
\text{For } n = 0, c_1 + c_2 = 0 \\
\text{For } n = 1, -0.14133c_1 + 1.06133c_2 = 50 \\
\text{Solving the system gives } c_1 = -41.57451 \text{ and } c_2 = 41.57451 \\
\text{Therefore, } x_n = -41.57451(-0.14133)^n + 41.57451(1.06133)^n \\
b) \text{ Since } 0.92 + 0.15 > 1, \text{ the population will grow} \\
\)

39. \( x_{n+1} = ax_n + bz_n \\
= ax_n + abM - ax_{n-1} \\
= ax_n - abx_{n-1} + abM \\
\)

41. \( x_{n+1} = 0.95x_n - 0.095x_{n-1} + 0.95 \\
a) \ c = 0.95c - 0.095c + 0.95 \rightarrow c = 955.17241 \\
\text{Homogeneous case:} \\
r^2 - 0.95r + 0.95 = 0 \\
r = 0.115368 \text{ and } r = 8.36912 \\
x_n = c_1(0.115368)^n + c_2(8.36912)^n + 655.17241 \\
b) \text{For } n = 0, c_1 + c_2 = 655.17241 \rightarrow 50 \\
\text{For } n = 1, 0.11536c_1 + 8.36912c_2 = 655.17241 \rightarrow 130 \\
\text{Solving the system gives } c_1 = 26.27632 \text{ and } c_2 = -631.4873 \\
\text{Therefore, } x_n = 26.27632(0.115368)^n - 631.4873(8.36912)^n + 655.17241 \\
\)
Chapter 7

Functions of Several Variables

Exercise Set 7.1

1. \( f(x, y) = x^2 - 2xy \)
   \[
   f(0, -2) = 0^2 - 2(0)(-2) = 0 - 0 = 0
   
   f(2, 3) = 2^2 - 2(2)(3) = 4 - 12 = -8
   
   f(10, -5) = 10^2 - 2(10)(-5) = 100 + 100 = 200
   
2. \( f(x, y) = 3x + 7xy \)
   \[
   f(0, -2) = 3(0) + 7(0)(-2) = 0 + 0 = 0
   
   f(-2, 1) = 3(-2) + 7(-2)(1) = -6 + 14 = 8
   
   f(2, 1) = 3(2) + 7(2)(1) = 6 + 14 = 20
   
3. \( f(x, y) = \sin x \tan y \)
   \[
   f(\pi/2, 0) = \sin \pi/2 \tan 0 = 1(0) = 0
   
   f(3\pi/4, 3\pi/4) = \sin 3\pi/4 \tan 3\pi/4 = (-1)(\sqrt{2}) = -\sqrt{2}
   
   f(\pi/6, \pi/1) = \sin \pi/6 \tan \pi/1 = (1/2)(1) = 1/2
   
4. \( f(x, y, z) = x^2 - y^2 + z^2 \)
   \[
   f(-1, 2, 3) = (-1)^2 - (2)^2 + (3)^2 = 1 - 4 + 9 = 6
   
   f(2, -1, 3) = (2)^2 - (-1)^2 + (3)^2 = 4 - 1 + 9 = 12
   
5. \( S(h, w) = \sqrt{\frac{3}{2}} \)
   \[
   S(165, 80) = \sqrt{\frac{3}{2}} \cdot 165 \cdot 80
   
   \approx 1.915 \text{ m}^2
   
6. \( w(x, y, z, m, v, p) = x(9.38 + 0.264s + 0.000233h(0 + 1.62x)p + 1) \)
   
   a) \( w(275, 1, 160, 71, 0.068, 0) \)
   \[
   w(275, 1, 160, 71, 0.068, 0) = 275(9.38 + 0.264(1)
   
   + 0.000233(160)(71) + 1.62(0.068)(0 + 1))
   
   \approx 3466.386
   
   b) \( w(282, -1, 171, 76, 0.085, 3) \)
   \[
   w(282, -1, 171, 76, 0.085, 3) = 282(9.38 + 0.264(-1)
   
   + 0.000233(171)(71) + 4.62(0.085)(3 + 1))
   
   \approx 3771.569
   
13. \( S(d, V, a) = \frac{d}{\sqrt{a^2}} \)
   \[
   S(100, 160000, 0.78) = \frac{100}{\sqrt{0.78^2(160000)^2}}
   
   = 244.766
   
15. \( V(L, R, x, R) = \frac{1}{4\pi}(R^2 - x^2) \)
   \[
   V(1, 100, 0.0075, 50.05) = \frac{100}{1(1)(0.05)} \times
   
   ((0.0075)^2 - (0.0025)^2)
   
   = 0.025
   
17. Left to the student (answer may vary)

19. \( W: (-10, 20) \) \(-22\)

21.
Exercise Set 7.2

1. \(z = 2x - 3xy\)
   \[
   \frac{\partial z}{\partial x} = 2 - 3y \\
   \frac{\partial z}{\partial y} = -3x \\
   \frac{\partial z}{\partial y} |_{(x=-2, y=3)} = 2 - 3(-3) = -11 \\
   \frac{\partial z}{\partial y} |_{(x=0, y=0)} = -3(0) = 0
   \]

3. \(z = 3x^2 - 2xy + y\)
   \[
   \frac{\partial z}{\partial y} = 6x - 2y \\
   \frac{\partial z}{\partial x} = -2x + 1 \\
   \frac{\partial z}{\partial x} |_{(x=-2, y=3)} = 6(-2) - 2(-3) = -12 + 6 = -6 \\
   \frac{\partial z}{\partial x} |_{(x=0, y=0)} = -2(0) + 1 = 0 + 1 = 1
   \]

5. \(f(x, y) = 2x - 3y\)
   \[f_x = \text{2 (a constant independent of \(x\) or \(y\))} \\
   f_y = -3 \text{ (a constant independent of \(x\) or \(y\))} \\
   f_x(-2, 1) = 2 \\
   f_y(-3, -2) = -3
   \]

7. \(f(x, y) = (x^2 + y^2)^{1/2}\)
   \[f_x = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}} \\
   f_y = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}} \\
   f_x(-2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2}{\sqrt{5}} \\
   f_y(-3, -2) = \frac{-2}{\sqrt{5}} = \frac{-2}{\sqrt{5}}
   \]

9. \(f(x, y) = 2x - 3y\)
   \[f_x = 2 \\
   f_y = -3
   \]

11. \(f(x, y) = \sqrt{x} + \sin(xy)\)
   \[f_x = \frac{1}{2\sqrt{x}} + \cos(xy)(y) = \frac{1}{2\sqrt{x}} + y \cos(xy) \\
   f_y = \cos(xy)(x) = x \cos(xy)
   \]

13. \(f(x, y) = x \ln y\)
   \[f_x = \ln y \\
   f_y = x \cdot \frac{1}{y} = \frac{x}{y}
   \]

15. \(f(x, y) = x^3 - 3xy + y^2\)
   \[f_x = 3x^2 - 4y \\
   f_y = -4x + 2y
   \]

17. \(f(x, y) = (x^2 + 2y + 2)^1\)
   \[f_x = 8(x^2 + 2y + 2)^0(2x) = 8x(x^2 + 2y + 2)^3 \\
   f_y = 8(x^2 + 2y + 2)^0(2) = 8(x^2 + 2y + 2)
   \]

19. \(f(x, y) = \sin(e^{x+y})\)
   \[f_x = \cos(e^{x+y})(e^{x+y})(1) = e^{x+y} \cos(e^{x+y}) \\
   f_y = \cos(e^{x+y})(e^{x+y})(1) = e^{x+y} \cos(e^{x+y})
   \]

21. \(f(x, y) = \frac{\ln x}{\sqrt{y}}\)
   \[f_x = \frac{\frac{1}{x} \cdot e^{e^y}}{\sqrt{y}} = \frac{xe^{e^y}}{y} \\
   f_y = \frac{\ln x}{\sqrt{y}} \cdot \frac{1}{e^y(2y^2 + 1)}
   \]

23. \(f(x, y) = |x^5 + \tan(y^2)|^1\)
   \[f_x = 4|x^5 + \tan(y^2)|^0(5x^4) = 20x^4|x^5 + \tan(y^2)|^3 \\
   f_y = 4|x^5 + \tan(y^2)|^0(2y \cos^2(y^2)(2y)) = 8y \cos^2(y^2) \cdot |x^5 + \tan(y^2)|^3
   \]

25. \(f(x, y) = \frac{2 \ln x}{y^2 - 1}\)
   \[f_x = \frac{2 \cdot \frac{1}{x} \cdot 1}{y^2 - 1} = \frac{2}{xy^2 - 1} \\
   f_y = \frac{2 \cdot \frac{1}{x} \cdot x - y \ln x}{{y^2 - 1}^2} = \frac{-2y^2 - 1 \ln x}{y^2 - 1} \]
27. \( f(b, m) = (m+b-4)^2 + (2m+b-5)^2 + 3m + b + 30 \)

\[ \frac{\partial f}{\partial b} = \frac{2(3m+b-6)(1)}{2m+2b-8+8m+12m+2b-54+12m+6b-30} \]

\[ = \frac{2m+2b-8+8m+12m+2b-54+12m+6b-30}{28m+12b-64} \]

29. \( z = \frac{x^2+4y}{2x+y} \)

\[ z_x = \frac{(x^2+4y)(2x) - (x^2+y^2)(2x)}{(2x+y)^2} = \frac{2x^3 - 2x^2y - 2x^2y - 2y^2x}{(2x+y)^2} \]

\[ = \frac{4x^2y}{(2x+y)^2} \]

\[ z_y = \frac{(x^2+y^2)(2y) - (x^2+y^2)(2y)}{(x^2+y^2)^2} = \frac{2x^2y - 2x^2y + 2y^2x - 2y^2y}{(x^2+y^2)^2} \]

\[ = \frac{4y^2x}{(x^2+y^2)^2} \]

31. \( z = \frac{2\sqrt{2}-2\sqrt{3}}{1+2\sqrt{3}} \)

\[ z_x = \frac{2}{1+\sqrt{3}} \frac{1}{1+2\sqrt{3}} \]

\[ z_y = \frac{2\sqrt{3} + 2\sqrt{2} + 2\sqrt{3}}{1+2\sqrt{3}} \]

\[ = \frac{2\sqrt{3} + 2\sqrt{2} - 1 - 2\sqrt{3}}{1+2\sqrt{3}} \]

\[ = \frac{\sqrt{3} - 1 + 2\sqrt{2}}{2\sqrt{3} + 2\sqrt{2}} \]

33. \( z = (x^2+y^2)^{1/4} \)

\[ z_x = \frac{1}{4}(x^2+y^2)^{-3/4}(2x^2) \]

\[ z_y = \frac{1}{4}(x^2+y^2)^{-3/4}(2y^2) \]

35. \( f(x, y) = x + 3y, g(x, y) = x - 2y \)

\[ J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \]

37. \( f(x, y) = \sqrt{x^2+y^2}, g(x, y) = e^{x+y} \)

\[ J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+3y)^{-1/2}(1) & \frac{1}{2}(x+3y)^{-1/2}(3) \\ e^{x+y}(-1) & e^{x+y}(-1) \end{bmatrix} \]

39. We will use the results from Exercise 9.

\[ f_x = 2 \to f_{yx} = 0 \text{ and } f_{xy} = 0 \]

\[ f_y = -3 \to f_{yx} = 0 \text{ and } f_{xy} = 0 \]

41. We will use the results from Exercise 11.

\[ f_x = \frac{1}{2}x^{-1/2} + y \cos(xy) \to \]

\[ f_{yx} = \frac{1}{4}e^{-3/2} - y(\sin(xy)(y) = \frac{1}{4\sqrt{e}} - y^2\sin(xy) \cos(xy) \]

\[ f_{xy} = y[-\sin(xy)(x) + \cos(xy)(1) = -xy \sin(xy) + \cos(xy) \]

\[ f_y = x \cos(xy) \to \]

\[ f_{yx} = -x[-\sin(xy)(y) + \cos(xy)(1) = -xy \sin(xy) + \cos(xy) \]

\[ f_{xy} = x[-\sin(xy)(x) + x^2 \sin(xy) \]

43. We will use the results from Exercise 13.

\[ f_x = \ln y \to f_{yx} = 0 \]

\[ f_{yx} = \frac{1}{y} \]

\[ f_y = \frac{z}{y} \to f_{xy} = \frac{1}{y} \]

\[ f_{yy} = x - 1y^{-2} - \frac{z}{y} \]
45. We will use the results from Exercise 15.

\[ f_x = 3x^2 - 4y \Rightarrow f_{xx} = 6x \text{ and } f_{xy} = -4 \]
\[ f_y = -4x + 2y \Rightarrow f_{yx} = -4 \text{ and } f_{yy} = 2 \]

47. \( f(x, y, z) = x^2y^3z^4 \)

\[ f_x = 3x^2y^3z^4 \]
\[ f_y = 6x^2y^2z^4 \]
\[ f_z = 4x^2y^3z^3 \]

49. \( f(x, y, z) = e^{x+y+z+t} \)

\[ f_x = e^{x+y+z+t}(1) = e^{x+y+z+t} \]
\[ f_y = e^{x+y+z+t}(2y) = 2y e^{x+y+z+t} \]
\[ f_z = e^{x+y+z+t}(3z^2) = 3z^2 e^{x+y+z+t} \]

51. \( z = f(x, y) = xy^2 \)

\[ f(2, 3) = (2)(3)^2 = 18 \cdot 9 = 36 \]
\[ f_x = y^2, \text{ so } f_x(2, 3) = 36 \]
\[ f_y = 2xy, \text{ so } f_y(2, 3) = 2(2)(3) = 12 \]
\[ z(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]
\[ z(2, 1.02) = f(2, 3) + f_x(2, 3)(2.01 - 2) + f_y(2, 3)(0.02) \]
\[ = 36 + 9(0.01) + 12(0.02) \]
\[ = 38.34 \]

53. \( z = f(x, y) = x \sin(xy) \)

\[ f(1, 0) = (1) \sin(0) = 0 \]
\[ f_x = x |\cos(xy)| \cdot \sin(xy), \text{ so } f_x(1, 0) = 1 |\cos(0)| \cdot \sin(0) = 0 \]
\[ f_y = x |\cos(xy)| \cdot x \sin(xy), \text{ so } f_y(1, 0) = x^2 \cos(0) = 1 \]
\[ z(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]
\[ z(0, 0, 0.02) = f(1, 0) + f_x(1, 0)(0.02) = 0 + 0(0) + 0(0.02) \]
\[ = 1.02 \]

55. \( S(h, w) = \sqrt{hw}/60 \)

a) \( S(100, 28) \)

\[ S(100, 28) = \frac{\sqrt{100}(28)}{60} \]
\[ = \frac{\sqrt{2800}}{60} \]
\[ = 0.881917 \]

b) Use linearization to estimate \( S(102, 30) \)

\[ S(100, 28) = 0.881917 \]
\[ S_h = \frac{a}{150} \sqrt{100}, \text{ so } \]
\[ S_h(100, 28) = \frac{28}{150} \sqrt{2800} = 0.00441 \]
\[ S_w = \frac{b}{30} \sqrt{2800}, \text{ so } \]
\[ S_w(100, 28) = \frac{100}{30} \sqrt{2800} = 0.05175 \]
\[ S(102, 30) = S(100, 28) + S_h(100, 28)(102 - 100) + S_w(100, 28)(30 - 28) \]
\[ = 0.881917 + 0.00441(2) + 0.05175(2) \]
\[ = 0.9294 \]

57. \( w(x, s, h, m, r, p) = x(0.38 + 0.264s + 0.000233hm + 4.62r[p + 1]) \)

a) \( w(280, 1, 150, 65, 0.08, 0) \)

\[ w = 280(0.38 + 0.264(1) + 0.000233(150)(65) + 4.62(0.08)[0 + 1]) \]
\[ = 280(0.38 + 0.264 + 2.27175 + 0.3696) \]
\[ = 3439.9 \]

b) \( w_r = (9.38 + 0.264s + 0.000233hm + 4.62r[p + 1]) \)

\[ w_r = 4.62e[p + 1] \]

59. \( P(m, T) = \frac{1}{1 + e^{(m - 31.660)(0.155) + 0.0882(T - 20)}} \)

a) \( P(0.15, 20) \)

\[ P(0.15, 20) = \frac{1}{1 + e^{(0.15 - 31.660)(0.155) + 0.0882(20)}} \]
\[ = 0.74773 \]

b) \( P_m = -\frac{1}{1 + e^{(m - 31.660)(0.155) + 0.0882(T - 20)}} \)

\[ P_m = -\frac{1}{1 + e^{4.5322 - 31.660(0.155) + 0.0882(20)}} \]
\[ = -\frac{1}{1 + e^{0.839}} \]
$P_m(0.15, 20) = \frac{-31.669}{\left(1 + e^{3.272 - 31.669(0.15) + 6.91065(20)}\right)^2}$

$P_r(0.15, 20) = \frac{-0.083}{\left(1 + e^{3.272 - 31.669(0.15) + 6.91065(20)}\right)^2}$

$\Delta T = 0.5107$

61.

$T_0 (0.1) = 1.98(90) - 1.10(1 - 1)(90 - 58) - 56.9 = 124.3$

63. $\frac{d}{dT} 1.09(T - 58)$

This means that for every 1 point change in humidity at a specific temperature $T$, the Temperature-Humidity index changes by 1.09($T - 58$).

65.

$L(146, 5) = 206.835 - 0.846(146) - 1.015(5) = 206.835 - 123.516 - 5.075 = 78.244$

67. $\frac{dK}{dx} = -0.846$. This means that for every increase of one syllable, there is a decrease of 0.846 in the reading ease of a 100-word section.

69. $f(x, y) = \ln(x^2 + y^2)$

\[ f_x = \frac{x^2}{x^2 + y^2} \]

\[ f_y = \frac{y^2}{x^2 + y^2} \]

71. a)

\[ \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \to 0} \frac{0.04(1)}{h} = \frac{0.04(1)}{h} \]

b)

\[ \lim_{h \to 0} \frac{f(x, h) - f(x, 0)}{h} = \lim_{h \to 0} \frac{x^2(1 - h^2)}{h} = \frac{x^2(1 - h^2)}{h} \]

c) Using the results from the previous two parts we have $f_y(x, 0) = -y$, which means $f_{yx}(x, 0) = -y$. Also, $f_y(0, y) = -y$, which means $f_{xy}(0, y) = -y$, and thus $f_{xy}(0, y) = -1$. We see that $f_{xy}(0, y) = -f_{yx}(0, y)$.

Exercise Set 7.3

1. $f(x, y) = x^2 + xy + y^2 - y$

- Find the partial derivatives:
  \[ f_x = 2x + y, \quad f_y = 2y \]

  \[ f_y = x + 2y - 1, \quad f_{xy} = 2 \]

  \[ f_{xy} = 1 \]

- We solve $f_x = 0$ and $f_y = 0$. We use the substitution method. From $f_x = 0$, we can write $y = -2x$. Thus:

  $x + 2y - 1 = 0$

  $x + 2(-2x) - 1 = 0$

  $x - 4x = 1$

  $x = \frac{-1}{3}$

  and therefore:

  $y = \frac{2}{3}$
Chapter 7: Functions of Several Variables

- Find $D$ for $(-1/3, -2/3)$

\[ D = f_{xx}(-1/3, -2/3) \cdot f_{yy}(-1/3, -2/3) - \left| f_{xy}(-1/3, -2/3) \right|^2 \]

\[ = 2 \cdot 2 - \left| 1 \right|^2 \]

\[ = 4 - 1 \]

\[ \approx 3 \]

- Since $D > 0$ and $f_{xx}(-1/3, -2/3) > 0$, $f(x, y)$ has a relative minimum at $(-1/3, -2/3)$

3. $f(x, y) = 2xy - x^3 - y^2$

- Find the partial derivatives:

\[ f_x = 2y - 3x^2, \quad f_{xx} = -6x \]

\[ f_y = 2x - 2y, \quad f_{yy} = -2 \]

\[ f_{xy} = 2 \]

- We solve $f_x = 0$ and $f_y = 0$. We use the substitution method, from $f_y = 0$ we can write the $y = x$. Thus

\[ 2y - 3x^2 = 0 \]

\[ 2x - 3x^2 = 0 \]

\[ x(2 - 3x) = 1 \]

\[ x = 0 \]

and therefore

\[ y = 0 \]

and therefore

\[ y = 2/3 \]

- Find $D$ for $(0, 0)$ and $(1, 1)$

\[ D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - \left| f_{xy}(0, 0) \right|^2 \]

\[ = 0 \cdot 0 - \left| 0 \right|^2 \]

\[ = -9 \]

\[ D = f_{xx}(1, 1) \cdot f_{yy}(1, 1) - \left| f_{xy}(1, 1) \right|^2 \]

\[ = 6 \cdot 6 - \left| 3 \right|^2 \]

\[ = 27 \]

- Since $D < 0$ at $(0, 0)$, $f(x, y)$ has a saddle at $(0, 0)$.

- Since $D > 0$ and $f_{xx}(1, 1) > 0$, $f(x, y)$ has a relative minimum at $(1, 1)$

7. $f(x, y) = x^2 + y^2 - 2x + 4y - 2$

- Find the partial derivatives:

\[ f_x = 2x - 2, \quad f_{xx} = 2 \]

\[ f_y = 2y + 4, \quad f_{yy} = 2 \]

\[ f_{xy} = 0 \]

- We solve $f_x = 0$ and $f_y = 0$.

\[ 2x - 2 = 0 \]

\[ x = 2 \]

\[ 2y + 4 = 0 \]

\[ y = -2 \]

- Find $D$ for $(1, -2)$

\[ D = f_{xx}(1, -2) \cdot f_{yy}(1, -2) - \left| f_{xy}(1, -2) \right|^2 \]

\[ = 2 \cdot 2 - \left| 0 \right|^2 \]

\[ = 4 \]

- Since $D > 0$ and $f_{xx}(1, -2) > 0$, $f(x, y)$ has a relative minimum at $(1, -2)$
9. \( f(x, y) = x^2 + y^2 \mid 2x - 4y \)
   - Find the partial derivatives:
     \[ f_x = 2x, \quad f_{xx} = 2 \]
     \[ f_y = 2y - 1, \quad f_{yy} = 2 \]
     \[ f_{xy} = 0 \]
   - We solve \( f_x = 0 \) and \( f_y = 0 \).
     \[
     \begin{align*}
     2x & \mid 0 \\
     x & = -2 \\
     y & = 0
     \end{align*}
     \]
   - Find \( D \) for \((-1, 2)\)
     \[
     D = f_{xx}(-1, 2) \cdot f_{yy}(-1, 2) - |f_{xy}(-1, 2)|^2
     = 2 \cdot 2 - 0^2
     = 4
     \]
   - Since \( D > 0 \) and \( f_{xx}(-1, 2) > 0 \), \( f(x, y) \) has a relative minimum at \((-1, 2)\).

11. \( f(x, y) = 4x^2 - y^2 \)
   - Find the partial derivatives:
     \[ f_x = 8x, \quad f_{xx} = 8 \]
     \[ f_y = -2y, \quad f_{yy} = -2 \]
     \[ f_{xy} = 0 \]
   - We solve \( f_x = 0 \) and \( f_y = 0 \).
     \[
     \begin{align*}
     8x & \mid 0 \\
     x & = 0 \\
     -2y & \mid 0 \\
     y & = 0
     \end{align*}
     \]
   - Find \( D \) for \((0, 0)\)
     \[
     D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - |f_{xy}(0, 0)|^2
     = 8 \cdot (-2) - 0^2
     = -16
     \]
   - Since \( D < 0 \) at \((0, 0)\), \( f(x, y) \) has a saddle at \((0, 0)\).

13. \( f(x, y) = e^{x^2 + y^2 + 1} \)
   - Find the partial derivatives:
     \[ f_x = 2xe^{x^2 + y^2 + 1}, \quad f_{xx} = 2e^{x^2 + y^2 + 1} \]
     \[ f_y = 2ye^{x^2 + y^2 + 1}, \quad f_{yy} = 2e^{x^2 + y^2 + 1} \]
     \[ f_{xy} = 2xye^{x^2 + y^2 + 1} \]
   - We solve \( f_x = 0 \) and \( f_y = 0 \).
     \[
     \begin{align*}
     2xe^{x^2 + y^2 + 1} & \mid 0 \\
     2x & \mid 0 \\
     x & = 0 \\
     2ye^{x^2 + y^2 + 1} & \mid 0 \\
     2y & \mid 0 \\
     y & = 0
     \end{align*}
     \]
   - Find \( D \) for \((0, 0)\)
     \[
     D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - |f_{xy}(0, 0)|^2
     = 2 \cdot 2 - 0^2
     = 4
     \]
   - Since \( D > 0 \) and \( f_{xx}(0, 0) > 0 \), \( f(x, y) \) has a relative minimum at \((0, 0)\).

15. We need to find the point at which \( P_x = 0 \) and \( P_y = 0 \)
   \[
   P_x = 0.0345 - 0.000230x + 0.109y \\
   P_y = 25.6 \times 0.109x \approx 26.4y
   \]
   Solve \( P_x = 0 \) and \( P_y = 0 \). We can use the substitution method. From \( P_x \):
   \[
   y = 0.0021t - 0.31651
   \]
   \[
   25.6 \times 0.109x \approx 26.4y
   \]
   \[
   -0.157t + 65.60996 = 0
   \]
   \[
   x = 116.0127
   \]
   and therefore
   \[
   0.00211(116.0127) - 0.31651 = y
   \]
   \[
   y \approx 0.56218
   \]

17. \( P = 5a^2 - 3\lambda^2 \mid 4\lambda - 4a + 2\lambda + 300 \)
   \[
   P_\alpha = -10a \times 18 \times 2a
   P_{\alpha} = -10
   \]
   \[
   P_{\alpha} = 2
   \]
   \[
   P_\beta = -6\lambda - 4 \times 2a
   \]
   \[
   V_{\alpha} = -6
   \]
   Solve \( P_\alpha = 0 \) and \( P_\beta = 0 \). From \( P_\alpha \):
   \[
   a = 55a - 24
   \]
   \[
   -6a - 4 + 2a = 0
   \]
   \[
   -6(5a - 21) - 4 + 2a = 0
   \]
   \[
   -30a + 141 - 4 + 2a = 0
   \]
   \[
   -28a = -140
   \]
   \[
   a = 5
   \]
   therefore
   \[
   \lambda = 5(5) - 24
   \]
   \[
   \lambda = 1
   \]
Find $D$ for $(5, 1)$

\[
D = -10 - 6 - |2|^2
\]

\[
= 60 - 4
\]

\[
= 56
\]

Since $D > 0$ and $P_{0w}(5, 1) < 0$ then $P(a, b)$ has a maximum at $(5, 1)$. To find the maximum value we have to find $P(5, 1)$

\[
P(5, 1) = -5(5)^2 - 3(1)^2 + 48(5) +
\]

\[
-4(1) + 2(5)(1) + 300
\]

\[
= -125 - 3 + 240 - 4 + 10 + 300
\]

\[
= 418
\]

19. \(T(x, y) = x^2 + 2y^2 - 8x + 4y\)

\[T'_x = 2x - 8\]

\[T''_x = 2\]

\[T'_y = 4y + 4\]

\[T''_y = 4\]

Solve $T'_x = 0$ and $T'_y = 0$

\[
2x - 8 = 0
\]

\[
2x = 8
\]

\[
x = 4
\]

\[
4y + 4 = 0
\]

\[
4y = -4
\]

\[
y = -1
\]

Find $D$ for $(4, -1)$

\[
D = 2 \cdot 4 - |0|^2
\]

\[
= 8
\]

Since $D > 0$ and $T_{xx}(4, -1) > 0$ then $T(x, y)$ has a minimum at $(4, -1)$. To find the value of the minimum we need to find $T(4, -1)$

\[
T(4, -1) = (4)^2 + 2(-1)^2 - 8(4) + 4(-1)
\]

\[
= 16 + 2 - 32 - 4
\]

\[
= -18
\]

There is no maximum value.

21. \(f(x, y) = e^x + e^y - e^{x+y}\)

\[f_x = e^x - e^{x+y}\]

\[f_{xx} = e^x - e^{x+y}\]

\[f_{xy} = -e^{x+y}\]

\[f_y = e^y - e^{x+y}\]

\[f_{yy} = e^y - e^{x+y}\]

Solve $f_x = 0$ and $f_y = 0$

\[e^x - e^{x+y} = 0\]

\[e^x = e^{x+y}\]

\[x = x + y\]

\[0 = y\]

\[e^y - e^{x+y} = 0\]

\[e^y = e^{x+y}\]

\[y = x + y\]

\[0 = x\]

Find $D$ for $(0, 0)$

\[
D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2
\]

\[
= 0 \cdot 0 - |1|^2
\]

\[
= -1
\]

Since $D < 0$ at $(0, 0)$ then $f(x, y)$ has a saddle point at $(0, 0)$

23. \(f(x, y) = 2y^2 + x^2 - x^2 y\)

\[f_x = 2x - 2xy\]

\[f_{xx} = 2 - 2y\]

\[f_{xy} = -2x\]

\[f_{yy} = -x^2\]

Solve $f_x = 0$ and $f_y = 0$

\[2x - 2xy = 0\]

\[2x(1 - y) = 0\]

\[x = 0\]

and

\[y = 1\]

From $f_y = 0$ we get $y = x^2 / 4$ which means when $x = 0$, $y = 0$ and when $y = 1, x = ±2$. Find $D$ for $(0, 0)$

\[
D = 2 \cdot 4 - |0|^2
\]

\[
= 8
\]

Since $D > 0$ and $f_{xx}(0, 0) > 0$ then $f(x, y)$ has a relative minimum at $(0, 0)$ Find $D$ for $(2, 1)$

\[
D = 2 \cdot 4 - |1|^2
\]

\[
= -16
\]

Since $D < 0$ at $(2, 1)$ then $f(x, y)$ has a saddle point at $(2, 1)$ Find $D$ for $(-2, 1)$

\[
D = 0 \cdot 4 - |1|^2
\]

\[
= -16
\]

Since $D < 0$ at $(-2, 1)$ then $f(x, y)$ has a saddle point at $(-2, 1)$
25. The D-Test is a method similar to the second derivative test for functions of one variable. It computes the points where the first partial derivatives are zero and then computes the value of D to determine the nature of the zeros of the first partial derivatives.

27. \( R = e^{-1.29} \cdot \cos \theta \cos \theta \cos \theta \sin \sin \theta \)

a) \( \frac{\partial R}{\partial \theta} \)

\[
\frac{\partial R}{\partial \theta} = -e^{-1.29} \cdot 1.35 \cdot \cos \theta \cdot \cos \theta \cdot \sin \sin \theta
\]

b) The sign of \( \frac{\partial R}{\partial \theta} \) is determined by the sign of the term in parenthesis in part a) since the other terms is positive for all values permissible. Since the trigonometric functions that are positive in the first quadrant (the permissible values for \( \theta \) and \( \phi \)) fall in the first quadrant the term in parenthesis will always be negative. Thus, \( \frac{\partial R}{\partial \theta} \) is always negative for the permissible values of \( \theta \) and \( \phi \).

c - d) Since \( \frac{\partial R}{\partial \phi} \) is negative for \( 0 \leq \phi \leq 90 \) then the function is decreasing in the \( \phi \)-direction which means that the maximum occurs when \( \phi = 90 \) (the beginning of the interval).

29. Relative minimum of -5 at \((0,0)\)

Exercise Set 7.4

1. \( \bar{x} = \frac{0 + 1 + \ldots + 6}{7} = 3 \)

\[
\bar{y} = \frac{33.49 + 31.72 + \ldots + 47.70}{7} = 40.75
\]

\[
m = \frac{(0 - 3)(33.49 - 40.75) + \ldots + (6 - 3)(47.70 - 40.75)}{(0 - 3)^2 + (1 - 3)^2 + \ldots + (6 - 3)^2} = 2.618
\]

a) The regression line is

\[
y - \bar{y} = m(x - \bar{x})
\]

\[
y - 40.75 = 2.618(x - 3)
\]

\[
y = 2.618x - 7.854 + 40.75
\]

\[
y = 2.618x + 32.896
\]

b) Find \( y \) when \( x = 16 \)

\[
y = 2.618(16) + 32.896 = 74.78
\]

Find \( y \) when \( x = 21 \)

\[
y = 2.618(21) + 32.896 = 87.87
\]

3. \( \bar{x} = \frac{0 + 10 + \ldots + 50}{6} = 25 \)

\[
\bar{y} = \frac{71.1 + 73.1 + \ldots + 79.5}{6} = 75.8
\]

\[
m = \frac{(0 - 25)(71.1 - 75.8) + \ldots + (50 - 25)(79.5 - 75.8)}{(0 - 25)^2 + (20 - 25)^2 + \ldots + (50 - 25)^2} = 0.177
\]

a) The regression line is

\[
y - \bar{y} = m(x - \bar{x})
\]

\[
y - 75.8 = 0.177(x - 25)
\]

\[
y = 0.177x - 1.325 + 75.8
\]

\[
y = 0.177x + 71.375
\]

b) Find \( y \) when \( x = 60 \)

\[
y = 0.177(60) + 71.375 = 81.995
\]

Find \( y \) when \( x = 65 \)

\[
y = 0.177(65) + 71.375 = 82.88
\]

5. \( \bar{x} = \frac{0 + 1 + \ldots + 4}{5} = 2 \)

\[
\bar{y} = \frac{86 + 83.6 + \ldots + 79.5}{5} = 82.28
\]

\[
m = \frac{(0 - 2)(86 - 82.28) + \ldots + (4 - 2)(79.5 - 82.28)}{(0 - 2)^2 + (2 - 2)^2 + \ldots + (4 - 2)^2} = -1.63
\]

a) The regression line is

\[
y - \bar{y} = m(x - \bar{x})
\]

\[
y - 82.28 = -1.63(x - 2)
\]

\[
y = -1.63x + 3.26 + 82.28
\]

\[
y = -1.63x + 85.54
\]

b) Find \( y \) when \( x = 11 \)

\[
y = -1.63(11) + 85.54 = 62.72
\]

7. \( \bar{x} = \frac{70 + 60 + \ldots + 85}{3} = 71.67 \)

\[
\bar{y} = \frac{75 + 60 + \ldots + 85}{3} = 75.34
\]
\[ m = \frac{(70 - 71.67)(75 - 75.34) + (60 - 71.67)(62 - 75.34)}{(70 - 71.67)^2 + (60 - 71.67)^2 + (85 - 71.67)^2} \\
= 1.07 \]

a) The regression line is
\[ y - \bar{y} = m(x - \bar{x}) \]
\[ y - 75.34 = 1.07(x - 71.67) \]
\[ y = 1.07x - 76.69 + 75.34 \]
\[ y = 1.07x - 1.35 \]

b) Find \( y \) when \( x = 81 \)
\[ y = 1.07(81) - 1.35 \]
\[ = 85.32 \]

9. Linear regression is a method for finding an equation to model a data set obtained from an experiment.

11. a)
\[
\begin{array}{|c|c|}
\hline
X & Y = \log y \\
\hline
1.4771 & 1.3979 \\
2.301 & 1.4771 \\
4.301 & 1.3931 \\
7.3979 & 2.2301 \\
9 & 2.3979 \\
\hline
\end{array}
\]

\[ \bar{x} = 4.89542 \text{ and } \bar{y} = 1.88131 \]

\[ m = \frac{(1.4771 - 4.89542)(1.3979 - 1.88131)}{(1.4771 - 4.89542)^2 + \cdots (9 - 4.89542)^2} \\
= 0.13568 \]

The regression line is
\[ Y - \bar{Y} = m(X - \bar{X}) \]
\[ Y - 1.88131 = 0.13568(X - 4.89542) \]
\[ Y = 0.13568X - 0.66421 + 1.88131 \]
\[ Y = 0.13568X + 1.21710 \]

c) \[
Y = 0.13568X + 1.21710 \\
\log(y) = 0.13568 \log(x) + 1.21710 \]
\[
\begin{align*}
\log(y) &= \log(x^{0.13568}) + 1.21710 \\
\log(y) &= \log(x^{0.13568}) + 1.21710 \\
\frac{y}{x^{0.13568}} &= 10^{0.13568} \\
\frac{y}{x^{0.13568}} &= 10^{0.13568} \\
y &= 16.48542 x^{0.13568}
\end{align*}
\]

d) Find \( y \) when \( x = 190000 \)
\[ y = 16.48542 (190000)^{0.13568} \]
\[ y = 197 \]

13. a) \( y = -0.005938x + 15.571914 \)

b) In 2010
\[ y = -0.005938(2010) + 15.571914 = 3.636534 = 3 : 38 : 19 \]
In 2015
\[ y = -0.005938(2015) + 15.571914 = 3.006844 = 3 : 36 : 41 \]

c) In 1999, the predicted value for the record is
\[ y = -0.005938(1999) + 15.571914 = 3.701852 = 3 : 42 : 11 \]

---

**Exercise Set 7.5**

1. \[
\int_0^1 \int_0^1 2y \, dx \, dy = \int_0^1 2y(1 - 0) \, dy \\
= \int_0^1 2y \, dy \\
y = \frac{y^2}{2} \bigg|_0^1 \\
= 1 - 0 \\
y = 1
\]

3. \[
\int_{-1}^1 \int_{-1}^1 xy \, dy \, dx = \int_{-1}^1 \frac{x^2}{2} \bigg|_{-1}^1 \, dx \\
= \int_{-1}^1 \left( \frac{x^2}{2} - \frac{x^2}{2} \right) \, dx \\
= \left( \frac{x^3}{4} - \frac{x^3}{4} \right) \bigg|_{-1}^1 \\
= \left( \frac{1}{4} - \frac{1}{4} \right) - \left( \frac{-1}{4} - \frac{-1}{4} \right) \\
= 0
\]

5. \[
\int_0^1 \int_{-1}^1 (x + y) \, dy \, dx = \int_0^1 \int_{-1}^1 (x + y)^2 \, dy \, dx \\
= \int_0^1 \left( 3x + \frac{9}{2} - x - \frac{1}{2} \right) \, dx \\
= \int_0^1 \left( 2x + \frac{4}{3} \right) \, dx \\
= (x^2 + \frac{4}{3}x) \bigg|_0^1 \\
= 1 + \frac{4}{3} \\
= \frac{7}{3}
\]
7. 
\[ \int_0^1 \int_{x^2}^{x} (xy + y^2) \, dy \, dx = \int_0^1 \left( xy \left|_{x^2}^{x} \right. \right) \, dx \\
= \int_0^1 \left( x^2 + \frac{x^2}{2} - x^3 - \frac{x^3}{2} \right) \, dx \\
= \int_0^1 \left( \frac{3x^2}{2} - x^3 - \frac{x^3}{2} \right) \, dx \\
= \left( \frac{x^2}{2} - \frac{x}{3} + \frac{x^3}{10} \right)_0^1 \\
= \frac{1}{2} - \frac{1}{3} - \frac{1}{10} \\
= \frac{3}{20} \]

9. 
\[ \int_0^1 \int_{y^2}^y \frac{1}{y} \, dx \, dy = \int_0^1 \left[ \ln y \right]_{y^2}^y \, dx \\
= \int_0^1 \left( \ln y \right)_0^1 \, dx \\
= \int_0^1 x \, dx \\
= \left( \frac{x^2}{2} \right)_0^1 \\
= \frac{1}{2} - 0 \\
= \frac{1}{2} \]

11. 
\[ \int_0^2 \int_0^x (x + y) \, dy \, dx = \int_0^2 \left( xy + \frac{y^2}{2} \right)_0^x \, dx \\
= \int_0^2 \left( x^2 + \frac{x^2}{2} \right) \, dx \\
= \left( \frac{x^3}{3} + \frac{x^3}{6} \right)_0^2 \\
= \frac{8}{3} + \frac{4}{3} - \frac{1}{3} \\
= 4 \]

13. 
\[ \int_0^1 \int_0^{x^2} (1 - y - x^2) \, dy \, dx = \int_0^1 \left( y - \frac{y^2}{2} - x^2 y \right)_0^{x^2} \, dx \\
= \int_0^1 \left( \frac{1 - x^2}{2} \right) \, dx \\
= \int_0^1 \left( \frac{1 - x^2}{2} \right) \, dx \\
= \frac{1}{2} \int_0^1 \left( 1 - x^2 \right) \, dx \\
= \frac{1}{2} \left[ \frac{3x - 3x^3}{3} \right]_0^1 \\
= \frac{1}{2} \left( -6(3) \right) - \frac{3(3)^3}{3} \\
= \frac{1}{2} \left( -9 + 9 \right) - 0 \\
= 16.2 \]

15. 
\[ \int_0^1 \int_{e^x}^{e^x} x \, e^{x^2} \, dy \, dx = \int_0^1 \left( \frac{y^2}{2} - e^{x^2} \right)_e^{e^x} \, dx \\
= \int_0^1 \left( \frac{y^2}{2} - e^{x^2} \right)_e^{e^x} \, dx \\
= \int_0^1 \left( \frac{y^2}{2} - e^{x^2} \right)_e^{e^x} \, dx \\
= \int_0^1 \left( \frac{y^2}{2} - e^{x^2} \right)_e^{e^x} \, dx \\
= \left( \frac{4}{3} e^{x^2} \right)_0^1 \\
= \frac{4}{3} e^1 - \frac{4}{3} \\
= \frac{4}{3} e - \frac{4}{3} \]

17. 
\[ \int_0^1 \int_0^{e^{-x^2}} (7 - 2x) y \, dy \, dx = \int_0^1 \left( \frac{7 - 2x}{2} \right)_0^{e^{-x^2}} \, dx \\
= \int_0^1 \left( \frac{7 - 2x}{2} \right) \left( 3x - x^3 \right)_0^{e^{-x^2}} \, dx \\
= \int_0^1 \left( \frac{7 - 2x}{2} \right) \left( 3x - x^3 \right)_0^{e^{-x^2}} \, dx \\
= \int_0^1 \left( \frac{7 - 2x}{2} \right) \left( 3x - x^3 \right)_0^{e^{-x^2}} \, dx \\
= \frac{1}{2} \int_0^1 (3x - x^3) \, dx \\
= \frac{1}{2} \int_0^1 (3x - x^3) \, dx \\
= \frac{1}{2} \int_0^1 (3x - x^3 + 4x^2) \, dx \\
= \frac{1}{2} \int_0^1 (12x^2 - 6x^3 + 4x^2) \, dx \\
= \frac{1}{2} \int_0^1 \left[ (3x - x^3) \right]_0^1 \, dx \\
= \frac{1}{2} \int_0^1 \left[ (3x - x^3) \right]_0^1 \, dx \\
= \frac{1}{2} \int_0^1 \left[ 3x^2 - 2x^2 + x \right] \, dx \\
= \frac{1}{2} \left( \frac{x^3}{3} - \frac{x^2}{3} + x \right)_0^1 \\
= \frac{1}{2} \left( 1 - \frac{1}{3} + 1 \right) \\
= \frac{1}{2} \frac{5}{3} \\
= \frac{5}{6} \\
= 0.833 \]
\[ = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} \]
\[ = \frac{1}{60} \]
\[ = 0.016 \]

21. \[ \int_0^{\pi/2} \int_{-\sin x}^{\sin x} y^2 \cos x \, dy \, dx \]
\[ = \int_0^{\pi/2} \left( \frac{y^3}{3} \cos x \right)_{-\sin x}^{\sin x} \, dx \]
\[ = \int_0^{\pi/2} \frac{2}{3} \sin^3(x) \cos x \, dx \]
\[ = \left( \frac{\sin^4(x)}{6} \right)_{0}^{\pi/2} \]
\[ = \frac{1}{6} - 0 \]
\[ = \frac{1}{6} \]

23. a) \[ A = \int_0^1 (x^3 - x) \, dx \]
\[ = \left( \frac{x^4}{4} - \frac{x^2}{2} \right)_{0}^{1} \]
\[ = (64 - 8) - (4 - 2) \]
\[ = 54 \]

b) \[ V = \int_0^3 \int_{-x}^{x} t \, dy \, dx \]
\[ = \int_0^3 (y^2)_{-x}^{x} \, dx \]
\[ = \int_0^3 (x^3 - x) \, dx \]
\[ = \left( \frac{x^4}{4} - \frac{x^2}{2} \right)_{0}^{1} \]
\[ = (64 - 8) - (4 - 2) \]
\[ = 54 \]

c) The area found in part (a) equals the volume in part (b) because the “thickness” is 1. The volume equals the product of the area and the thickness.

25. \[ \int_0^1 \int_0^1 \int_{-1}^{1} (2x + 3y - z) \, dx \, dy \, dz \]
\[ = \int_0^1 \int_0^1 (x^2 + 3x - xz)^2 \, dy \, dz \]
\[ = \int_0^1 \int_0^1 (x^2 + 3y - yz)^2 \, dy \, dz \]
\[ = \int_0^1 \int_0^1 (3y + \frac{3}{2}y - yz)^2 \, dy \, dz \]
\[ = \int_0^1 \int_0^1 (15 - 2z) \, dy \, dz \]
\[ = 14 \]

27. \[ \int_0^1 \int_0^{1-x} \int_0^{3-x} xyz \, dz \, dy \, dx \]
\[ = \int_0^1 \left( \frac{xy^2z}{2} \right)_{0}^{3-x} \, dx \]
\[ = \int_0^1 \left( \frac{x(2-x)^2y}{2} \right)_{0}^{1-x} \, dy \]
\[ = \int_0^1 \left( \frac{x(2-x)^2(1-x)^2}{4} \right)_{0}^{1-x} \, dx \]
\[ = \int_0^1 \left( \frac{2x - 8x^2 + 11x^3}{4} \right) - \int_0^1 \frac{(-6x^2 + x^4)}{4} \, dx \]
\[ = \left( \frac{x^2 - \frac{9}{2}x^3 + \frac{11}{4}x^4}{4} \right)_{0}^{1} \]
\[ = \frac{1}{80} \]

29. The geometric meaning of the multiple integral of a function of two variables is the volume of the solid generated from the function bounded by the limits of the multiple integrals.

31. 2.697335366
33. 0.353157821
Chapter 8

First Order Differential Equations

Exercise Set 8.1

1. $y' = 3x^2$

\[ y = \int y' \, dx = \int 3x^2 \, dx = 3 \int x^2 \, dx = x^3 + C \]

3. $y' = \frac{3}{x} - x^2 + 3x^4$

\[ y = \int y' \, dx = \int \left( \frac{3}{x} - x^2 + 3x^4 \right) \, dx = \ln |x| + \frac{1}{3}x^3 - \frac{1}{2}x^5 + \frac{3}{6}x^6 + C \]

5. $y' = 4e^{3x} + \sqrt{x}$

\[ y = \int y' \, dx = \int 4e^{3x} + \sqrt{x} \, dx = \frac{4}{3}e^{3x} + \frac{2}{3}x^{3/2} + C \]

7. $y' = x^2 \sqrt{3x^3 - 5}$

\[ y = \int y' \, dx = \int x^2 \sqrt{3x^3 - 5} \, dx = \frac{2}{27} \left( 3x^3 - 5 \right)^{3/2} + C \]

9. $y' = \frac{\sin 2x}{(1 + \cos 2x)^3}$

\[ y = \int y' \, dx = \int \frac{\sin 2x}{(1 + \cos 2x)^3} \, dx = \frac{1}{4} (1 + \cos 2x)^{-2} + C \]

11. $y' = \frac{1}{1 - x^2}$

\[ y = \int y' \, dx = \int \frac{1}{1 - x^2} \, dx = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + C \]

13. $y' = x^2 + 2x - 3$

\[ y = \int y' \, dx = \int x^2 + 2x - 3 \, dx = \frac{1}{3}x^3 + x^2 - 3x + C \]

15. $y' = e^{3x} + 1$

\[ y = \int y' \, dx = \int e^{3x} + 1 \, dx = \frac{1}{3}e^{3x} + x + C \]

17. $f(x)' = x^{5/3} - x$

\[ f(x) = \int f'(x) \, dx = \int x^{5/3} - x \, dx = \frac{3}{5}x^{5/3} - \frac{1}{2}x^2 + C \]

\[ -6 = \frac{3}{5} - \frac{1}{2} + C ; \quad \frac{61}{10} = C \]

\[ f(x) = \frac{3}{5}x^{5/3} - \frac{1}{2}x^2 - \frac{61}{10} \]
19. \( y' = x\sqrt{x^2 + 1} \)

\[
y = \int y' \, dx
= \int x\sqrt{x^2 + 1} \, dx
= \frac{1}{3} \left[ (x^2 + 1)^{3/2} + C \right]
3 = \frac{1}{3} + C
-\frac{5}{2} = C
y = -\frac{1}{2} \cos(x^2) - \frac{5}{2}
\]

20. \( y' = \frac{x^3}{(x^2 + 1)^2} \)

\[
y = \int y' \, dx
= \int \frac{x^3}{(x^2 + 1)^2} \, dx
= \frac{1}{3} (x^2 + 1)^{-1} + C
-\frac{2}{8} = \frac{1}{8} + C
y = \frac{1}{4x^2 + 1} - \frac{15}{8}
\]

21. \( y' = xe^x \)

\[
y = \int y' \, dx
= \int xe^x \, dx
= xe^x - e^x + C
2 = -1 + C
3 = C
y = xe^x - e^x + 3
\]

22. \( y' = \ln x \)

\[
y = \int y' \, dx
= \int \ln x \, dx
= x \ln x - x + C
2 = 0 - 1 + C
3 = C
y = x \ln x - x + 3
\]

23. \( y' = x \sin(x^2) \)

\[
y = \int y' \, dx
= \int x \sin(x^2) \, dx
= -\frac{1}{2} \cos(x^2) + C
\]

24. \( y'' = 2 \)

\[
f''(x) = \int f''(x) \, dx
= \int 2 \, dx
= 2x + C
4 = 0 + C
4 = C
f'(x) = 2x + 4
f(x) = \int f'(x) \, dx
= \int (2x + 4) \, dx
= x^2 + 4x + K
3 = 0 + 0 + K
3 = K
f(x) = x^2 + 4x + 3
\]

25. \( y'' = x + 1/x^2 \)

\[
f''(x) = \int f''(x) \, dx
= \int x + x^{-3} \, dx
= \frac{1}{2} x^2 + \frac{1}{4} x^{-2} + C
0 = 2 + \frac{1}{8} + C
\]

26. \( y'' = \sin x \)

\[
f''(x) = \int f''(x) \, dx
= \int \sin x \, dx
= -\frac{1}{2} \cos x + \frac{1}{8} \cos x - \frac{15}{8}
\]

27. \( y' = x \sin(x^2) \)

\[
f'(x) = \int f'(x) \, dx
= \int \sin x \, dx
= \frac{1}{6} x^3 + \frac{1}{2x} - \frac{15}{8} x + \frac{19}{6}
\]

28. \( y'' = (x^2 + 1)^2 \)

\[
f''(x) = \int f''(x) \, dx
= \int (x^2 + 1)^2 \, dx
= \frac{2}{3} (x^2 + 1)^{3/2} + C
\]

29. \( y'' = x + 1/x \)

\[
f''(x) = \int f''(x) \, dx
= \int x + x^{-1} \, dx
= \frac{1}{2} x^2 - \frac{1}{2} x^{-2} + C
0 = 2 - \frac{1}{8} + C
\]

30. \( y'' = x + 1/x^2 \)

\[
f''(x) = \int f''(x) \, dx
= \int x + x^{-2} \, dx
= \frac{1}{2} x^2 - \frac{1}{2} x^{-2} + C
\]

31. \( y'' = \sin(3x) \)

\[
f''(x) = \int f''(x) \, dx
= \int \sin(3x) \, dx
= -\frac{1}{3} \cos(3x) + C
\]

32. \( y'' = x^2 + 1/x^2 \)

\[
f''(x) = \int f''(x) \, dx
= \int x^2 + x^{-2} \, dx
= \frac{1}{3} x^3 + \frac{1}{x} + C
\]

33. \( y'' = \sin(x) \)

\[
f''(x) = \int f''(x) \, dx
= \int \sin x \, dx
= -\frac{1}{2} \cos x + \frac{1}{8} \cos x - \frac{15}{8} x + \frac{19}{6}
\]

34. \( y'' = \cos(x) \)

\[
f''(x) = \int f''(x) \, dx
= \int \cos x \, dx
= \sin x + C
\]
33. \( y = -2e^x + 7e^{3x} \)
\( y' = -2e^x + 21e^{3x} \)
\( y'' = 2e^x - 61e^{3x} \)
\( -4y' + 3y = 2e^x - 61e^{3x} - 8e^{3x} + 84e^{5x} + 6e^{x} - 21e^{3x} = 0 \)
Thus, \( y \) is a solution to the differential equation.

45. \( y = e^x \sin 2x \)
\( y' = 2e^x \cos 2x + e^x \sin 2x \)
\( y'' = 4e^x \cos 2x - 3e^x \sin 2x \)
\( y'' = 2y' + 5y \)
\( -4e^x \cos 2x - 3e^x \sin 2x - 4e^x \cos 2x - 2e^x \sin 2x + 5e^x \sin 2x = 0 \)
Thus, \( y \) is a solution to the differential equation.

47.
\[
\begin{align*}
\frac{dR}{dS} &= \frac{k}{S} \\
\frac{dR}{dS} &= \frac{k}{S} \\
R &= \int \frac{k}{S} dS \\
&= k \ln |S| + C \\
0 &= -k \ln |S_0| + C \\
R(S) &= k \ln |S| - k \ln |S_0| \\
\end{align*}
\]
Thus, \( R(S) = k \ln \left| \frac{S}{S_0} \right| \)

49. a) \( y = 0, y' = 0, y = \sqrt{6} \)
Thus, \( y = 0 \) is a solution of the initial value problem.

b) \( y = x^2/4, y' = x/2, y(0) = \sqrt{x/2} \)
Thus, \( y = x^2/4 \) is a solution of the initial value problem.

c) \( \frac{d}{dy} y = \frac{1}{2\sqrt{y}} \)
The initial value problem does not have a unique solution.

d) \( x^2/4 = 0 \Rightarrow x = 0 \Rightarrow y = 0 \)
The function does not satisfy the continuity criteria near the origin since the function is not defined for “real” values to the left of the origin.
51. a) \( x^2 - 1 = 0 \rightarrow x = -1 \text{ and } x = 1 \)
   
   b) The slope of the tangent line at the point where \( y' = 0 \)
      is 0
   
   c) Left to the student

53. a) \( 2x/3 + y = 0 \rightarrow y = -2x/3 \)
   
   b) The slope of the tangent line at the point where \( y' = 0 \)
      is 0
   
   c) Left to the student

Exercise Set 8.2

1. Both \(-x^2\) and \(x^3\) are continuous for all real numbers. Therefore the unique solution will exist on all real numbers

3. Discontinuities at \( x = \pm \pi/2 \). Therefore the unique solution will exist on \((-\pi/2, \pi/2)\).

5. Discontinuities at \( x = 2 \) and \( x = -1 \). Therefore the unique solution will exist on \((2, \infty)\)

7. \[
\begin{align*}
\int -3 \, dx & = -3x + C \\
G(x) & = e^{-3x} \\
e^{-3x} y & = \int 0 \, dx \\
y & = C e^{-3x}
\end{align*}
\]

9. \[
\begin{align*}
\int \cos 2x \, dx & = \frac{1}{2} \sin 2x + C \\
G(x) & = e^{\sqrt{2} \sin 2x} \\
e^{\sqrt{2} \sin 2x} y & = \int \cos 2x \, e^{\sqrt{2} \sin 2x} \, dx \\
y & = C e^{-\sqrt{2} \sin 2x} + 1
\end{align*}
\]

11. \[
\begin{align*}
\int -2t \, dt & = -t^2 + C \\
G(t) & = e^{-t} \\
e^{-t} y & = \int e^{-t} (-2t) \, dt \\
e^{-t} y & = -e^{-t^2} + C \\
y & = e^{-t^2} + C
\end{align*}
\]

13. \[
\begin{align*}
\int -dt & = -t + C \\
G(t) & = e^{-t} \\
e^{-t} y & = \int e^{-t} e^t \, dt \\
e^{-t} y & = t + C \\
y & = t e^t + Ce^t
\end{align*}
\]

15. \[
y' - \frac{4}{x} y = x^5 e^{2x} + 3x^3 - 6x^{-3}
\]
\[
\int \frac{4}{x} \, dx = -4 \ln |x| + C = \ln x^{-4} + C \\
G(x) = x^{-4} \\
x^{-4} y = \int x^5 e^{2x} + 3x^3 - 6x^{-3} \, dx \\
x^{-4} y = \frac{1}{2} x^5 e^{2x} + 3 \ln |x| + 6 \frac{1}{5} x^{-5} + C \\
y = \frac{1}{2} x^5 e^{2x} 3x \ln |x| + \frac{6}{5} x^{-5} + C e^x
\]

17. \[
y' + \frac{3}{x} y = \ln x
\]
\[
\int \frac{3}{x} \, dx = 3 \ln |x| + C = \ln |x^3| + C \\
G(x) = x^3 \\
x^3 y = \int x^3 \ln x \, dx \\
x^3 y = \frac{1}{3} x^3 \ln x - \frac{1}{16} x^4 + C \\
y = \frac{1}{3} x^3 \ln x - \frac{1}{16} x^4 + C e^{-x}
\]

19. \[
\int \arctan t \, dt = \arctan t + C
\]
\[
G(t) = \arctan t \\
e^{\arctan t} y = \int e^{\arctan t} \, dt \\
e^{\arctan t} y = \ln(1 + t^2) + C \\
y = \ln(1 + t^2) + C e^{\arctan t}
\]

21. \[
\int 4 \, dx = 4x + C
\]
\[
G(x) = e^{4x} \\
e^{4x} y = \int 6 e^{4x} \, dx \\
e^{4x} y = \frac{3}{2} e^{4x} + C \\
y = \frac{3}{2} e^{4x} + C \\
y = \frac{3}{2} \frac{1}{2} e^{-4x}
\]

The solution exists for all real numbers

23. \[
y' + \frac{\cos x}{1 + \sin x} y = \cos x
\]
\[
\int \frac{\cos x}{1 + \sin x} \, dx = \ln |1 + \sin x| + C \\
G(x) = \ln |1 + \sin x|
\]
25. \( y' + \frac{1}{t} \) \( y = t^2 \)

\[
\int \frac{1}{t} \, dt = \ln |t| + C
\]
\( G(t) = t \)
\( ty = \frac{1}{4} t^3 + C \)
\( ty = \frac{1}{4} t^3 + C \)
\( y = \frac{1}{4} t^3 + \frac{C}{t} \)
\( 5 = 2 + C \)
\( 6 = C \)
\( 6 = \frac{C}{2} \)
\( 6 = \frac{C}{2} \)
\( y = \frac{1}{4} t^3 - \frac{C}{t} \)

The solution exists on \([0, \infty)\).

29. \( y' \left( \frac{e^{t}}{(e^{t} - 2)} \right) y = \frac{2e^{-2t} - e^{-3t}}{e^t - 2} \)

\[
\int \frac{e^{t}}{(e^t - 2)} \, dt = \ln |e^t - 2| + C
\]
\( G(t) = e^{t} \)
\( (e^t - 2)y = \int 2e^{-2t} - e^{-3t} \, dx \)
\( (e^t - 2)y = -e^{-2t} \left( \frac{1}{3} e^{-3t} + C \right) \)
\( y = \frac{e^{-3t}}{3(e^t - 2)} - \frac{e^{-2t}}{e^t - 2} + \frac{C}{3(e^t - 2)} \)
\( 3 = -\frac{1}{3} + 1 - C \)
\( \frac{7}{3} = C \)
\( y = \frac{e^{-3t}}{3(e^t - 2)} - \frac{e^{-2t}}{e^t - 2} + \frac{7}{3(e^t - 2)} \)
\( = \frac{3e^{-3t} - 3e^{-2t} - 7}{3e^t - 6} \)

The solution exists on \((\pi/2, 3\pi/2)\).

31. \( y' + \frac{e^t}{t} \) \( y = e^t \)

\[
\int 3t^2 \, dt = \frac{t^3}{3} + C
\]
\( G(t) = e^t \)
\( e^ty = \int \frac{3}{e^t} e^{3t} \, dt \)
\( e^ty = \frac{1}{6} t^2 e^{3t} + C \)
\( 17 \quad 6 \quad C \)
\( y = \frac{1}{6} t^2 e^{3t} + \frac{17}{6} e^{-t^2} \)

The solution exists for all real numbers.

33. \( y' + \frac{2t^2}{t} \) \( y = 1 + C \)

\[
\int 4t^2 \, dt = \frac{2t^3}{3} + C
\]
\( G(t) = e^{2t^2} \)
\( e^{2t^2} y = \int t \, e^{2t^2} \, dt \)
\( e^{2t^2} y = \frac{1}{4} t e^{2t^2} + C \)
\( y = \frac{1}{4} t e^{2t^2} + C \)
\( 3 \quad \frac{1}{4} + C \)
\( \frac{11}{4} = C \)
\( y = \frac{1}{4} t + \frac{11}{4} e^{-2t^2} \)

The solution exists for all real numbers.
35. \( P' = -0.2P + 3 \rightarrow P' + 0.2P = 3 \)

\[
\int 0.2 \, dt = 0.2t + C
\]

\[
G(t) = e^{0.2t}
\]

\[
e^{0.2t}P = \int 3e^{0.2t} \, dt
\]

\[
e^{0.2t}P = 15e^{0.2t} + C
\]

\[
P' = 15 + C \, e^{-0.2t}
\]

37. \( Q' = -0.1Q - 5 \rightarrow Q' + 0.1Q = -5 \)

\[
\int 0.1 \, dt = 0.1t + C
\]

\[
G(t) = e^{0.1t}
\]

\[
e^{0.1t}Q = \int -5e^{0.1t} \, dt
\]

\[
e^{0.1t}Q = -50e^{0.1t} + C
\]

\[
Q = -50 + C \, e^{-0.1t}
\]

39. \( P' = kP \rightarrow P' - kP = 0 \)

\[
\int -k \, dt = -kt + C
\]

\[
G(t) = e^{-kt}
\]

\[
e^{-kt}P = \int 0 \, dt
\]

\[
e^{-kt}P = C
\]

\[
P' = C \, e^{kt}
\]

\[
P_0 = C
\]

\[
P = P_0e^{kt}
\]

41. \( S' = -0.005S + 4 \rightarrow S' + 0.005S = 4 \)

\[
\int 0.005 \, dt = 0.005t + C
\]

\[
G(t) = e^{0.005t}
\]

\[
e^{0.005t}S = \int 4e^{0.005t} \, dt
\]

\[
e^{0.005t}S = 800e^{0.005t} + C
\]

\[
S = 800 + C \, e^{-0.005t}
\]

\[
100 = 800 + C
\]

\[
-700 = C
\]

\[
S = 800 - 700e^{-0.005t}
\]

After two hours, \( t = 120 \)

\[
S(120) = 800 - 700e^{-0.005(120)}
\]

\[= 417.63 \text{ lbs}\]

43. \( y' + ky = c \)

a)

\[
\int k \, dt = kt + C
\]

\[
G(t) = e^{kt}
\]

\[
e^{kt}y = \int kCe^{kt} \, dt
\]

\[
e^{kt}y = C \, e^{kt} + K
\]

\[
y' = C + Ke^{-kt}
\]

\[
y_0 = C + K
\]

\[
k = \frac{1}{-30} \ln \frac{47}{73}
\]

\[
k = 0.016771
\]

\[
y' = C + (y_0 - C)e^{-0.016771t}
\]

\[
y_0 = 70 + (143 - 70)e^{-0.016771t}
\]

\[
\ln \frac{47}{73} = -30k
\]

\[
k = 0.016771
\]

\[
y = \frac{1}{-0.016771} \ln \frac{20}{73}
\]

\[= 88.2 \text{ min}\]

45. \( g' + \frac{2}{x} y = 5e^x \)

a) The solution will exist on \((0, \infty)\)

b)

\[
\int \frac{2}{x} \, dx = 2 \ln x + C = \ln x^2 + C
\]

\[
C(x) = x^2
\]

\[
\int x^2y = \int 5x^4 \, dx
\]

\[
x^2y = x^5 + C
\]

\[
y = x^3 + Cx^{-2}
\]

\[= 1 + C
\]

\[= C
\]

\[y = x^3\]

c) The domain of \( x^3 \) is all real numbers where the domain of \( x^3 + \frac{1}{4x^2} \) is \((0, \infty)\)

d) Left to the student
47. There is a net flow into the tank of 3 gallons per minute, and no salt is added. Therefore
\[ S' = 0 - \frac{2S}{500 + 3t} = -\frac{2}{500 + 3t} S \]
is the differential equation associated with this problem.
\[ \int \frac{dS}{S} = \int \frac{2}{500 + 3t} \, dt \]
\[ \ln S = -\frac{2}{3} \ln(500 + 3t) + C \]
\[ \ln S = \ln(500 + 3t)^{-2/3} + C \]
\[ S = \frac{C}{(500 + 3t)^{2/3}} \]
\[ 200 = \frac{C}{(500)^{2/3}} \]
\[ S = \frac{200(500)^{2/3}}{(500 + 3t)^{2/3}} \]
It will take 2000 - 500 + 3t = 500 minutes to fill the tank.
\[ S(500) = \frac{200(500)^{2/3}}{(2000)^{2/3}} \]
\[ = 79.37 \text{ pounds} \]

49.
\[ P' = 1.2(1 - e^{-1.2t}) - P \]
\[ P' + 1.2P = 1.2 - 1.2e^{-1.2t} \]
\[ \int 1.2 \, dt = 1.2t + C \]
\[ C(t) = e^{1.2t} \]
\[ e^{1.2t}P = \int 1.2e^{1.2t} - 1.2 \, dt \]
\[ e^{1.2t}P = 1.2t e^{1.2t} - 1.2t + C \]
\[ P = 1 - 0.2t + C e^{-1.2t} \]
\[ C = -1 \]
\[ P = 1 - 0.2t e^{-1.2t} - e^{-1.2t} \]

51. a) Left to the student.
b) \( Q' + bQ = a \)
\[ \int b \, dt = \frac{a}{b} + C \]
\[ G(t) = e^{bt} \]
\[ e^{bt}Q = \int ae^{bt} \, dt \]
\[ e^{bt}Q = \frac{a}{b} e^{bt} + C \]
\[ Q = \frac{a}{b} + C e^{-bt} \]
\[ 0 = \frac{a}{b} + C \]
\[ C = \frac{-a}{b} \]
\[ Q = \frac{a}{b} - \frac{a}{b} e^{-bt} \]

c) \( Q' + bQ = a \)
\[ 0 + bQ = a \]
\[ Q = \frac{a}{b} \]

Exercise Set 8.3
1. a) \( 2 - y = 0 \rightarrow y = 2 \)
b) \( y'' < -1 < 0 \) Therefore the equilibrium point is asymptotically stable.
c) No inflection points since \( y'' \) does not change signs.
d) \( y'' = \frac{y}{(y-1)(y-4)} \)
\[ y = 1 \]
\[ y = 4 \]

3. a) \( y'' = 5y \)
\[ 14 \rightarrow 0 \]
\[ (y-1)(y-4) \geq 0 \]
\[ y = 1 \]
\[ y = 4 \]

b) \( y'' = 2y - 5 \)
\( y''(1) < -3 \) Therefore \( y = 1 \) is an asymptotically stable equilibrium point.
\( y''(4) = -3 \) Therefore \( y = 4 \) is an unstable equilibrium point.
c) \( 2y - 5 \rightarrow y = \frac{5}{2} \) Inflection point at \( y = \frac{5}{2} \)
d) \( y'' = y \)
\[ 14 \rightarrow 0 \]
\[ 6 \rightarrow 0 \]
\[ (y-1)(y-2) \leq 0 \]
\[ y = 0 \]
\[ y = 2 \]
b) $y'' = 3y^2 - 4y$
   $y''(0) = 0$ Therefore $y = 0$ is a semistable equilibrium point
   $y''(2) = 4$ Therefore $y = 2$ is an unstable equilibrium point

9. a) $e^{2y} - e^y = 0$
   $e^y(e^y - 1) = 0$
   $y = 0$

13. $y' + k_L \left(1 - \left[\frac{y}{L}\right]^q\right)$

   a) Left to the student
   b) The per capita growth rate is $k \left(1 - \left[\frac{y}{L}\right]^q\right)$
   c) $k_L \left(1 - \left[\frac{y}{L}\right]^q\right) = 0$
   $y = 0$
   $\left(t - \left[\frac{y}{L}\right]^q\right) = 0$
   $y = L$
\[ y'' = k - k \left( \frac{x}{2} \right) y - k \left( \frac{x}{2} \right)^3 \]
\[ y'(0) = k > 0 \]

Therefore \( y = 0 \) is an unstable equilibrium point.
\[ y''(L) = -kL < 0 \]

Therefore \( y = L \) is an asymptotically stable equilibrium point.

15. \( y' = ky(P - y) \)

\[ ky(P - y) = 0 \]
\[ y = 0 \]
\[ y' = P \]
\[ y'' = ky + 2ky \]
\[ y''(0) = kP > 0 \]

Therefore \( y = 0 \) is an unstable equilibrium point.
\[ y''(P) = -kP < 0 \]

Therefore \( y = P \) is an asymptotically stable equilibrium point.

17. a) \( y'' - k - s - 2 \left( \frac{k}{2} \right) y \)
\[ y''(0) = k - s > 0 \], thus \( y = 0 \) is an unstable equilibrium point.
\[ y''(c) = -2(k-s) < 0 \], thus \( y = c \) is an asymptotically stable equilibrium point.

b) \[ H = \frac{sk}{L}s - \frac{Ls^2}{k} \]

19. \( y' = y^2 - 2y \)
\[ y'' = 0 \]
\[ y''(0) = 2y - 6 \]

\( y''(0) = -6 < 0 \), stable
\[ y''(0) = 6 > 0 \), unstable

\[ \lim_{t \to \infty} y(t) = 0 \]

21. a) By Theorem 4 it follows that \( k \) is semistable.

b) The result follows from Part (a) and the definition on page 564.

23. \( y' = \frac{kq^2}{T} - ky - \frac{kq^2}{IT} \quad \frac{kq^2}{L} \cdot \frac{ky}{L} \]
\[ y'' = \frac{2ky}{T} - k - \frac{3ky^2}{T} + \frac{2ky}{L} \]
\[ y''(0) = -k < 0 \), asymptotically stable.
\[ y''(L) = \frac{2kL}{T} - k - \frac{3kL}{T} + 2k \]
\[ k \cdot \frac{mL}{T} < 0 \), asymptotically stable.

25.
\[ \frac{2ky}{T} - k - \frac{3ky^2}{T} + \frac{2ky}{L} \]
\[ \frac{3k}{L} \cdot y'' = \left( \frac{2y}{T} + \frac{2ky}{L} \right) y + k \]
\[ y'' = \frac{3}{T} + \frac{3k}{L} \pm \sqrt{\left( \frac{3k}{L} \right)^2 - \frac{2k^2}{L^2}} \]
\[ \frac{3k}{L} \pm \frac{3k}{T} \]
\[ \frac{3k}{L} \pm \frac{3k}{T} \]
\[ L + T = \sqrt{3L^2 - LT + T^2} \]

27. a) \[ \frac{k}{L} \cdot y'' - ky + s = 0 \]
\[ k \pm \sqrt{k^2 - \frac{4ks}{L}} = g \]

The radicand is negative and therefore there are no real equilibrium points.

b) \( s > kL/A \rightarrow s - kL/A > 0 \), but from Part (a) we know that there are no equilibrium points therefore the most the right hand side can get is \(-(s - kL/A)\).

c) Because 0 is the only physical value the population can approach.
29. \[
    y'' = nk y^{n-1} - k \\
    y''(0) = -k < 0
\]

\(y = 0\) is an asymptotically stable equilibrium point.

31. \[
    nk y^{n-1} - k = 0 \\
    y^{n-1} = \frac{1}{n} \\
    y = \left(\frac{1}{n}\right)^{1/(n-1)}
\]

33. a)

\[
\begin{align*}
    y(1) &= (1 + 0.0006624e^{15})^{-1/3} \\
    &= 0.3092
\end{align*}
\]

b)

\[
\begin{align*}
    0.5 &= (1 + 0.0006624e^{15})^{-1/3} \\
    0.5 - 1 &= 0.0006624e^{15} \\
    0.5 - 1 &= e^{15t} \\
    \ln\left(\frac{0.5 - 1}{0.0006624}\right) &= 15t \\
    \frac{1}{15} \ln\left(\frac{0.5 - 1}{0.0006624}\right) &= t
\end{align*}
\]

c)

\[
\begin{align*}
    \frac{dy}{dx} &= \frac{x^2 \sec y}{(x^3 + 1)^{3/2}} \\
    \cos y \frac{dy}{dx} &= \frac{x^2}{(x^3 + 1)^{3/2}} dx \\
    \int \cos y \frac{dy}{dx} &= \int \frac{x^2}{(x^3 + 1)^{3/2}} dx \\
    \sin y + \frac{3}{2(x^3 + 1)^{3/2}} &= C
\end{align*}
\]

7.

\[
\begin{align*}
    \frac{dy}{dt} &= \frac{y^2 + 1}{y^2} \\
    \int \frac{y^2}{y^2 + 1} \frac{dy}{dt} &= \int dt \\
    \frac{1}{3} \ln |y^3 + 1| &= t + C \\
    \ln |(y^3 + 1)^{1/3}| &= t + C \\
    (y^3 + 1)^{1/3} &= Ce^t \\
    y^3 + 1 &= C e^{3t} - 1
\end{align*}
\]

Exercise Set 8.4

1. \[
\begin{align*}
    \frac{dy}{dx} &= 4x^3 y \\
    \frac{dy}{y} &= 4x^3 dx \\
    \int \frac{dy}{y} &= \int 4x^3 dx \\
    \ln y &= x^4 + C \\
    y &= Ce^{x^4}
\end{align*}
\]
11. \[
\frac{dy}{dx} = x \cos^2 y \\
\sin^2 y \, dy = x \, dx \\
\int \sec^2 y \, dy = \int x \, dx \\
\tan y = \frac{1}{2} x^2 + C \\
\tan y - \frac{1}{2} x^2 = C
\]

NOTE: \( y = \frac{(2n+1)\pi}{2} \) are constant solutions as well.

13. \[
\frac{dy}{dc} = \frac{x}{\sin y + \cos y} \\
\sin y \, \cos y \, dy = \sqrt{x} \, dx \\
\int \sin y \, \cos y \, dy = \int \sqrt{x} \, dx \\
-\cos y \, \sin y = \frac{2}{3} x^{3/2} + C \\
\sin y - \cos y - \frac{2}{3} x^{3/2} = C
\]

15. \[
\frac{dy}{dt} = \frac{1}{(t^2 + 1)(g^2 + 1)} \\
y^2 + 1 \, dy = \frac{1}{t^2 + 1} \, dt \\
\frac{1}{5} y^5 + y - \frac{1}{2} \ln(t^2 + 1) = C
\]

17. \[
\frac{dy}{dx} = 3x^2(y - 2)^2 \\
(y - 2)^3 \, dy = 3x^2 \, dx \\
\int (y - 2)^3 \, dy = \int 3x^2 \, dx \\
-1 \frac{1}{y - 2} = x^3 + C \\
\frac{1}{y - 2} = -x^3 + C \\
y - 2 = -\frac{1}{x^3} + C \\
y = 2 - \frac{1}{x^3} + C
\]

NOTE: \( y - 2 \) is a constant solution as well.

19. \[
\int 3y^3 \, dy = \int 2x \, dx \\
y^4 = x^2 + C \\
125 = 4 + C \\
C = 121 \\
y^4 = x^2 + 121 \\
y = \sqrt{x^2 + 121}
\]

21. \[
\int \csc^2 y \, dy = \int t^n \, dt \\
-\cot y = \frac{1}{2} t^{2n} + C \\
-1 \frac{1}{2} = C \\
\cot y = \frac{1}{2} t^{2n} = \frac{3}{2}
\]

23. \[
\int y^{-2} \, dy = \int \ln t \, dt \\
-\frac{1}{y} = \frac{1}{2} \ln t + C \\
\frac{1}{y} = C \\
\frac{1}{y} = -\frac{1}{2} \ln t + \frac{1}{2} \\
y = \frac{-4}{2 \ln t + 1}
\]

25. \[
\int 3y^2(y^3 + 2)^{-2} \, dy = \int x \, dx \\
-(y^3 + 2)^{-1} = \frac{1}{2} x^2 + C \\
-1 \frac{1}{10} = \frac{1}{2} x^2 + C \\
\frac{1}{y^3 + 2} = C \\
\frac{1}{y^3 + 2} = \frac{1}{10} \\
y^3 + 2 = -\frac{10}{6} \sqrt{x^2 + \frac{3}{5}} \\
y^3 + 2 = \frac{-10}{6} \sqrt{x^2 + \frac{3}{5}} \\
y^3 = \frac{-10 + 10x^2 - 12}{6 - 5x^2} \\
y = \sqrt{\frac{10x^2 - 2}{6 - 5x^2}}
\]

19. \[
\int 3y^3 \, dy = \int 2x \, dx \\
y^4 = x^2 + C \\
125 = 4 + C \\
C = 121 \\
y^4 = x^2 + 121 \\
y = \sqrt{x^2 + 121}
\]

21. \[
\int \csc^2 y \, dy = \int t^{2n} \, dt \\
-\cot y = \frac{1}{2} t^{2n} + C \\
-1 \frac{1}{2} = C \\
-\frac{3}{2} = C \\
\cot y = \frac{1}{2} t^{2n} = \frac{3}{2}
\]

23. \[
\int y^{-2} \, dy = \int \ln t \, dt \\
-\frac{1}{y} = \frac{1}{2} \ln t + C \\
\frac{1}{y} = C \\
\frac{1}{y} = -\frac{1}{2} \ln t + \frac{1}{2} \\
y = \frac{-4}{2 \ln t + 1}
\]

25. \[
\int 3y^2(y^3 + 2)^{-2} \, dy = \int x \, dx \\
-(y^3 + 2)^{-1} = \frac{1}{2} x^2 + C \\
-\frac{1}{10} = \frac{1}{2} x^2 + C \\
\frac{1}{y^3 + 2} = C \\
\frac{1}{y^3 + 2} = \frac{1}{10} \\
y^3 + 2 = -\frac{10}{6} \sqrt{x^2 + \frac{3}{5}} \\
y^3 + 2 = \frac{-10}{6} \sqrt{x^2 + \frac{3}{5}} \\
y^3 = \frac{-10 + 10x^2 - 12}{6 - 5x^2} \\
y = \sqrt{\frac{10x^2 - 2}{6 - 5x^2}}
\]
27.

\[
\int \frac{\cos y}{(2 + \sin y)^2} dy = \int t^2 dt
\]

\[
- \frac{1}{2 + \sin y} = \frac{1}{3} t^3 + C
\]

\[
- \frac{1}{5/2} = \frac{1}{3} t^3 + C
\]

\[
\frac{1}{15} = C
\]

\[
\frac{1}{2 + \sin y} = \frac{1}{3} t^3 + \frac{1}{15}
\]

\[
\frac{1}{2 + \sin y} = \frac{-5t^3 + 1}{15}
\]

\[
\sin y + 2 = \frac{15}{1 - 5t^3}
\]

29.

\[
\int e^{xy} + e^{xy} dy = \int \sqrt{t}
\]

\[
\frac{1}{4} e^{xy} + \frac{1}{5} e^{xy} = \frac{2}{3} t^{3/2} + C
\]

\[
\frac{1}{4} + \frac{1}{5} = \frac{2}{3} t^{3/2} + C
\]

\[
-\frac{13}{60} = C
\]

\[
\frac{1}{4} e^{xy} + \frac{1}{5} e^{xy} = \frac{2}{3} t^{3/2} - \frac{13}{60}
\]

31.

\[
\int y dy = \int x dx
\]

\[
\frac{1}{2} y^2 - \frac{1}{2} x^2 + C
\]

\[
\frac{21}{2} - \frac{25}{2} = C
\]

\[
-\frac{2}{2} = C
\]

\[
\frac{1}{2} y^2 = \frac{1}{2} x^2 - 2
\]

\[
y = \sqrt{x^2 - 4}
\]

Which has a domain of \((2, \infty)\)

33. a)

\[
\int \frac{dy}{y} = \int k dt
\]

\[
\ln |y| = kt + C
\]

\[
y = Ce^{kt}
\]

\[
y_0 = C
\]

\[
y = y_0 e^{kt}
\]

b) Left to the student

35.

\[
y = y_0 e^{kt}
\]

\[
\frac{4.404}{5} = 0.8888
\]

\[
\frac{4.404}{5} = e^{11k}
\]

\[
\frac{1}{5} \ln 4.404 = 11k
\]

\[-0.01154 = k
\]

The decay rate is 1.154% per day

37. - 39. Left to the student

41. Left to the student

43. a)

\[
\int \frac{L}{R(t - R)} dR = \int k dt
\]

\[
\frac{1}{L} \ln \left( \frac{R}{R - t} \right) = kt + C
\]

\[
\frac{1}{L} \ln \left( \frac{R_0}{R - R_0} \right) = C
\]

\[
k + \frac{1}{L} \ln \left( \frac{R_0}{R - R_0} \right) = \frac{1}{L} \ln \left( \frac{R}{L - R} \right)
\]

\[
\frac{1}{L} \left[ \ln \left( \frac{R}{L - R} \right) - \ln \left( \frac{R_0}{L - R_0} \right) \right] = kt
\]

\[
\frac{1}{L} \ln \left( \frac{\frac{R}{L - R}}{\frac{R_0}{L - R_0}} \right) = klk
\]

\[
\left( \frac{R}{L - R} \right) = e^{klk}
\]

\[
R(t - R_0) + RR_0 e^{klk}
\]

\[
\frac{L}{L(1 - R)} e^{klk} = L R_0 e^{klk}
\]

\[
\frac{L}{L(1 - R)} e^{klk} + R_0 e^{klk}
\]

\[
\frac{L}{L - R_0} e^{klk} + R_0 e^{klk}
\]

b) Left to the student
Exercise Set 8.5

1. a) $y(1) \approx -1.2$
b) $y(1) \approx -1$
c)\[
\int \frac{dy}{y} = \int 2x \, dx
\]
\[
y = x^2 + C
\]
\[
-2 = C
\]
\[
y = x^2 - 2
\]
d) The exact value is $y(1) = -1$

3. a) $y(2) \approx 18.235$
b) $y(2) \approx 40.067$
c)\[
\int \frac{dy}{y} = \int 2e \, dx
\]
\[
\ln y = x^2 + C
\]
\[
y = Ce^{x^2}
\]
\[
2 = C e^{2}
\]
\[
C = \frac{2}{e}
\]
\[
y = 2e^{x^2 - 1}
\]
d) The exact value is $y(2) = 40.17$.

5. a) $y(2) \approx 28.898$
b) $y(2) \approx 31.171$
c) $y' - y = 2x$
\[
\int -1 \, dx = -x + C
\]
\[
(y(x)) = e^{-x}
\]
\[
e^{-x}y = \int 2xe^{-x} \, dx
\]
\[
e^{-x}y = -2xe^{-x} - 2e^{-x} \cdot C
\]
\[
y = -2x - 2 \cdot Ce^{x}
\]
\[
2 = -2 - 2 \cdot C
\]
\[
C = -1
\]
\[
y = 2e^{x+1} - 2xe^{-x} - 2e^{-x}
\]
d) The exact value is $y(2) = 34.171$

7. a) $y(2) \approx 9.304$
b) $y(2) \approx 14.390$
c)\[
\int \frac{dy}{y} = \int x^2 \, dx
\]
\[
\ln y = \frac{x^3}{3} + C
\]
\[
0 = C
\]
\[
\ln y = \frac{x^3}{3}
\]
\[
y = e^{x^3/3}
\]
d) The exact value is $y(2) = e^{9.304} \approx 1.392$

9. a) $P(5) \approx 101.782$
\[
P(5) \approx 301.5586472
\]
b) $P(5) \approx 108.397$
\[
P(5) \approx 436.419
\]
c) $P(5) = 108.397$
\[
P(5) = 436.419
\]

11. a) $P(2) \approx 0.6975$
\[
P(4) \approx 0.9589
\]
b) $P(2) \approx 0.6916$
\[
P(4) \approx 0.9523
\]
c) $P(2) = P(4) \approx 0.6916$
\[
P(4) = 0.9523
\]

13. a) $Q(6) \approx 3.0746$
\[
Q(6) \approx 3.1553
\]

15. a) No
b) No
c) Yes
d) $\Delta x = \frac{1}{n}$ where $n$ is an integer
e) $\Delta x = \frac{a}{n}$ where $n$ is an integer

17. a) $y(2) = 2.47523873$
b) $y(2) = 2.47522970$
c) $y(2) = 2.47522913$
d) $|2.47522913 - 2.47523873| = 0.000009577$
\[
|2.47522913 - 2.47522970| = 0.000000577
\]
e) 16
Chapter 9

Higher Order and Systems of Differential Equations

Exercise Set 9.1

1. 
\[ r^2 - 6r + 5 = 0 \]
\[ (r - 1)(r - 5) = 0 \]
\[ r = 1 \]
\[ r = 5 \]
\[ y = C_1e^r + C_2e^{5r} \]

3. 
\[ r^2 - r - 2 = 0 \]
\[ (r - 2)(r + 1) = 0 \]
\[ r = 2 \]
\[ r = -1 \]
\[ y = C_1e^{2r} + C_2e^{-r} \]

5. 
\[ r^2 + 3r + 2 = 0 \]
\[ (r + 1)(r + 2) = 0 \]
\[ r = -1 \]
\[ r = -2 \]
\[ y = C_1e^{-r} + C_2e^{-2r} \]

7. 
\[ 2r^2 - 5r + 2 = 0 \]
\[ (2r - 1)(r - 2) = 0 \]
\[ r = \frac{1}{2} \]
\[ r = 2 \]
\[ y = C_1e^{r/2} + C_2e^{2r} \]

9. 
\[ r^2 - 9 = 0 \]
\[ (r - 3)(r + 3) = 0 \]
\[ r = 3 \]
\[ y = C_1e^{-3r} + C_2e^{3r} \]

11. 
\[ r^2 + 10r + 25 = 0 \]
\[ (r + 5)^2 = 0 \]
\[ r = -5 \text{ repeated} \]
\[ y = C_1e^{-5r} + C_2e^{-5r} \]

13. 
\[ 4r^2 + 12r + 9 = 0 \]
\[ (2r + 3)^2 = 0 \]
\[ r = -\frac{3}{2} \text{ repeated} \]
\[ y = C_1e^{-3r/2} + C_2xe^{-3r/2} \]

15. 
\[ r^3 + r^2 + 4r + 4 = 0 \]
\[ r^2(r + 1) + 4(r + 1) = 0 \]
\[ (r + 1)(r^2 + 4) = 0 \]
\[ r = -1 \]
\[ r = \pm 2i \]
\[ y = C_1e^{-r} + C_2 \sin 2x + C_3 \cos 2x \]

17. 
\[ r^3 + 6r^2 + 12r + 8 = 0 \]
\[ (r + 2)^3 = 0 \]
\[ r = -2 \text{ repeated twice} \]
\[ y = C_1e^{-2r} + C_2xe^{-2r} + C_3x^2e^{-2r} \]

19. 
\[ r^3 - 6r^2 + 3r - 18 = 0 \]
\[ (r - 6)(r^2 - 3) = 0 \]
\[ r = 6 \]
\[ r = \pm \sqrt{3} \]
\[ y = C_1e^{6r} + C_2 \sin \sqrt{3}x + C_3 \cos \sqrt{3}x \]

21. 
\[ r^4 - 5r^3 + 4r^2 = 0 \]
\[ r^2(r - 4)(r - 1) = 0 \]
\[ r = 0 \text{ repeated} \]
\[ r = 4 \]
\[ r = 1 \]
\[ y = C_1 + C_2r + C_3xe^r + C_4e^{-r} \]

23. 
\[ r^2 + 36 = 0 \]
\[ r = \pm 6i \]
\[ y = C_1 \sin 6x + C_2 \cos 6x \]
25.

\[ r^2 + 8r + 41 = 0 \]
\[ r = -4 \pm 5i \]
\[ y = e^{4t} \left( C_1 \sin 5x + C_2 \cos 5x \right) \]

27.

\[ r^3 + 2r^2 + 5r = 0 \]
\[ r(r^2 + 2r + 5) = 0 \]
\[ r = 0 \]
\[ r = -1 \pm 2i \]
\[ y = C_1 e^{-t} \left( C_1 \sin 2x + C_2 \cos 2x \right) \]

39.

\[ (r + 2)^2 = 0 \]
\[ r = -2 \text{ repeated} \]
\[ y = C_1 e^{-2x} + C_2 xe^{-2x} \]
\[ y' = -2C_1 e^{-2x} - 2C_2 xe^{-2x} + C_2 t e^{-2x} \]
\[ y(0) = 2 \rightarrow C_1 = 2 \]
\[ y'(0) = 3 \rightarrow C_2 = 7 \]
\[ y = e^2 + xe^2 \]

31.

\[ 2r^2 + 2r - 5 = 0 \]
\[ r = -1 \pm \frac{\sqrt{11}}{2} \]
\[ y = C_1 e^{-1-\sqrt{11}/2} + C_2 e^{-1+\sqrt{11}/2} \]

33.

\[ 3r^2 - 2r + 10 = 0 \]
\[ r = \frac{-1 \pm i \sqrt{11}}{6} \]
\[ y = e^{r/3} \left( C_1 \sin \frac{\sqrt{116}x}{6} + C_2 \cos \frac{\sqrt{116}x}{6} \right) \]
\[ = e^{r/3} \left( C_1 \sin \frac{2\sqrt{29}x}{3} + C_2 \cos \frac{2\sqrt{29}x}{3} \right) \]

35.

\[ r(r-1) = 0 \]
\[ r = 0 \]
\[ r = 1 \]
\[ y = C_1 + C_2 e^x \]
\[ y' = C_2 e^x \]
\[ y(0) = 0 \rightarrow C_1 = -C_2 \]
\[ y'(0) = -1 \rightarrow C_2 = -1 \]
\[ \rightarrow \]
\[ C_1 = 1 \]
\[ y = 1 - e^x \]
45. \[2.7\cdot 3.91 + 2(0.0576) = 6.7562
0.0576(2.7 + 3.91 + 0.0576) = 0.36578176\]
\[r^2 = 6.7562 r^2 + 0.36578176 = 0\]
\[r = -0.0576\]
\[r = 0.0576\]
\[N(t) = C_1 e^{-0.0576t} + C_2 e^{0.0576t}\]

47. \[0.31 \cdot 1.61 + 2(0.54) = 2.97\]
\[0.54(0.31 + 1.61 + 0.54) = 1.2546\]
\[r^2 + 2.97r + 1.2546 = 0\]
\[r = -0.54\]
\[r = -2.45\]
\[N(t) = C_1 e^{-0.54t} + C_2 e^{-2.45t}\]

49.
\[g(x) = C_1 e^{2x} + C_2 e^{3x} + C_3 e^{4x} \sin 4x + C_4 e^{2x} \cos 4x + C_5 e^{2x} \sin 4x + C_6 e^{2x} \cos 4x\]

51. \[g = C_1 y_1 + C_2 y_2\]
\[g' = C_1 y_1' + C_2 y_2'\]
\[g'' = C_1 y_1'' + C_2 y_2''\]
\[ay'' + by' + cy = a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2)\]
\[= aC_1 y_1'' + bC_1 y_1' + cC_1 y_1 + aC_2 y_2'' + bC_2 y_2' + cC_2 y_2\]
\[= 0 + 0\]
\[= 0\]

53. a) If the quadratic equation has only one root then the discriminant must equal 0. Thus
\[x = \frac{-b \pm \sqrt{0}}{2a}\]
\[x = \frac{-b}{2a}\]

b) Since \(x = r\) is a solution to the equation then substitution \(r\) for \(x\) in the equation \(a e^{2r} + b r + c\) yields the desired result.

c)
\[r = \frac{-b}{2a}\]
\[2ar = -b\]
\[2ar + b = 0\]

d)
\[y = x e^{rt}\]
\[y' = r e^{rt} + e^{rt}\]
\[y'' = r^2 e^{rt} + r e^{rt} + r e^{rt}\]
\[ay'' + by' + cy = a(r^2 e^{rt} + r e^{rt} + r e^{rt}) + b(r e^{rt} + r e^{rt} + r e^{rt}) + c e^{rt}\]
\[= (ar^2 + br + c) e^{rt} + (2ar + b) e^{rt}\]
\[= 0 + 0\]
\[= 0\]

Exercise Set 9.2

1. \(r^2 + 1 = 0\)
\[r = \pm i\]
\[y_0 = C_1 \sin x + C_2 \cos x\]
\[y_p = A\]
\[y_p' = 0\]
\[y_p'' = 0\]
\[y_p'' + y_p = 7\]
\[0 + A = 7\]
\[y_p = 7\]
\[y = C_1 \sin x + C_2 \cos x + 7\]

3. \(r^2 - 2r + 1 = 0\)
\[r = 1\text{ repeated}\]
\[y_0 = C_1 e^x + C_2 x e^x\]
\[y_p = A\]
\[y_p' = 0\]
\[y_p'' = 0\]
\[y_p'' - 2y_p + y_p = 3\]
\[0 - A + 3 = 3\]
\[y_p = 3\]
\[y = C_1 e^x + C_2 x e^x + 3\]

5. \(r^2 + 4r + 4 = 0\)
\[r = -2\text{ repeated}\]
\[y_0 = C_1 e^{-2x} + C_2 x e^{-2x}\]
\[y_p = Ax + B\]
\[y_p' = 0\]
\[y_p'' = 0\]
\[0 + 4(A) + 4(Ax + B) = 8 - 12x\]
\[4A = -12 \rightarrow A = -3\]
\[4B = 8 - 12x\]
\[y_p = -3 - 3x\]
\[y = C_1 e^{-2x} + C_2 x e^{-2x} - 3x + 5\]
7.
\[ r^2 - 4r + 3 = 0 \]
\[ r = -3 \]
\[ r = 1 \]
\[ C_1 e^{-3x} + C_2 e^{-x} = y_b \]
\[ y_p = A2x^2 + Bx + C \]
\[ y_p' = 2Ax + B \]
\[ y_p'' = 2A \]
\[ 2A - 4(2Ax + B) + 3(4Ax^2 + Bx + C) = 6x^2 - 4 \]
\[ 3Ax^2 + (-8A + 3B)x + (2A - 4B + 3C) = 6x^2 - 4 \]
\[ 3A = 6 \rightarrow A = 2 \]
\[ -8A + 3B = 0 \rightarrow B = -\frac{16}{3} \]
\[ 2A - 4B + 3C = -4 \rightarrow C = \frac{40}{9} \]
\[ 2e^{-3} - \frac{16}{3}x + \frac{40}{9} = y_p \]
\[ C_1 e^{-3x} + C_2 e^{-x} + 2e^{-3} - \frac{16}{3}x + \frac{40}{9} = y \]

Note: The particular solution had to be \( Ax + B \) since the homogeneous solution already contained the constant solution.

9.
\[ r^2 - r - 2 = 0 \]
\[ r = 2 \]
\[ r = -1 \]
\[ C_1 e^{-x} + C_2 e^{3x} = y_b \]
\[ Ax^3 + Bx^2 + Cx + D = y_p \]
\[ 3Ax^2 + 2Bx + C = y_p' \]
\[ 6Ax + 2B = y_p'' \]
\[ x^3 - 1 = 5Ax + 2B - 3Ax^2 - 2Bx - C + 2(4Ax^3 + Bx^2 + Cx + D) \]
\[ x^3 - 1 = 2Ax^3 + (2B - 3A)x^2 + (2C - 2B + 6A)x + (2D + 2B - C) \]
\[ A = \frac{1}{2} \]
\[ 2B - 3A = 0 \rightarrow B = \frac{3}{4} \]
\[ 2C - 2B + 6A = 0 \rightarrow C = \frac{-9}{4} \]
\[ 2D - C + 2B = -1 \rightarrow D = \frac{19}{8} \]
\[ y_p = \frac{x^3}{2} + \frac{3x^2}{4} - \frac{9x}{4} + \frac{19}{8} \]
\[ y = \frac{x^3}{2} + \frac{3x^2}{4} - \frac{9x}{4} + \frac{19}{8} \]

Note: The particular solution had to be \( Ax^3 + Bx + C \) since the homogeneous solution already contained the linear solution.

13.
\[ r^3 + r^2 = 0 \]
\[ r = 0 \text{ repeated} \]
\[ r = -1 \]
\[ y_b = C_1 + C_2x + C_3 e^{-x} \]
\[ y_p = A2x^2 + Bx + C \]
\[ y_p' = 2Ax + B \]
\[ y_p'' = 2A \]
\[ y_p''' = 0 \]
\[ 0 + 2A = -2 \]
\[ A = -1 \]
\[ y_p = -x^2 \]
\[ y = C_1 + C_2x + C_3 e^{-x} - x^2 \]

Note: The particular solution had to be \( Ax^2 + Bx + C \) since the homogeneous solution already contained the linear solution.

15.
\[ r^2 + 4x^2 + 20x = 0 \]
\[ r = 0 \]
\[ r = -2 \pm 4i \]
\[ y_b = C_1 e^{-2x} + C_2 \sin 4x + C_3 \cos 4x \]
\[ y_p = Ax^3 + Bx + C \]
\[ y_p' = 2Ax + B \]
\[ y_p'' = 2A \]
\[ y_p''' = 0 \]
\[ 0 + 8A + 40Ax + 20B = 40x - 12 \]
\[ 40Ax + 40B = 40x - 12 \]
\[ 40A = 40 \rightarrow A = 1 \]
\[ 8A + 20B = -12 \rightarrow B = -1 \]
\[ y_p = x^3 - x \]
\[ y = C_1 e^{-2x} + C_2 \sin 4x + C_3 \cos 4x \]
\[ + x^3 - x \]

Note: The particular solution had to be \( Ax^2 + Bx + C \) since the associated auxiliary equation has 0 as a root.
17. 
\[ r^2 - r - 2 = 0 \]
\[ r = 2, -1 \]
\[ y_n = C_1 e^{-x} + C_2 e^{2x} \]
\[ y_p = Ax + B \]
\[ y_p' = A \]
\[ y_p'' = 0 \]
\[ 0 = A - 2Ax + 2B = 2x - 1 \]
\[ -2Ax + (-A - 2B) = 2x - 1 \]
\[ A = -1 \]
\[ B = 1 \]
\[ y_p = -x + 1 \]
\[ y = C_1 e^{-x} + C_2 e^{2x} - x \]
\[ y_p' = -C_1 e^{-x} + 2C_2 e^{2x} - 1 \]
\[ y_p(0) = 0 \to C_1 + C_2 = 6 \]
\[ y_p'(0) = 0 \to -C_1 + 2C_2 = 0 \]

Solving the system

\[ C_1 = 4 \]
\[ C_2 = 2 \]
\[ y = 4e^{-x} - 2e^{2x} - x \]

19. 
\[ r^2 + 2r + 1 = 0 \]
\[ r = -1 \text{ repeated} \]
\[ y_n = C_1 e^{x} + C_2 xe^{x} \]
\[ y_p = Ax + B \]
\[ y_p' = 2Ax + B \]
\[ y_p'' = 2A \]

2A + 2Ax + 2B + Ac^2 + Bc + C = x^2

Ax^2 + (A + 2A)x + (2A + 2B + C) = x^2
A = 1

4A + 2B = 0 \to B = -2

2A + 2B + C = 0 \to C = -6

\[ y_p = x^2 - 4x + 6 \]
\[ -C_1 e^{-x} + C_2 xe^{x} + C_2 e^{x} = 2x - 1 \]
\[ C_1 = 5 \]
\[ y_p(0) = 2 \to C_1 - C_2 = 1 \]
\[ C_2 = 1 \]
\[ y_p'(0) = 1 \to C_1 - C_2 = 1 \]
\[ y = e^x (2 \sin x + \cos x) + 1 \]

21. 
\[ r^2 + 4r - 6 = 0 \]
\[ r = 0, -1 \]
\[ y_n = C_1 + C_2 e^{-x} \]
\[ y_p = Ax^2 + Bx + C \]

2A + 8Ax + 1B = 16x
8Ax + (2A + 4B) = 16x
8A = 16 \to A = 2
2A + 4B = 0 \to B = -1
y_p = 2x^2 - x
y_p(0) = 2 \to C_1 + C_2 = 2
y_p'(0) = -3 \to -4C_2 = -2

\[ C_1 = \frac{1}{2} \]
\[ C_2 = \frac{3}{2} \]
\[ y = \frac{3}{2}x - 2 \]

Note: \( r = 0 \) is a root to the auxiliary equation and the effect of our selection of \( y_p \).
29. $x'' + 2x' + 5x = 10$

$r^2 + 2r + 5 = 0$

$r = -1 \pm 2i$

$e^{-t} (C_1 \sin 2t + C_2 \cos 2t) = x_h$

$x_p = A$

$x_p' = 0$

$x_p'' = 0$

$0 + 5 \lambda = 10 \rightarrow \lambda = 2$

$x_p = 2$

$y = C_1 e^{-2x} (C_2 \sin \frac{\pi}{2} x + C_3 \cos \frac{\pi}{2} x)$

$+ \frac{5}{2} x^2 + 5x$

$y' = -2 C_2 \cos \frac{\pi}{2} x + C_3 \sin \frac{\pi}{2} x)

$+ 5x$

$y'' = 4 e^{-2x} (C_2 \sin \frac{\pi}{2} x + C_3 \cos \frac{\pi}{2} x)$

$- 4 e^{-2x} (C_2 \cos \frac{\pi}{2} x - C_3 \sin \frac{\pi}{2} x)$

$+ 5$

$y(0) = 0 \rightarrow C_1 + C_2 = 0$

$y'(0) = 0 \rightarrow C_2 - 2C_3 = 0$

$y''(0) = 1 \rightarrow 4C_3 - 4C_2 - C_3 = 1$

Solving the system

$C_1 = \frac{21}{16}$

$C_2 = -\frac{7}{16}$

$C_3 = \frac{1}{5}$

$y = \frac{21}{5} e^{-2x} \left( \frac{-7}{5} \sin \frac{x}{2} - \frac{16}{5} \cos \frac{x}{2} \right)$

$+ \frac{5}{2} x^2 + 5x$

27. $x'' + 16x = 1$

$r^2 + 16 = 0$

$r = \pm 4i$

$x_h = C_1 \sin 4t + C_2 \cos 4t$

$x_p = A$

$x_p' = 0$

$x_p'' = 0$

$0 + 16A = 1 \rightarrow A = \frac{1}{16}$

$x = C_1 \sin 4t + C_2 \cos 4t + \frac{1}{16}$

$x' = 4C_1 \cos 4t - 4C_2 \sin 4t$

$x(0) = 0 \rightarrow C_2 + \frac{1}{16} \rightarrow C_2 = -\frac{1}{16}$

$x'(0) = 0 \rightarrow C_1 = 0$

$x = \frac{1 - \frac{1}{16}}{\cos 4t}$

$\frac{1 - \cos \frac{t}{\sqrt{c}}}{\sqrt{c}}$

31. $cm x'' + x = cpd + x_0$

a)

$cm r^2 + 1 = 0$

$r = \pm \frac{1}{\sqrt{cm}} i$

$x_h = C_1 \sin \frac{t}{\sqrt{cm}} + C_2 \cos \frac{t}{\sqrt{cm}}$

$x_p = A$

$x_p' = 0$

$x_p'' = 0$

$A = cpd + x_0$

$x = C_1 \sin \frac{t}{\sqrt{cm}} + C_2 \cos \frac{t}{\sqrt{cm}}$

b)

$C_1 \cos \frac{t}{\sqrt{cm}} - C_2 \sin \frac{t}{\sqrt{cm}} = x'$

$C_2 = cpd - x_0 = 0 \rightarrow x(0) = 0$

$-cpd - x_0 = C_2$

$0 = \frac{C_1}{\sqrt{cm}} \rightarrow x' = 0$

$C_1 = 0$

$cpd + x_0 = (cpd + x_0) \cos \frac{t}{\sqrt{cm}} = x$

$(cpd + x_0) \left( 1 - \cos \frac{t}{\sqrt{cm}} \right)$
33. \( F'' + 1.05F' + 0.05F = 0 \)
\[
F'' = -1.05F' - 0.05F
\]
\[
r^2 + 1.05r + 0.05 = 0
\]
\[
r = -\frac{1}{20}
\]
\[
r = -1
\]
\[
F_h = C_1 e^{-t} + C_2 e^{-t/20}
\]
\[
F_p = A
\]
\[
F''_p = 0
\]
\[
F'_p = 0
\]
\[
0.05 \rightarrow A = 1
\]
\[
F = C_1 e^{-t} + C_2 e^{-t/20} + 1
\]
\[
f'' = -C_1 e^{-t} - \frac{C_2}{20} e^{-t/20}
\]
\[
f'(0) = 0 \rightarrow C_1 + C_2 = -1
\]
\[
f''(0) = 0.05 \rightarrow -C_1 - \frac{C_2}{20} = 0.05
\]
Solving the system
\[
C_1 \quad 0
\]
\[
C_2 \quad -1
\]
\[
A \quad 1 - e^{-1/20}
\]

35. \( y' + 2y = x \)
\[
r + 2 = 0
\]
\[
r = -2
\]
\[
y_h : C_1 e^{-2x}
\]
\[
y_p : \text{Ax}^2 + Bx + C
\]
\[
y''_p = 2Ax + B
\]
\[
y'''_p = 2A
\]
\[
x^2 = 2Ax + B + 2Ax^2 + 2Bx + 2C
\]
\[
A = -\frac{1}{2}
\]
\[
B + 2C' = 0 \rightarrow C = \frac{1}{4}
\]
\[
y = C_1 e^{-2x} + \frac{1}{2} x^2 + \frac{1}{4} x + \frac{1}{4}
\]

3. \( y' = -2x \) and \( x'' = 2x + y' \)
\[
x'' = 2x' + 3x + 4(x' - 2x)
\]
\[
x'' - 6x' + 5x = 0
\]
\[
r^2 - 6r + 5 = 0
\]
\[
r = 5
\]
\[
r = 1
\]
\[
x = C_1 e^t + C_2 e^{5t}
\]
\[
x' = C_1 e^t + 5C_2 e^{5t}
\]
\[
y = x' - 2x = 3C_2 e^{5t} - C_1 e^t
\]

5. \( x' = -0.5x + y \rightarrow y = x' + 0.5x \) and \( y' = 0.5x \)
\[
x'' = -0.5x' + y'
\]
\[
x'' = -0.5x' + 0.5x
\]
\[
x'' + 0.5x' - 0.5x = 0
\]
\[
r^2 - 0.5r - 0.5 = 0
\]
\[
r = 1/2
\]
\[
r = -1
\]
\[
x = C_1 e^{t/2} + C_2 e^{-t}
\]
\[
x' = \frac{C_1}{2} e^{t/2} - C_2 e^{-t}
\]
\[
y = x' + 0.5x = C_1 e^{t/2} - C_2 e^{-t}
\]

7. \( y = x' - 4x \)
\[
x'' = 4x' + y'
\]
\[
x'' = 4x' - x + 2y
\]
\[
x'' - 6x' + 9x = 0
\]
\[
r^3 - 6r + 9 = 0
\]
\[
r = \text{3 repeated}
\]
\[
x = C_1 e^{-3t} + C_2 e^{-3t}
\]
\[
x' = -3C_1 e^{-3t} - 3C_2 e^{-3t} + C_2 e^{-3t}
\]
\[
y = x' - 4x
\]
\[
y = (C_2 - C_1) e^{-3t} - 3C_2 e^{-3t}
\]

9. \( y = \frac{x'}{z} \)
\[
x'' = 2y' + 2(-18z)
\]
\[
x'' + 36x = 0
\]
\[
r^3 + 36 = 0
\]
\[
r = \pm 6i
\]
\[
x = C_1 \sin 6t + C_2 \cos 6t
\]
\[
x' = 6C_1 \cos 6t - 6C_2 \sin 6t
\]
\[
g = \frac{x}{2}
\]
\[
g = 3C_1 \cos 6t - 3C_2 \sin 6t
\]

Exercise Set 9.3

1.
\[
x'' = -2x + 3x'
\]
\[
x'' = 3x' - 2x = 0
\]
\[
r^2 - 3r + 2 = 0
\]
\[
r = 2
\]
\[
r = 1
\]
\[
x = C_1 e^t + C_2 e^{3t}
\]
\[
y = x' = C_1 e^t + 2C_2 e^{3t}
\]
11. \( 3x' - x'' \)

\[
x'' = 3x' - 5(x - y)
= 3x' - 5x - 3x - x'
\]

\[
x'' - 2x' + 2 = 0
r^2 - 2r + 2 = 0
r = 1 \pm i
x = e^{i(C_1 \sin t + C_2 \cos t)}
\]

\[y = \frac{e^{i(2C_1 - C_2) \sin t + (2C_2 - C_1) \cos t}}{5}\]

13. \( y' - 5x - x'' \)

\[
x'' - 5x' - (2x + 10x - 2x') + 4 )
\]

\[
x'' - 7x' + 12 = -4
r^2 - 7r + 12 = 0
r = 4
\]

\[x = C_1 e^{4t} + C_2 e^{4t} \]

\[y = e^{4t} - 4 e^{4t}\]

19.

\[
x'' = 2x' + 3(-3x + 8y)
\]

\[
x'' = 2x' - 9x + 8x' - 16x
\]

15.

\[
x'' = y' + 2
-\quad -t + 4t - 2 + 2
\]

\[
x'' + x = 4t
r^2 + 1 = 0
r = \pm i
\]

\[x = C_1 \sin t + C_2 \cos t
\]

\[x' = At + B
\]

\[y = 0 + At + B = 4t\]

21.

\[
x'' = -4x
\]

\[
x'' + 4x = 0
r^2 + 4 = 0
r = \pm 2i
\]

\[x = C_1 \sin 2t + C_2 \cos 2t
\]

\[y = 2x' + 2C_1 \cos 2t - 2C_2 \sin 2t
x(0) = 1 \rightarrow C_2 = 1
y(0) = 2 \rightarrow C_1 = 1
\]

\[x = \sin 2t \cos 2t
y = 2 \cos 2t - 2 \sin 2t
\]
23.  
\[ x'' = 2x' + (5x - 2y + 12) \]
\[ x'' = 2x' + 5x - 2x' + 4x + 6 + 12 \]
\[ r^2 - 9r = 0 \]
\[ r = -3 \]
\[ r = 3 \]
\[ x_0 = C_1 e^{-3t} + C_2 e^{3t} \]
\[ x_p = A \]
\[ x_p' = 0 \]
\[ x_p'' = 0 \]
\[ 0 - 9A = 18 \]
\[ A = -2 \]
\[ x = C_1 e^{-3t} + C_2 e^{3t} + 2 \]
\[ x' = -3C_1 e^{-3t} + 3C_2 e^{3t} \]
\[ y = x' - 2x - 3 \]
\[ y = -5C_1 e^{-3t} + C_2 e^{3t} + 1 \]
\[ x(0) = 6 \rightarrow C_1 + C_2 = 8 \]
\[ y(0) = -3 \rightarrow -5C_1 + C_2 = -1 \]

Solving the system

\[ C_1 = 2 \]
\[ C_2 = 6 \]
\[ x = 2 e^{-3t} + 6 e^{3t} - 2 \]
\[ y = -10 e^{-3t} + 6 e^{3t} + 1 \]

25.  
\[ P'' = -3P + 2Q \] and \[ Q' = 3P - 2Q \]

\[ P'' = -3P' + 2(3P - 2Q) \]
\[ P'' = -3P' + 6P - 2(P' + 3P) \]
\[ P'' \| 5P'' = 0 \]
\[ r^2 + 5r = 0 \]
\[ r = 0 \]
\[ r = -5 \]
\[ P = C_1 e^{-5t} \]
\[ P' = -5C_1 e^{-5t} \]
\[ Q = \frac{P' + 3P}{2} \]
\[ = \frac{3}{2} C_1 e^{-5t} \]

27.  
\[ x' = -2x + 2x - 10 \] and \[ y' = 2x - 2y \]

\[ x'' = -2x' + 2y' \]
\[ = -2x' + 4x - 2(x' + 2x + 10) \]
\[ x'' + 4x' = -20 \]
\[ r^2 + 4r = 0 \]
\[ r = 0 \]
\[ r = -4 \]
\[ c_0 = C_1 + C_2 e^{-4t} \]
\[ x_p = A + B \]
\[ x_p' = A \]

29.  
a) \( A(t)/2 \) per hour and \( B(t)/2 \) per hour
b) Left to the student
c) It follows from the statement of the problem
d) \[ A'' = -0.5A' + 0.5B' \]
\[ = -0.5A' + 0.5(0.5A - 0.5B) \]
\[ = -0.5A' + 0.25A - 0.25A \]

\[ A'' \| A' = 0 \]
\[ r^2 + r = 0 \]
\[ r = 0 \]
\[ r = -1 \]
\[ A = C_1 + C_2 e^{-t} \]
\[ A' = -C_2 e^{-t} \]
\[ B = 2A' + A \]
\[ C_1 - C_2 e^{-t} \]

\[ A(0) = 2000 \rightarrow C_1 + C_2 = 2000 \]
\[ B(0) = 1000 \rightarrow C_1 - C_2 = 1000 \]

Solving the system

\[ C_1 = 1500 \]
\[ C_2 = 500 \]
\[ A(t) = 1500 + 500 e^{-t} \]
\[ B(t) = 1500 - 500 e^{-t} \]

f) As the limit approaches infinity both functions approach 1500, which is the equilibrium point.
31. a) Left to the student
   
b) \( P'' = -3P + 0.6L - 2.4P \)
   \[ \Rightarrow P'' = -5.4P + 0.6L \text{ and } L' = 3P - 0.6L \]

c) 
\[
-5.4P' + 0.6(3P - 0.6L) = P''
\]
\[
-5.4P' + 1.8P - 0.6(-5.4P - P') = 0
\]
\[
-4.8P' + 5.04P = 0
\]
\[
r^2 + 4.8r - 5.04 = 0
\]
\[
0.886 = r
\]
\[
-5.086 = r
\]
\[
C_1 e^{-5.086t} + C_2 e^{-0.886t} = P
\]
\[-5.086C_1 e^{-5.086t} - 0.886C_2 e^{-0.886t} = P'
\]
\[
P' + 5.4P = l
\]
\[
-0.47667C_1 e^{-5.086t} + 7.52333C_2 e^{-0.886t} = 0
\]
\[
C_1 + C_2 = 0 \quad \Rightarrow \quad P(0) = 0
\]
\[
-0.47667C_1 + 7.52333C_2 = -214/15 \quad \Rightarrow \quad L(0) = 4.9
\]

Solving the system:
\[
\begin{align*}
2.28004 &= C_1 \\
-2.28004 &= C_2 \\
2.28004 e^{-5.086t} - 2.28004 e^{-0.886t} &= P \\
1.083927 e^{-5.086t} + 17.15319 e^{-0.886t} &= L
\end{align*}
\]

33. a) Left to the student
   
b) Left to the student

c) 
\[
L' = -aL - mL + cM + p \\
M' = mL - aM - cM \\
L' = -(a + m)L + cM + p \\
M' = mL - (a + c)M
\]

35. 
\[ L' = -aL - mL + cM + p \text{ and } M' = mL - aM - cM \]
\[ L' = -(a + m)L + cM + p \text{ and } M' = mL - (a + c)M \]

37. 
\[ L_p = A \]
\[ L_p' = 0 \]
\[ L_p'' = 0 \]
\[ 0 + 0 + a(a + c + m)A = (a + c)p \]
\[ A = \frac{(a + c)p}{a(a + c + m)} \]

39. \( N'' + (c + m + 2a)N' + a(c + m + a)N = 0 \)
\[ L(t) - \frac{(c + a)p}{a(c + m + a)} = N(t) \]
\[ N''(t) = L'(t) \]
\[ N'' = L''(t) \]

Thus, \( L'' + (c + m + 2a)L' + a(c + m + a)L = (c + a)p \)

Which has \( L_p \) as a solution

41. Left to the student

43. 
\[ b(t) = \frac{C_1 + C_2 e^{(q - p)t}}{V_B} \]
\[ = \frac{C_1}{V_B} + \frac{C_2 e^{(q - p)t}}{V_B} \]
\[ = C_1 + C_2 e^{-rt} \]

45. 
\[ C_1 + C_2 = \frac{qD}{rV_B} + \frac{pD}{rV_B} \]
\[ C_1 + C_2 = \frac{D(p - q)}{rV_B} \]
\[ C_1 + C_2 = \frac{V_B}{C_1 + C_2} \]
\[ q = \frac{C_1 rV_B}{D} \]
\[ = \frac{C_1}{C_1 + C_2} \]
\[ p = \frac{C_2 rV_B}{D} \]
\[ = \frac{C_2}{C_1 + C_2} \]

Exercise Set 9.4

1. \[ \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

3. \[ \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

5. \( x' = x + 3y \text{ and } y' = 5x + 7y \)

7. \( x' = 3y \text{ and } y' = x - 2y \)
9. \[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2e^{3t} - e^{2t} \\ -2e^{3t} + 2e^{2t} \end{pmatrix} = \begin{pmatrix} 6e^{3t} - 2e^{2t} \\ -6e^{3t} + 4e^{2t} \end{pmatrix}
\]

\[
\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 2e^{3t} - e^{2t} \\ -2e^{3t} + 2e^{2t} \end{pmatrix} = \begin{pmatrix} 6e^{3t} - 2e^{2t} \\ -6e^{3t} + 4e^{2t} \end{pmatrix}
\]

11. \[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} (-2e^t \sin 2t) \\ (3e^t \sin 2t + e^t \cos 2t) \end{pmatrix}
\]

\[
-4e^t \cos 2t - 2e^t \sin 2t
\]

\[
6e^t \cos 2t + 3e^t \sin 2t + e^t \sin t + e^t \cos t
\]

\[
-3 -1
\]

\[
10 -7
\]

\[
-4e^t \cos 2t - 2e^t \sin 2t
\]

\[
6e^t \cos 2t - 3e^t \sin 2t - e^t \sin t + e^t \cos t
\]

13. \[
A = \begin{pmatrix} 0 & -2 \\ 3 & 3 \end{pmatrix}
\]

\[
\det(A) = 2 \quad \text{and} \quad \text{trace}(A) = 3
\]

\[
r^2 - 3r + 2 = 0
\]

\[
(r - 1)(r - 2) = 0
\]

\[
r = 1
\]

\[
r = 2
\]

Then (by Theorem 9 of Chapter 6) the eigenvectors

For \( r = 1 \) are \[
\begin{pmatrix} -2 \\ 1 \end{pmatrix}
\]

For \( r = 2 \) are \[
\begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

Therefore, the general solution is given by

\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}
\]

15. \[
A = \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}
\]

\[
\det(A) = -16 \quad \text{and} \quad \text{trace}(A) = 0
\]

\[
r^2 + 16 = 0
\]

\[
(r - 4)(r + 4) = 0
\]

\[
r = -4
\]

\[
r = 4
\]

Then (by Theorem 9 of Chapter 6) the eigenvectors

For \( r = -4 \) are \[
\begin{pmatrix} -1 \\ 6 \end{pmatrix}
\]

For \( r = 4 \) are \[
\begin{pmatrix} 6 \\ 3 \end{pmatrix}
\]

Therefore, the general solution is given by

\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-4t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

17. \[
A = \begin{pmatrix} -2 & 4 \\ -1 & 1 \end{pmatrix}
\]

\[
\det(A) = 18 \quad \text{and} \quad \text{trace}(A) = -9
\]

\[
r^2 - 9r + 18 = 0
\]

\[
r = 3
\]

\[
r = 6
\]

Then (by Theorem 9 of Chapter 6) the eigenvectors

For \( r = 3 \) are \[
\begin{pmatrix} -1 \\ -1 \end{pmatrix}
\]

For \( r = 6 \) are \[
\begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

Therefore, the general solution is given by

\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} -1 \\ -1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

\[
-3C_1 e^{3t} - C_2 e^{6t}
\]

\[
-C_1 e^{3t} + C_2 e^{6t}
\]

19. \[
A = \begin{pmatrix} -5 & 10 \\ -4 & 7 \end{pmatrix}
\]

\[
\det(A) = 5 \quad \text{and} \quad \text{trace}(A) = 2
\]

\[
r^2 - 2r + 5 = 0
\]

\[
r = 1 \pm 2i
\]

\[
x = e^t (C_1 \sin 2t + C_2 \cos 2t)
\]

\[
x' = e^t (2C_1 \cos 2t - 2C_2 \sin 2t)
\]

\[
y = e^t (C_1 \sin 2t + C_2 \cos 2t)
\]

\[
y' = e^t (3C_1 \sin 2t - 3C_2 \cos 2t)
\]

21. \[
A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}
\]

\[
\det(A) = -1 \quad \text{and} \quad \text{trace}(A) = 2
\]

\[
r^2 - 2r + 1 = 0
\]

\[
r = 1 \text{ repeated}
\]

\[
x = C_1 e^t + C_2 t e^t
\]

\[
x' = C_1 e^t + C_2 e^t + C_2 e^t
\]

\[
y = -x' + (C_1 + 3C_2) e^t \cos 2t
\]

23. Since the eigenvalues have different signs, then the origin is an unstable saddle point.
25. Since the eigenvalues are both positive, then the origin is an unstable node.

27. Since the eigenvalues are both negative, then the origin is an asymptotically stable node.

29. Since the eigenvalues are both positive, then the origin is an unstable node.

31. Since the eigenvalues have different signs, then the origin is an unstable saddle point.

33. Since the eigenvalues are both positive, then the origin is an asymptotically stable node.

35. Since the eigenvalues are complex with a positive real part, then the origin is an unstable spiral point.

37. Since the eigenvalues are positive and equal, then the origin is an unstable improper node.

39. \( A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \)

\[ x'' = -x + 2y \]
\[ y'' = -x + 2y' \]
\[ x'' - 2x + x = 0 \]
\[ r^2 - 2r + 1 = 0 \]
\[ (r - 1)^2 = 0 \]
\[ r = 1 \] repeated

\[ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[
\begin{bmatrix}
  x \\
  y 
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]

b) Left to the student.

41. 
\[
\begin{align*}
\dot{x}' &= ax' + by' \\
&= ax' + b(cx' + dy) \\
&= ax' + b(cx + d(y' - ax')) \\
&= ax' + bx' + dy' - adx' - (ad + bc)x \\
\dot{r}' &= (n + d)x' + (ad - bc)x \\
&= 0 \\
r' &= (a + d)r + (ad - bc)x \\
&= 0
\end{align*}
\]

43. a) 
\[
\begin{align*}
x'' &= 2x' + 6(2x - 6y) \\
&= 2x' + 12x - 6x' - 2x \\
r' &= 12x - 6x' - 12x \\
r'' &= 8x' = 0 \\
&= 0 \\
r' &= -8 \\
r &= 0
\end{align*}
\]

b) The eigenvectors associated with \( r = -8 \) are 
\[
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
\]

and the eigenvectors associated with \( r = 0 \) are 
\[
\begin{bmatrix}
  3 \\
  1
\end{bmatrix}
\]

Thus 
\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \begin{bmatrix}
  3 \\
  1
\end{bmatrix} C_1 e^{t} + \begin{bmatrix}
  -1 \\
  1
\end{bmatrix} C_2 e^{-8t}
\]

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \begin{bmatrix}
  3C_1 - C_2 e^{-8t} + C_1 + C_2 e^{-8t}
\end{bmatrix}
\]

c) 
\[
\begin{align*}
y + x &= 3C_1 - C_2 e^{-8t} + C_1 + C_2 e^{-8t} \\
y &= 4C_1 \\
y &= b - x
\end{align*}
\]

d) 
\[
\begin{align*}
x' &= 0 \\
-2x + 6y &= 0 \\
y &= x \\
3
\end{align*}
\]

Similarly, 
\[
\begin{align*}
g' &= 0 \\
2x - 6y &= 0 \\
g &= x
\end{align*}
\]

45. 
\[
A = \begin{bmatrix}
  1 & -3 & 0 \\
  -3 & -3 & -4 \\
  3 & 5 & 6
\end{bmatrix}
\]

The eigenvalues of matrix \( A \) are \( r = -2, r = 2 \) and \( r = 4 \) which have the following eigenvectors 
\[
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
\]
and 
\[
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]
respectively. Therefore, the general solution is given by 
\[
\begin{bmatrix}
  -C_1 e^{-2t} - 3C_2 e^{2t} + C_3 e^{4t} \\
  -C_1 e^{-2t} + C_2 e^{2t} - C_3 e^{4t} \\
  C_1 e^{-2t} + C_2 e^{2t} + C_3 e^{4t}
\end{bmatrix}
\]

47. 
\[
A = \begin{bmatrix}
  3 & 2 & 0 \\
  -1 & 2 & -2 \\
  -1 & 0 & 0
\end{bmatrix}
\]

The eigenvalues of matrix \( A \) are \( r = 1, r = 2 \) and \( r = 4 \) which have the following eigenvectors 
\[
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
\]
and 
\[
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]
respectively. Therefore, the general solution is given by 
\[
\begin{bmatrix}
  C_1 e^t - 2C_2 e^{2t} + C_3 e^{4t} \\
  C_1 e^t + C_2 e^{2t} - C_3 e^{4t} \\
  C_1 e^t + C_3 e^{4t}
\end{bmatrix}
\]

49. 
\[
A = \begin{bmatrix}
  2 & 0 & 6 \\
  1 & 1 & 0 \\
  -1 & 1 & -1
\end{bmatrix}
\]

The eigenvalues of matrix \( A \) are \( r = -2, r = -1 \) and \( r = 2 \) which have the following eigenvectors 
\[
\begin{bmatrix}
  -3 \\
  1
\end{bmatrix}
\]
and 
\[
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]
respectively. Therefore, the general solution is given by 
\[
\begin{bmatrix}
  -3C_1 e^{-2t} - 2C_2 e^{-t} + C_3 e^{2t} \\
  C_1 e^{-2t} + C_2 e^{-t} - C_3 e^{2t} \\
  2C_1 e^{-2t} + C_2 e^{-t}
\end{bmatrix}
\]
51. \[ S' = -aS \]
\[ B' = aS - (b + c)B + dB \]
\[ H' = cB - dH \]

53. Since all the eigenvalues are negative then the origin will be an asymptotically stable node.

55. The eigenvalues \(-8, -5, \text{ and } -\frac{1}{2}\) have the eigenvectors
\[
\begin{bmatrix}
0 \\
-2 \\
1
\end{bmatrix}
, \quad
\begin{bmatrix}
1 \\
40/27 \\
-35/27
\end{bmatrix}
, \quad
\text{and}
\begin{bmatrix}
0 \\
0 \\
7
\end{bmatrix}
\]
respectively. Therefore, the general solution is given by
\[
\begin{bmatrix}
C_2 e^{-8t} \\
-2C_1 e^{-8t} + 40/27 C_2 e^{-5t} + C_3 e^{-t/2} \\
C_1 e^{-5t} - 35/27 C_2 e^{-5t} + 7C_3 e^{-t/2}
\end{bmatrix}
\]

Applying the initial conditions
\[
\begin{align*}
1 &= C_2 \\
-2C_1 + 40/27 C_2 + C_3 &= 0 \\
C_1 - 35/27 C_2 + 7C_3 &= 0
\end{align*}
\]
Solving the system
\[
\begin{align*}
C_1 &= \frac{7}{9} \\
C_2 &= 1 \\
C_3 &= \frac{2}{27}
\end{align*}
\]
Thus the solution is
\[
S(t) = e^{-\frac{7}{9} t}
\]
\[
B(t) = -\frac{14}{9} e^{-8t} + 40/27 e^{-5t} + \frac{2}{7} e^{-t/2}
\]
\[
H(t) = \frac{7}{9} e^{-5t} - 35/27 e^{-5t} + 14/27 e^{-t/2}
\]

3. Jacobian
\[
\begin{bmatrix}
2x & 2y \\
1 & 1
\end{bmatrix}
\]

The equilibrium points are:
\((0, 0) \rightarrow \) Jacobian has two positive eigenvalues which means the equilibrium point is an unstable node
\((-1, 0) \rightarrow \) Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node
\((-3, -1) \rightarrow \) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point
\((0, 1/2) \rightarrow \) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point.

5. Jacobian
\[
\begin{bmatrix}
-\sqrt{y} & (2-x)/\sqrt{y} \\
\frac{y}{x} & 1
\end{bmatrix}
\]

The equilibrium point is \((2, 1) \rightarrow \) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point.

7. Jacobian
\[
\begin{bmatrix}
-1 & 2y \\
1 & 3y
\end{bmatrix}
\]

The equilibrium points are:
\((1, 1) \rightarrow \) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point
\((1, -1) \rightarrow \) Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node.
9. Jacobian \[
\begin{bmatrix}
1 & -e^y \\
1 & 2-2e^y
\end{bmatrix}
\]

\[x - e^y = 0\]
\[x + e^y - 2 = 0\]
\[e^y - 2 = 0\]
\[(e^y - 1)(e^y - 2) = 0\]
\[x = 0 \text{ no solution}\]
\[e^y = -2 \text{ or } x = 1\]

The equilibrium point is \((1, 0)\). Jacobian has complex eigenvalues with positive real part which means the equilibrium point is an unstable spiral point.

11. \[
J = \begin{bmatrix}
0.1 - 0.02x & -0.005y \\
-0.001y & 0.05 - 0.001x - 0.004y
\end{bmatrix}
\]

a) \[
x(0.1 - 0.01x - 0.005y) = 0
\]
\[y(0.05 - 0.001x - 0.002y) = 0
\]
\[x = 0\]
\[y = 0\]
\[y = -25\]
\[y = -10\]
\[x(0.1 - 0.01x) = 0\]
\[x = 0\]
\[x = 0\]
\[0.1 - 0.01x - 0.005y = 0\]
\[20 - 2x = y\]
\[(20 - 2x)(0.05 - 0.001x - 0.002(20 - 2x)) = 0\]
\[y = 0 \leftrightarrow x = 10\]

Only non-negative values accepted \(x = -\frac{3}{4}\).

The non-negative equilibrium points are:
\((0, 0)\) \(\rightarrow\) Jacobian has two positive eigenvalues which means the equilibrium point is an unstable node.
\((0, 25)\) \(\rightarrow\) Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node.
\((10, 0)\) \(\rightarrow\) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point.
\((12.5, 18.75)\) \(\rightarrow\) Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node.

b) The stable equilibrium point indicates there is coexistence.

15. \[
J = \begin{bmatrix}
0.1 - 0.005x & -0.0008y \\
-0.002y & 0.1 - 0.002x - 0.01y
\end{bmatrix}
\]

a) The non-negative equilibrium points are:
\((0, 0)\) \(\rightarrow\) Jacobian has two positive eigenvalues which means the equilibrium point is an unstable node.
\((0, 20)\) \(\rightarrow\) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point.
\((100, 0)\) \(\rightarrow\) Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node.
\((300, -100)\) \(\rightarrow\) Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point.

b) The stable equilibrium point indicates only the first species survives.
\[ 0 = (0.1 - 0.01x - 0.005y) \]
\[ y = 20 - 2x \]
\[ 0 = (20 - 2x)(0.2 - 0.015x - 0.02(20 - 2x)) \]
\[ x = 10 \]
\[ y = 8 \rightarrow y = 4 \]

(0, 0) gives an unstable node
(0, 10) gives an unstable node
(10, 0) gives an unstable saddle point
(8, 4) gives an asymptotically stable node, and it indicates coexistence

19. \[ J = \begin{bmatrix} 0.1 - 0.2x & -0.005y \\ 0.05y & 0.2 - 0.015x - 0.4y \end{bmatrix} \]

a) \[ x(0.1 - 0.01x - 0.005y) - 0.08y - 0 \]
\[ y(0.2 - 0.015x - 0.02y) = 0 \]

\[ 0.2 - 0.015x - 0.02y = 0 \]
\[ 10 - 0.75x = y \]
\[ x(0.1 - 0.01x - 0.02(10 - 0.75x)) = 0 \]
\[ x[0.1 - 0.01x - 0.2 + 0.015x - 0.08] = 0 \]
\[ x = -4.8 \]
not acceptable
\[ y = 0 \]
\[ x = 0 \]
\[ x = 2 \]

There are no coexistence equilibrium points

b) (0, 0) gives an unstable node
(0, 10) gives an asymptotically stable node
(2, 0) gives an unstable saddle point

c) The excessive fishing (of the first species) helps eliminate the first species

21. \[ J = \begin{bmatrix} 0.6 - 0.3xy & -0.3y \\ 0.2y & -1 + 0.2x \end{bmatrix} \]

\[ x(0.6 - 0.3y) = 0 \]
\[ -y(1 - 0.2x) = 0 \]
\[ x = 0 \rightarrow y = 0 \]
\[ y = 2 \rightarrow x = 5 \]

(0, 0) gives an unstable saddle point
(5, 2) gives a center

25. \[ J = \begin{bmatrix} 0.6 - 0.06x & -0.3y \\ -0.3x & 0.2y \end{bmatrix} \]

\[ x[0.6 - 0.06x - 0.3y] = 0 \]
\[ -y(1 - 0.2x) = 0 \]
\[ x = 0 \rightarrow y = 0 \]
\[ y = 3 \rightarrow y = 1.5 \]
\[ x = 20 \rightarrow y = 0 \]

(0, 0) gives an unstable saddle point
(5, 1.5) gives a center
(20, 0) gives an unstable node

27. \[ J = \begin{bmatrix} 0.5 - 0.1x & -0.2y \\ 0.1y & -0.4 + 0.1x \end{bmatrix} \]

\[ x[0.5 - 0.05x - 0.2y] = 0 \]
\[ -y(0.4 - 0.1x) = 0 \]
\[ x = 0 \rightarrow y = 0 \]
\[ x = 4 \rightarrow y = 1.5 \]
\[ x = 10 \rightarrow y = 0 \]

(0, 0) gives an unstable saddle point
(4, 1.5) gives an asymptotically stable spiral point
(10, 0) gives an unstable saddle point

29. a) When \( x = 0 \) and \( y = 0 \), then \( ax - by = 0 \) and 
\[ -dy + \frac{cey}{\sqrt{c^2x^2 + 1}} = 0 \]

b) \[ x = \frac{\sqrt{65}}{12}, y = \frac{\sqrt{65}}{9} \]

c) Left to the student

31. \[ J = \begin{bmatrix} 0.2 - 0.1x & -0.02y \\ 0.1 - 0.01x - 0.04y \end{bmatrix} \]

a) The eigenvalues for (0, 5) are -0.1 and 0.4 which have the corresponding eigenvectors \([0, 1]\) and \([4, -1]\) respectively. The trajectories seem to approach parallel to \([0, 1]\) and parallel to \([4, -1]\) as they leave

b) The eigenvalues for (1, 0) are -0.2 and 0.0 which have the corresponding eigenvectors \([1, 0]\) and \([-4, 13]\) respectively. The trajectories seem to approach parallel to \([1, 0]\) and parallel to \([-4, 13]\) as they leave

c) The eigenvalues for (2, 5, 3.75) are -0.05 and -0.15 which have the corresponding eigenvectors \([-2, 3]\) and \([2, 1]\) respectively. The trajectories seem to approach parallel to \([-2, 3]\)

33. a) In Exercises 11, 13, 15 and 16, \( a_1a_2 \) is larger and in Exercises 12 and 14, \( b_1b_2 \) is larger

b) Left to the student

35. If \( a < 0 \) then

\[ x' = pr \left( 1 - \frac{x}{L} \right) - \frac{axy}{1 + 0} \]

\[ y' = -cy + \frac{axy}{1 + 0} \]

\[ x' = -cy + s2xy \]
37. \( x' = 0.2x \left( 1 - \frac{x}{10} \right) - \frac{0.014y}{1 + 0.1x} \)
\( y' = -0.06y + \frac{0.03xy}{1 + 0.1x} \)

The coexistence equilibrium point is \((2, 5.7)\) which is an unstable spiral point.

39. 
   a) Left to the student.
   b) When \( \frac{r}{x} \) is plugged into the formula the numerator becomes zero.
   c) When \( \frac{\mu}{y} \) is plugged into the formula the denominator becomes zero.
   d) One population is maximized when the other population agrees with the coexistence equilibrium point value.

41. \( x' = 2x - xy \) and \( y' = -y + 0.4xy \)

This means \( \mu = 2, \lambda = 1, \) and \( s = 0.4 \rightarrow \)

\[
\begin{align*}
2 \ln y - y + \ln x - 0.4x & = C' \\
2 \ln 1 - 1 + \ln 5 - 0.4(5) & = C' \\
-1 + \ln 5 - 2 & = C' \\
\ln 5 - 3 & = C' \\
2 \ln y - y + \ln x - 0.4x & = \ln 5 - 3 \\
\end{align*}
\]

43. \( x = 1.0153393 \) and \( x = 5 \)

45. \( x = 1.6317622 \) and \( x = 3.6330767 \)

---

**Exercise Set 9.6**

1. \( x(1) \approx 2.42368 \)
   \( y(1) \approx 1.018 \)

3. \( x(2) \approx 38.21927 \)
   \( y(2) \approx 83.17852 \)

5. \( x(3) \approx 75.15387 \)
   \( y(3) \approx 212.32262 \)

7. a)
13. a) 

b) 

c) About 7.4 years

15. a) 

17. The trajectory is a spiral and no longer a closed circle

19. a) \( W(20) \approx 0.00064 \)  
\( E(20) \approx 0.8020 \)  
\( U(20) \approx 0.19816 \)

b) \( W(t) \)
Exercise Set 9.6

21. \[ J = \begin{bmatrix}
  0.1 - 0.02x - 0.004y & -0.004x \\
  -0.004y & 0.5 - 0.001x - 0.0032y
\end{bmatrix} \]

Equilibrium points are:
(0, 0) -> Jacobian has two positive eigenvalues which means the equilibrium point is an unstable node
(0, 25) -> Jacobian has two positive eigenvalues which means the equilibrium point is an unstable node
(10, 20) -> Jacobian has two negative eigenvalues which means the equilibrium point is an asymptotically stable node
(\frac{30}{7}, 0) -> Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point

25. \[ J = \begin{bmatrix}
  0.8 - 0.2y & -0.2x \\
  0.1y & -0.6 - 0.1x
\end{bmatrix} \]

The equilibrium points are:
(0, 0) -> Jacobian has eigenvalues with opposite signs which means the equilibrium point is an unstable saddle point
(6, 4) -> Jacobian has two imaginary eigenvalues which means the equilibrium point is a center

27. Left to the student.
Chapter 10

Probability

Exercise Set 10.1

1. $\frac{2}{50} = 0.06$

2. $\frac{11}{20} = 0.55$

3. $\frac{13}{70} = 0.19$

4. $\frac{14}{25} = 0.56$

5. $\frac{21}{50} = 0.46$

6. $\frac{1}{14} \cdot \frac{24}{50} = 0.072$

7. $\frac{30-14}{50} = \frac{16}{50} = 0.32$

8. $\frac{30+14}{50} = \frac{44}{50} = 0.88$

9. $\frac{30-14}{50} = \frac{16}{50} = 0.32$

11. The possible outcomes are $HH, HT, TH, TT$

   a. $P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

   $P(A \& B) = \frac{1}{4}$

   Therefore, events $A$ and $B$ are independent

   b. Events $A$ and $B$ are not disjoint. The outcome $HT$ is common between them.

13. The possible outcomes are $\{1, 2, 3, 4, 5, 6\}$

   a. $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

   $P(A \& B) = \frac{1}{36}$

   Therefore, events $A$ and $B$ are independent

   b. Events $A$ and $B$ are disjoint

15. a. $P(A) \cdot P(B) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$

   $P(A \& B) = \frac{12}{1326} \cdot \frac{2}{221}$

   Therefore, events $A$ and $B$ are not independent.

   b. Events $A$ and $B$ are not disjoint.

17. a. $P(A) \cdot P(B) = \frac{1}{52} \cdot \frac{4}{52} = \frac{1}{169}$

   $P(A \& B) = \frac{1}{52}$

   Therefore, events $A$ and $B$ are not independent

   b. Events $A$ and $B$ are disjoint

19. $\frac{1}{x} > 0.2$

20. $\frac{1}{x} > 0.5$

21. $\frac{1}{x} < 0.8$

22. $\frac{1}{x} < 0.4$

25. $\frac{1}{x} \cdot \frac{1}{2} < 0.05$

27. $\frac{1}{x} \cdot \frac{1}{2} = 0.15$

29. $\frac{1}{x} \cdot \frac{1}{2} < 0.04$

31. $\frac{1}{x} = \frac{3}{x} = 0.16$

33. $\frac{1}{x} = 0.1$

35. $\frac{1}{x} = \frac{1}{x} + 0.1$

37. $\frac{1}{x} < \frac{1}{x} + 0.7$

39. $\frac{1}{x} < \frac{1}{x} + 0.7$

41. $\frac{1}{x} > \frac{1}{x} + 0.7$

43. $\frac{1}{x} < \frac{1}{x} + 0.7$

45. $\frac{1}{x} < \frac{1}{x} + 0.7$

47. $\frac{1}{x} < \frac{1}{x} + 0.7$

49. $\frac{1}{x} < \frac{1}{x} + 0.7$

51. $\frac{1}{x} < \frac{1}{x} + 0.7$

53. $\frac{1}{x} < \frac{1}{x} + 0.7$

55. $\frac{1}{x} < \frac{1}{x} + 0.7$

57. $\frac{1}{x} < \frac{1}{x} + 0.7$

59. $P(6 \text{ or } 4) = P(6) + P(4) - P(6, 4)$

   $= \frac{1}{36} \cdot \frac{6}{36} - \frac{1}{36}$

   $= \frac{6}{36} - \frac{1}{36}$

   $= \frac{5}{36}$
61. \[ P(\text{Ace or King}) = P(\text{Ace}) + P(\text{King}) - P(\text{Ace, King}) \]
\[ = \frac{4}{52} + \frac{4}{52} - \frac{4}{52} \cdot \frac{4}{52} \]
\[ = 0.14707 \]

63. Answers Vary.

Exercise Set 10.2

1. The possible outcomes are (1, 2) or (3, 2). Thus, the probability is
\[ \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{3} \]

3. The possible outcomes are (1, 3) or (3, 1). Thus, the probability is
\[ \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{3} \]

5. The possible outcomes are (1, 2), (2, 1), or (3, 2). Thus, the probability is
\[ \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{3} \]

7. The possible outcomes are (1, 3), (3, 2), or (3, 1). Thus, the probability is
\[ \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{3} \]

9. \[ \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{3} \]

11. \[ \frac{1}{2} \]

13. \[ \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \]

15. \[ \frac{10}{50} \cdot \frac{21}{49} + \frac{21}{50} \cdot \frac{10}{49} = \frac{11}{50} = 0.38235 \]

17. a. Possible genotype outcomes are \{FF, Ff, ff\}. Since FF cannot occur when Ff and ff are probability 0, and the other two outcomes have a probability of 1/2.

b. The genotypes in Example 2 did not allow for the ff type while in this Exercise the ff genotype is allowed. If a cross test is performed with the sperm allele transmitted from the ff plant and the ovum allele transmitted from the FF plant then we will be able to distinguish between the FF and the Ff genotypes.

19. a. \[ \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16} \]

b. \[ \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

c. \[ \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16} \]

d. \[ \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

21. a. \[ P(T) = \frac{1}{2} \]

b. \[ P(T) = \frac{1}{2} \]

c. \[ P(D) = \frac{1}{10} \]

23. a. \[ P(D) = \frac{1}{2} \]

Exercise Set 10.3

1. \[ \binom{4}{2} = 6 \]

3. \[ \binom{4}{1} = 1 \]

5. \[ \binom{4}{0} = 1 \]

7. \( \binom{4}{1} \) is not possible.

9. \( \binom{4}{3} \) is not possible.

11. \( \binom{6}{1} \) ~ 8008

13. \[ P(X = 2) = \binom{3}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \cdot \frac{4}{9} = \frac{2}{3} \]

15. \[ P(X = 3) = \binom{4}{3} \cdot \left( \frac{1}{3} \right)^2 \cdot \left( \frac{2}{3} \right)^1 \]

17. \[ P(X = 5) = \binom{6}{5} \cdot \left( \frac{1}{2} \right)^5 \cdot \left( \frac{1}{2} \right)^4 \]

25. - 27. Left to the student.
19. 

\[ P(X = 10) = \binom{20}{10}(0.6)^{10}(0.4)^{10} \]

\[ = 184756 \cdot 0.0660466176 \cdot 0.0001018576 \]

\[ = 0.11711 \]

21. \( P(X = 5) = 0 \) since we cannot choose 5 out of 4 items

33. 

\[ P(X = 2) = \binom{5}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 \]

\[ = 10 \cdot \frac{1}{36} \cdot \frac{125}{216} \]

\[ = 0.16075 \]

35. 

\[ P(X \geq 1) = 1 - P(X = 0) \]

\[ = 1 - \binom{10}{0} \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{10} \]

\[ = 1 - 0.16075 = 0.83925 \]

37. 

\[ P(X > 6) = P(X = 7) + P(X = 8) + P(X = 9) \]

\[ = \binom{7}{6} \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right)^1 + \binom{8}{6} \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right)^2 + \binom{9}{6} \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right)^3 \]

\[ = 0.000089336 + 0.000003153 + 0.0000009229 \]

\[ = 0.0000933829 \]

39. 

\[ P(X = 30) = \binom{30}{0} \cdot \left(\frac{1}{2}\right)^{30} \cdot \left(\frac{1}{2}\right)^{30} \]

\[ = 0.018172 \]

41. 

\[ P(X = 2) = \binom{3}{2} \cdot (0.80)^2 \cdot (0.20)^1 \]

\[ = 0.384 \]

43. a.) 

\[ P(X \leq 2) = P(X = 2) + P(X = 1) + P(X = 0) \]

\[ = \binom{12}{2} \cdot (0.346)^2 \cdot (0.654)^{10} + \binom{12}{1} \cdot (0.346)^1 \cdot (0.654)^{11} + \binom{12}{0} \cdot (0.346)^0 \cdot (0.654)^{12} \]

\[ = 0.11310 + 0.03867 + 0.0061226 \]

\[ = 0.15989 \]

b.) 

\[ P(X \geq 9) = P(X = 9) + P(X = 10) + \cdots \]

\[ = \binom{12}{9} \cdot (0.346)^3 \cdot (0.654)^9 + \binom{12}{10} \cdot (0.346)^4 \cdot (0.654)^8 \]
\[
\begin{align*}
\binom{12}{10} \cdot (0.316)^{10} \cdot (0.654)^2 + \\
\binom{12}{11} \cdot (0.316)^{11} \cdot (0.654)^1 + \\
\binom{12}{12} \cdot (0.316)^{12} \cdot (0.654)^0 \\
= 0.004374 + 0.00069416 + \\
0.000066772 + 0.00000294383 \\
= 0.005138 \\
\end{align*}
\]

45.

\[
P(X = CC) = \binom{2}{2} \cdot (0.08)^2 \cdot (0.92)^0 \\
= 0.0064 \\
P(X = CC) = \binom{2}{1} \cdot (0.08)^1 \cdot (0.92)^1 \\
= 0.1472 \\
P(X = CC) = \binom{2}{2} \cdot (0.08)^0 \cdot (0.92)^2 \\
= 0.8464
\]

47. We seek the value of \(2pq\)
\[
p^2 = \frac{1}{256} \Rightarrow p = \frac{1}{16} \\
q = 1 - \frac{1}{16} = \frac{15}{16} \\
\text{Therefore, } 2pq = 2\left(\frac{1}{16}\right)\left(\frac{15}{16}\right) = 0.0392
\]

49. We seek the value of \(2pq\)
\[
p^2 = \frac{1}{360000} \Rightarrow p = \frac{1}{600} \\
q = 1 - \frac{1}{600} = \frac{599}{600} \\
\text{Therefore, } 2pq = 2\left(\frac{1}{600}\right)\left(\frac{599}{600}\right) = 0.033278
\]

51.

\[
P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) \\
= \binom{5}{3} \cdot (0.40)^3 \cdot (0.6)^2 + \binom{5}{4} \cdot (0.40)^4 \cdot (0.6)^1 \\
+ \binom{5}{5} \cdot (0.40)^5 \cdot (0.6)^0 \\
= 0.2304 + 0.0768 + 0.01024 \\
= 0.31734
\]

53. a.) \(X = \{0, 1, 2\}\)

\[
P(X = 0) = \frac{39}{52} \cdot \frac{38}{51} = 0.598824 \\
P(X = 1) = \left(\frac{13}{52}\right) \cdot \left(\frac{39}{51}\right) + \left(\frac{39}{52}\right) \cdot \left(\frac{13}{51}\right) = 0.382353 \\
P(X = 2) = \frac{52}{51} \cdot \frac{51}{50} \cdot \frac{50}{49} = 0.0388235
\]
67. a. $\{SSSS, SSFS, SSFS, SSFF, FSSS, FSFF, SFSS, SFSS, FSFF, FSFF, FFSS, FFFF\}$

b.) 6
c.) 4
d.) $\binom{4}{2} = 6$
e.) $\binom{4}{3} = 4$

69.

\[
P(i, j, k) = \frac{\binom{n}{i,j,k}}{\binom{n}{n}} (p^i)(q^j)(r^k)
\]

\[
P(2,5,3) = \frac{\binom{9}{2,5,3}}{2592}\]

\[
= \frac{\binom{9}{2,5,3}}{2592} = \frac{2720 \cdot 0.02682761 \cdot 0.02658509 \cdot 0.01218053}{0.0796118}
\]

\[
= 0.00796118
\]

71. a. $P(Aa) = 2pq, P(aa) = q^2$

\[
P(A|Aa) = 1/2, P(a|Aa) = 1/2
\]

\[
P(A|aa) = 0, P(a|aa) = 1
\]

b. 

\[
P(A) = p^2 + 2pq + q^2 - 1
\]

\[
= p^2 + q(1-p) + q
\]

\[
= p^2 + q - p^2
\]

c.) $q = 1 - p$

\[
P(a) = p^2 + 2pq + \frac{1}{2} q^2 - 1
\]

\[
= p^2 + (1-q)p + q^2
\]

\[
= q - q^2 + q^2
\]

d.) Left to the student

3. From Exercise 1, $\mu = E(X) = 1.3$

\[
\text{Var}(X) = \sum (x - \mu)^2 P(X = x)
\]

\[
= [0 - 1.3]^2(0.2) + (1 - 1.3)^2(0.3) + (2 - 1.3)^2(0.5)
\]

\[
= 0.338 + 0.027 + 0.245
\]

\[
= 0.61
\]

\[
SD(X) = \sqrt{\text{Var}(X)}
\]

\[
= \sqrt{0.61}
\]

\[
= 0.781025
\]

5. a.) $E = 1 \cdot 0.25 + 1 \cdot 0.25 + 0.5$

b.) $E = np - 2 \cdot 0.25 - 0.5$

7. a.) $E = 1 \cdot 0.5 + 1 \cdot 0.5 + 0.5 = 1.5$

b.) $E = np - 3 \cdot 0.5 - 1.5$

9. $SD = \sqrt{npq} = \sqrt{2 \cdot 0.25 \cdot 0.75} = 0.61237$

11. $SD = \sqrt{npq} = \sqrt{2 \cdot 0.25 \cdot 0.75} = 0.61237$

13. $E(X) = np = 6 \cdot 0.2 = 1.2$

\[
SD(X) = \sqrt{npq} = \sqrt{6 \cdot 0.2 \cdot 0.8} = 0.979796
\]

15. $E(X) = np = 20 \cdot 0.1 = 2$

\[
SD(X) = \sqrt{npq} = \sqrt{20 \cdot 0.1 \cdot 0.9} = 1.34164
\]

17. $E(X) = np = 50 \cdot 0.1 = 5$

\[
SD(X) = \sqrt{npq} = \sqrt{50 \cdot 0.1 \cdot 0.9} = 3.46410
\]

19.

\[
z = \frac{x - E(X)}{SD(X)}
\]

\[
= \frac{20 - 16}{2} = 2
\]

21.

\[
z = \frac{x - E(X)}{SD(X)}
\]

\[
= \frac{13.1 - 13.5}{0.24} = -0.1
\]

\[
= \frac{0.21}{0.24}
\]

\[
= \frac{-5}{3}
\]

23.

\[
z = \frac{x - E(X)}{SD(X)}
\]

\[
= \frac{29.3 - 20.3}{4.5} = 2
\]
25. a) \( E(X) = np = 12(0.73) - 8.76 \)
\( SD(X) = \sqrt{npq} = \sqrt{12(0.73)(0.27)} = 1.53792 \)

b) 
\[
P(X = k) = \binom{n}{k} p^k q^{n-k}
\]
\[
P(X = 8) = \binom{12}{8} (0.73)^8 (0.27)^4
\approx 0.0896469092 \cdot 0.00531441
= 0.21215
\]

27. \( E(X) = 12 \cdot 0.638 = 7.656 \)
\( SD(X) = \sqrt{12(0.638)(0.362)} = 1.66477 \)

29. 
\[
Var(aX) = E(a^2X^2) - [E(aX)]^2
\]
\[
= a^2E(X^2) - E(aX)E(aX)
\]
\[
= a^2\mathbb{E}(X^2) - a^2\mathbb{E}(X)\mathbb{E}(X)
\]
\[
= a^2\mathbb{E}(X^2) - a^2\mathbb{E}^2(X)
\]
\[
= a^2 (\mathbb{E}(X^2) - \mathbb{E}^2(X))
\]
\[
= a^2 Var(X)
\]

Exercise Set 10.5

1. \( f(x) = 2x \geq 0 \text{ on } [0, 1/2] \)

\[ \int_0^{1/2} 2x \, dx = \left[ x^2 \right]_0^{1/2} = 1^2 - 0^2 = 1 \]

3. \( f(x) = \frac{3}{26} x^2 \geq 0 \text{ on } [1, 3] \)

\[ \int_1^3 \frac{3}{26} x^2 \, dx = \left[ \frac{1}{2} x^3 \right]_1^3 = \frac{27}{26} - \frac{1}{26} = 1 \]

5. \( P(1/4 \leq X \leq 3/4) = \int_{1/4}^{3/4} 4x^3 \, dx \)

\[ = \left[ x^4 \right]_{1/4}^{3/4} = \left( \frac{3}{4} \right)^4 - \left( \frac{1}{4} \right)^4 \]

\[ = \frac{81}{256} - \frac{1}{256} = \frac{80}{256} = \frac{5}{16} \]

10. \( P(X \geq \pi/6) = \int_{\pi/6}^{\pi/4} \sec^2 x \, dx \)

\[ = \left[ \tan x \right]_{\pi/6}^{\pi/4} = \tan(\pi/4) - \tan(\pi/6) = 0.12265 \]

12. \( P(0 \leq X \leq 1/2) = \int_0^{1/2} \frac{e^x}{e-1} \, dx \)

\[ = \left[ \frac{e^x}{e-1} \right]_0^{1/2} = \frac{e^{1/2}}{e-1} - \frac{1}{e-1} = \frac{1}{2} \cos(\pi/4) + \frac{1}{2} \cos(0) - 0.14645 \]

15. \( P(X \geq 3.4) = \int_{3.4}^{\infty} e^{-x} \, dx \)

\[ = \lim_{b \to \infty} \left[ -e^{-x} \right]_3^{b} = \lim_{b \to \infty} [-e^{-3} - (-e^{-3.4})] = 0.03337 \]
17. 
\[
P(X \geq 4) = \int_4^\infty \frac{2e}{(x^2 + 1)^2} \, dx
\]
\[
= \lim_{b \to \infty} \left[ -\frac{1}{(x^2 + 1)} \right]_b^4
\]
\[
= \lim_{b \to \infty} \left[ -\frac{1}{(b^2 + 1)} + \frac{1}{(4^2 + 1)} \right]
\]
\[
= 0.458924
\]

19. 
\[
P(X \geq c) = \int_c^\infty \frac{2}{x^2} \, dx
\]
\[
= \lim_{b \to \infty} \int_c^b \frac{2}{x^2} \, dx
\]
\[
= \lim_{b \to \infty} \left[ -\frac{2}{x} \right]_c^b
\]
\[
= \lim_{b \to \infty} \left[ -\frac{1}{b} + \frac{1}{c} \right]
\]
\[
= \frac{1}{c}
\]
Thus,
\[
\frac{1}{c} = 0.05
\]
\[
c = \frac{1}{0.05} = 20
\]

21. 
\[
P(X \geq c) = \int_c^\infty e^{-x} \, dx
\]
\[
= \lim_{b \to \infty} \int_c^b e^{-x} \, dx
\]
\[
= \lim_{b \to \infty} \left[ -e^{-x} \right]_c^b
\]
\[
= \lim_{b \to \infty} \left[ -e^{-b} + e^{-c} \right]
\]
\[
e^{-c}
\]
Thus,
\[
e^{-c} = 0.05
\]
\[
c = -\ln(0.05) = 2.99573
\]

23. 
\[
P(X \geq c) = \int_c^\infty \frac{(3.8/x^3 + 33.6/x^2)}{x} \, dx
\]
\[
= \lim_{b \to \infty} \int_c^b \frac{(3.8/x^3 + 33.6/x^2)}{x} \, dx
\]
\[
= \lim_{b \to \infty} \left[ -1.9 + 8.4/x + 8.4/x^2 \right]_c^b
\]
\[
= \lim_{b \to \infty} \left[ -1.9 + 8.4/x + 8.4/x^2 \right]_c^b
\]
\[
= \frac{1.9}{c^3} + \frac{8.4}{c^2}
\]

Thus,
\[
\frac{1.9}{c^3} + \frac{8.4}{c^2} = 0.05
\]
\[
0.05c^3 + 1.9c^2 - 8.4 \rightarrow 0
\]
\[
c^2 = 6.4807 \text{ not acceptable}
\]
\[
c^3 = 0.4807 \rightarrow c = 2.5457
\]

25. 
\[
f(x) \text{ is positive for all } x \text{ in } [1, 3]
\]
\[
l = \int_1^3 kx \, dx
\]
\[
= \left[ \frac{k}{2} x^2 \right]_1^3
\]
\[
= \frac{9k}{2} - \frac{k}{2}
\]

Thus,
\[
k = \frac{1}{4}
\]

27. 
\[
f(x) \text{ is positive for all } x \text{ in } [-1, 1]
\]
\[
l = \int_{-1}^1 kx^2 \, dx
\]
\[
= \left[ \frac{k}{3} x^3 \right]_{-1}^1
\]
\[
= \frac{k}{3} - \frac{k}{3}
\]

Thus,
\[
k = \frac{3}{2}
\]

29. Since \( f(x) \) is negative on \([0, 1]\), we cannot make a probability density function of \( f(x) \) on \([0, 2]\)

31. a) 
\[
f(x) \text{ is positive for all } x \text{ in } [0, 2]
\]
\[
l = \int_0^2 k(2 - x)^3 \, dx
\]
\[
= k \int_2^0 (8 - 12x + 6x^2 - x^3) \, dx
\]
\[
= k \left[ 8x - 6x^2 + 2x^3 - \frac{1}{4} x^4 \right]_0^2
\]
\[
l = k [16 - 24 + 16 - 1]
\]
\[
l = 4k
\]

Thus,
\[
k \frac{1}{4}
\]
\[
P(X = x) = \frac{(2 - x)^3}{4}
\]
b) 

\[ P(X \leq 3) = \int_0^1 \frac{1}{4} (2-x)^3 \, dx = \frac{1}{4} \int_0^1 (8 - 12x + 6x^2 - x^3) \, dx = \frac{1}{4} \left[ 8x - 12x^2 + 6x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{1}{4} \left[ 8 - 12 + 6 - \frac{1}{4} \right] = \frac{15}{16} \]

33. a) \( f(x) \) is positive for all \( x \) in \([\pi/6, \pi/2]\)

\[ 1 = \int_{\pi/6}^{\pi/2} k \sin(x) \, dx = -k \left[ \sin(x) \right]_{\pi/6}^{\pi/2} \]

\[ 1 = -k \left[ 0 - \frac{\sqrt{3}}{2} \right] = \frac{\sqrt{3}}{2} k \]

Thus, \( k = \frac{2\sqrt{3}}{3} \)

\[ P(X = x) = \frac{2\sqrt{3}}{3} \sin(x) \]

b) 

\[ P(X \leq 3) = \int_{\pi/3}^{\pi/2} \frac{2\sqrt{3}}{3} \sin(x) \, dx = \frac{2\sqrt{3}}{3} \int_{\pi/3}^{\pi/2} \sin(x) \, dx = \frac{2\sqrt{3}}{3} \left[ -\cos(x) \right]_{\pi/3}^{\pi/2} = \frac{2\sqrt{3}}{3} \left[ 0 - (-0.5) \right] = 0.57735 \]

35. a) \( f(x) \) is positive for all \( x \) in \([0, 5]\)

\[ 1 = \int_0^5 k \frac{2x + 3}{x^2 + 3x + 4} \, dx = k \left[ \ln(x^2 + 3x + 4) \right]_0^5 = k \ln(25) - \ln(4) = \ln(11) \]

Thus, \( k = \frac{1}{\ln(11)} \)

\[ P(X = x) = \frac{2x + 3}{\ln(11)(x^2 + 3x + 4)} \]

37. a) \( f(x) \) is positive for all \( x \) in \([-\pi, \pi]\)

\[ 1 = \int_{-\pi}^{\pi} kx \sin(x) \, dx = k \left[ \sin(x) - x \cos(x) \right]_{-\pi}^{\pi} = k \left[ 0 - \pi(1) - (0 - (\pi)(1)) \right] = 2\pi k \]

Thus, \( k = \frac{1}{2\pi} \)

\[ P(X = x) = \frac{x \sin(x)}{2\pi} \]

b) 

\[ P(X \leq 3) = \int_{-\pi}^{\pi/2} (x \sin(x)) \, dx = \int_{-\pi}^{\pi/2} x \sin(x) \, dx - \int_{-\pi}^{\pi/2} \sin(x) \, dx = \frac{1}{2\pi} \left[ x \sin(x) \right]_{-\pi}^{\pi/2} - \frac{1}{2\pi} \left[ \cos(x) \right]_{-\pi}^{\pi/2} = \frac{1}{2\pi} \left[ \left[ \sin(x) - x \cos(x) \right]_{-\pi}^{\pi/2} - 2 \right] = 0.06915 \]

39. \( f(v) = \frac{v^{1/2}}{\sqrt{\pi}} - \frac{1}{2} \)

\[ P(3 \leq X \leq 4) = \int_3^4 \frac{1}{2} \, dv = \left[ \frac{1}{2} v \right]_3^4 = 2 - \frac{3}{2} = \frac{1}{2} \]

41. \( f(v) \) \( \frac{1}{\ln(200) - 100} : \frac{1}{100} \)

\[ P(200 \leq X \leq 350) = \int_{200}^{350} \frac{1}{100} \, dv = \left[ \frac{1}{100} v \right]_{200}^{350} = \frac{350 - 200}{400} = \frac{3}{8} \]

43. \( E(X) = \int_0^3 \frac{2x^2}{9} \, dx = \left[ \frac{2x^3}{27} \right]_0^3 = \frac{27}{27}(27 - 0) - 2 \]
\[ \text{E}(X) = \frac{4}{21} \int_0^3 (x^2 + 2x^3) \, dx \]
\[ = \frac{4}{21} \left[ \frac{x^3}{3} + \frac{2}{5} x^5 \right]_0^3 \]
\[ = \frac{4}{21} \left[ \frac{27}{3} + \frac{486}{5} \right] \]
\[ = 1.652533625 \]

\[ \text{Var}(X) = \frac{4}{21} \int_0^3 (x - 1.625396825)(x + x^3) \, dx \]
\[ = \frac{4}{21} \int_0^3 (3.611914839 + x^3 + 3.25079365x^2 - 3.25079365x^4 + 2.614914839x) \, dx \]

\[ \text{SD}(X) = \sqrt{0.92} \approx 0.96 \]

\[ \text{E}(X) = \frac{1}{2} \int_0^\pi x \sin x \, dx \]
\[ = \frac{1}{2} \left[ x \cos x - \pi \sin x \right]_0^\pi \]
\[ = \frac{1}{2} [0 - (\pi) - 0 + 0] \]
\[ = -\frac{\pi}{2} \]

\[ \text{Var}(X) = \frac{1}{2} \int_0^\pi (x - \frac{\pi}{2})^2 \sin x \, dx \]
\[ = \frac{1}{2} \left[ x^2 \sin x - \pi x \sin x + \frac{\pi^2}{4} \cos x \right]_0^\pi \]
\[ = \frac{1}{2} \left[ 0 - (\pi) + \frac{\pi^2}{4} \right] \]
\[ = \frac{1}{2} \left[ \frac{\pi^2}{4} - \pi \right] \]
\[ = 0.476301 \]

\[ \text{SD}(X) = \sqrt{0.476301} \approx 0.69 \]

\[ \text{E}(X) = \frac{3}{14} \int_0^1 (x + 1)^{1/2} \, dx \]
\[ = \frac{3}{14} \left[ \frac{2}{3} (x + 1)^{3/2} \right]_0^1 \]
\[ = \frac{3}{14} \left[ \frac{2}{3} (2) - \frac{2}{3} (1) \right] \]
\[ = 1.05741028 \]

\[ \text{Var}(X) = \frac{3}{14} \int_0^1 (x - 1.05741028)^2 (x + 1)^{1/2} \, dx \]
\[ = \frac{3}{14} \left[ \frac{2}{3} (x + 1)^{3/2} - \frac{524}{75} (x + 1)^{1/2} \right]_0^1 \]
\[ = \frac{3}{14} \left[ \frac{3}{5} 0.775 - \frac{2}{3} (1) \right] \]
\[ = 0.14322 \]
53. \[ E(X) = \int_0^\infty 3x e^{-3x} \, dx \]
\[ = \lim_{b \to \infty} \int_0^b 3x e^{-3x} \, dx \]
\[ = \lim_{b \to \infty} \left[ -x e^{-3x} - \frac{1}{3} e^{-3x} \right]_0^b \]
\[ = \frac{1}{3} \]

\[ Var(X) = \int_0^\infty 3 e^{-3x} \left( x - \frac{1}{3} \right)^2 \, dx \]
\[ = \lim_{b \to \infty} \int_0^b 3 e^{-3x} \left( x - \frac{1}{3} \right)^2 \, dx \]
\[ = \lim_{b \to \infty} \left[ -e^{-3x} - \frac{x^2}{3} - \frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9} \right]_0^b \]
\[ = \frac{1}{9} \]

\[ SD(X) = \sqrt{\frac{1}{9}} \]
\[ = \frac{1}{3} \]

55. \[ E(X) = \frac{3 \cdot 9}{2} \]
\[ = 6 \]

\[ Var(X) = \frac{(9 - 3)^2}{12} \]
\[ = 3 \]

\[ SD(X) = \sqrt{3} \]
\[ = 1.73205 \]

57. \[ E(X) = \frac{10 + 20}{2} \]
\[ = 15 \]

\[ Var(X) = \frac{(20 - 10)^2}{12} \]
\[ = \frac{100}{12} \approx 8.3333333333 \]

\[ SD(X) = \sqrt{8.3333333333} \]
\[ \approx 2.88675 \]

59. \[ P = \int_0^a \frac{1}{5} \, dx \]
\[ = \left[ \frac{x}{5} \right]_0^a \]
\[ = \frac{3}{5} \]

61. \[ 70 - 60 = 10 \]

63. \[ \int_0^a x^3 \, dx = 1 \]
\[ b = \sqrt{a} = \sqrt{2} \]

65. a) \[ E(X) = \int_a^b x \, f(x) \, dx \]
\[ = \int_a^b \frac{1}{b - a} \, x \, dx \]
\[ = \frac{1}{2(b - a)} \left. x^2 \right|_a^b \]
\[ = \frac{1}{2(b - a)} \left( b^2 - a^2 \right) \]
\[ = \frac{b + a}{2} \]

b) \[ E(X^2) = \int_a^b x^2 \, f(x) \, dx \]
\[ = \int_a^b \frac{x^2}{b - a} \, dx \]
\[ = \left. \frac{x^3}{3(b - a)} \right|_a^b \]
\[ = \frac{b^2}{3(b - a)} - \frac{a^2}{3(b - a)} \]
\[ = \frac{b^2 + ab + a^2}{3(b - a)} \]
\[ = \frac{1}{3} \left( b^2 + ab + a^2 \right) \]
c) \[ V ar(X) = \int_{a}^{b} (x - \mu)^2 f(x) \, dx \]
\[ = \int_{a}^{b} \left( x - \left( \frac{b + a}{2} \right) \right)^2 \frac{1}{b - a} \, dx \]
\[ = \frac{1}{b - a} \int_{a}^{b} \left( \frac{(b - a)^2}{3} \right) \, dx \]
\[ = \frac{1}{b - a} \left( \frac{(b - a)^2}{3} \right) \]
\[ = \frac{(b - a)^2}{12} \]

69. a) Left to the student
b) \[ P(X \leq 2) \]
\[ = \int_{0}^{2} \frac{0.68}{0.89} \left( \frac{x}{0.89} \right)^{-0.32} e^{-\left(\frac{x}{0.89}\right)} \, dx \]
\[ = \frac{0.68}{0.89} \left[ -1.308823528 \, e^{-1.0832.073755} \right]_{0}^{2} \]
\[ = 0.82317 \]

73. 0.018897
75. 0.604701
77. 0.265526
79. 9.488
81. 12.592

Exercise Set 10.6

1. \[ P(X \leq 0) = \frac{2.2^{2}}{0!} \, e^{-2.2} \]
\[ = 0.1108 \]

3. \[ P(X = 1) = \frac{1.3^{1}}{1!} \, e^{-1.3} \]
\[ = 0.3654 \]

5. \[ P(X \geq 7) = \frac{5^{7}}{7!} \, e^{-5} \]
\[ = 0.05854 \]

7. \[ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) \]
\[ = 0.2200 + 0.1494 + 0.04979 \]
\[ = 0.4292 \]

9. \[ P(X \geq 3) = 1 - P(X \leq 2) \]
\[ = 1 - \left( P(X = 0) + P(X = 1) + P(X = 2) \right) \]
\[ = 1 - \left( 0.2200 + 0.1494 + 0.04979 \right) \]
\[ = 0.9656 \]
11. 
\[ P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = 0.08674 + 0.02602 + 0.009625 = 0.1368 \]

13. \[ E(Y) = 2.3 \]
15. \[ SD(X) = \sqrt{4} = 2 \]
17. 
\[ P(0 \leq X \leq 2) = \int_0^2 e^{-x} \, dx = -e^{-x}\bigg|_0^2 = 0.8647 \]

19. 
\[ P(0 \leq X \leq 2) = \int_0^2 3e^{-3x} \, dx = -e^{-3x}\bigg|_0^2 = 0.9975 \]

21. 
\[ P(2 \leq X) = 1 - P(2 > X) = 1 - \int_0^2 2e^{-2x} \, dx = 1 - e^{-2x}\bigg|_0^2 = 1 - 0.9817 = 0.0183 \]

23. 
\[ P(X \leq 0.5) = \int_0^{0.5} 2.5e^{-2.5x} \, dx = -e^{-2.5x}\bigg|_0^{0.5} = 0.7135 \]

53. \[ E(X) \approx 22 \Rightarrow \lambda = \frac{1}{22} \]
\[ P(X \leq 20) = \int_0^{20} \frac{1}{22} e^{-11x} \, dx = -e^{-11x}\bigg|_0^{20} \approx 0.5971 \]

55. a) \[ \lambda \approx 3, \mu = 3(2) = 6 \]
\[ P(Y = 1) = \frac{6!}{1!} e^{-6} = 0.01487 \]

b) \[ \mu \approx 3(9) = 27 \]
\[ P(Y = 5) = \frac{27!}{22!} e^{-22} = 0.0607 \]

c) \[ \mu = 3 \]
\[ P(Y = 0) = \frac{3!}{0!} e^{-3} = 0.0498 \]

57. a) \[ \lambda = 3, \mu = \frac{4}{3} \]
\[ P(Y \geq 1) = 1 - P(Y = 0) = 1 - 0.7364 = 0.2636 \]

b) \[ \mu = \frac{2}{3} \]
\[ P(Y = 0) = \frac{\left(\frac{2}{3}\right)!}{0!} e^{-2/3} = 0.5134 \]

c) \[ \mu = \frac{1}{3} \]
\[ P(Y = 0) = \frac{\left(\frac{1}{3}\right)!}{0!} e^{-1/3} = 0.7165 \]

d) \[ \mu = \frac{1}{3} \]
\[ P(Y = 1) = \frac{\left(\frac{1}{3}\right)!}{1!} e^{-1/3} = 0.2338 \]

e) \[ \mu = \frac{1}{3} \]
\[ P(Y \geq 1) = 1 - P(Y < 2) = 1 - (P(Y = 0) + P(Y = 1)) = 1 - (0.7165 + 0.2338) = 0.0447 \]
59. a) \[ E(X) = \frac{6}{\sqrt{6}} = 2.4495 \]

b) 
\[
P(Y = 0) = \frac{1^0}{0!} e^{-1} = 0.00248
\]

61. First, note that \( f(x) > 0 \) for all \( x \) values and positive \( \lambda > 0 \). Second,
\[
\int_0^\infty f(x) \, dx = \int_0^\infty \lambda e^{-\lambda x} \, dx
\]
\[
= -e^{-\lambda x} \bigg|_0^\infty
\]
\[
= 0 - (-1)
\]
\[
= 1
\]
Therefore the function \( f(x) = \lambda e^{-\lambda x} \) is a probability density function.

63.
\[
\int_0^\infty x \, dx = \frac{1}{2}
\]
\[
\frac{x^2}{2} \bigg|_0^\infty = \frac{1}{2}
\]
\[
m = \frac{1}{2}
\]
\[
m^2 = \frac{1}{2}
\]
\[
m^3 = \frac{1}{2}
\]
\[
m = \sqrt{2}
\]

65. \[ E(X) = \frac{3}{2} \]

\[
\int_0^\infty \frac{1}{3} \, dx = \frac{1}{2}
\]
\[
\frac{x^3}{3} \bigg|_0^\infty = \frac{1}{2}
\]
\[
m = \frac{1}{2}
\]
\[
m^2 = \frac{1}{2}
\]
\[
m = \frac{1}{2}
\]

The mean and median are equal.

67. a)
\[
\int_0^\infty e^{-x} \, dx = \frac{1}{2}
\]
\[
-e^{-x} \bigg|_0^\infty = \frac{1}{2}
\]
\[
-e^{-m} = \frac{1}{2}
\]
\[
e^{-m} = \frac{1}{2}
\]
\[
m = \ln \frac{1}{2} = -\ln 2
\]

b) The median is larger than the mean \( E(X) = 1 \). This makes sense since at \( m = \ln 2 \), for exponential given, the value of \( f(x) \) is 0.5.

69. Left to the student.
71. Binomial; \[ P(X = 1) = 0.168284 \]
    Poisson; \[ P(X = 1) = 0.168031 \]
73. Binomial; \[ P(X = 2) = 0.256561 \]
    Poisson; \[ P(X = 2) = 0.256516 \]
75. Binomial; \[ P(X = 2) = 0.183949 \]
    Poisson; \[ P(X = 2) = 0.183940 \]
13. a) \( \mu = 2 \)
   b) \( \sigma = 5 \)
   c) \( Z = \frac{4 - 2}{5} = 0.4 \)
       \[ P(Z \geq 0.4) = 1 - P(Z < 0.4) = 1 - 0.6554 = 0.3446 \]
   d) \( Z = \frac{3 - 2}{5} = 0.2 \)
       \[ P(Z \leq 0.3) = 0.5793 \]
   e) \( Z = \frac{-8 - 2}{5} = -2, \quad Z = \frac{7 - 2}{5} = 1 \)
       \[ P(-2 \leq Z \leq 1) = P(Z = -2) + P(Z = 1) = 0.4772 + 0.3413 = 0.8185 \]

15. a) \( Z = \frac{-0.2 - 0}{0.1} = -2 \)
       \[ P(Z \geq -2) = 1 - P(Z < -2) = 1 - 0.0228 = 0.9972 \]
   b) \( Z = \frac{-0.05 - 0}{0.1} = -0.5 \)
       \[ P(Z \leq -0.5) = 0.5085 \]
   c) \( Z = \frac{-0.08 - 0}{0.1} = -0.8 \)
       \[ Z = \frac{0.09 - 0}{0.1} = 0.9 \]
       \[ P(-0.8 \leq Z \leq 0.9) = P(Z = -0.8) + P(Z = 0.9) = 0.2881 + 0.3159 = 0.6041 \]

17. - 31. Left to the student (see answers to 1. - 15.)

33. \( \mu = 100(0.2) = 20 \)
   \[ \sigma = \sqrt{100(0.2)(0.8)} = 1 \]
   \[ Z = \frac{18 - 20}{4} = -0.5 \]
   \[ P(Z \geq -0.5) = 1 - P(Z < -0.5) = 1 - 0.3085 = 0.6915 \]

35. \( \mu = 64(0.5) = 32 \)
   \[ \sigma = \sqrt{64(0.5)(0.5)} = 4 \]
   \[ Z = \frac{30 - 32}{4} = -0.5 \]
   \[ P(Z \leq -0.5) = 0.3085 \]

37. \( \mu = 1000(0.3) = 300 \)
   \[ \sigma = \sqrt{1000(0.3)(0.7)} = 14.4914 \]
   \[ Z = \frac{280 - 300}{14.4914} = -1.38 \]
   \[ Z = \frac{300 - 300}{14.4914} = 0 \]
   \[ P(-1.38 \leq Z \leq 0) = P(Z < -1.38) + P(Z = 0) + 0.4162 + 0. = 0.4162 \]

39. a) \( Z = \frac{5 - 4}{1} = 1 \)
   \[ P(Z \geq 1) = 1 - P(Z < 1) = 1 - 0.6413 = 0.3587 \]
   b) \( Z = \frac{2 - 4}{1} = -2 \)
   \[ P(Z \geq 2) = 1 - P(Z < -2) = 1 - 0.0228 = 0.9772 \]
   c) \( Z = \frac{4}{1} = 3 \)
   \[ P(Z \leq 3) = 0.9913 \]

41. \( \mu = (300)(0.25) = 75 \)
   \[ \sigma = \sqrt{300(0.75)(0.25)} = 7.5 \]
   \[ Z = \frac{80 - 75}{7.5} = 0.67 \]
   \[ P(Z \geq 0.67) = 1 - P(Z < 0.67) = 1 - 0.7480 = 0.2514 \]

43. \( \mu = (70)(0.25) = 17.5 \)
   \[ \sigma = \sqrt{70(0.25)(0.75)} = 3.6228 \]
   \[ Z = \frac{20 - 17.5}{3.6228} = 0.69 \]
   \[ P(Z < 0.69) = 0.7549 \]

45. \( p = \frac{238351}{362740} = 0.6576 \)
   \( \mu = 120(0.6576) = 78.91 \)
   \[ \sigma = \sqrt{120(0.6576)(0.3424)} = 5.20 \]
   \[ Z = \frac{80 - 78.91}{5.20} = 0.21 \]
47. a) \( \mu = 1000 \) (9/19) = 473.7

\[
\sigma = \sqrt{\frac{100(9/19)}{10/19}} = 15.79
\]

\[
Z = \frac{501 - 473.7}{15.79} = 1.73
\]

\[
P(Z \geq 1.73) = 1 - P(Z < 1.73) = 1 - 0.9582 = 0.0418
\]

b) The probability in part a was larger. This makes sense since the probability of winning is less than the probability of losing.

49. a) 35% \( \approx \) \( X = -0.38 \)

\[
X' = 507
\]

b) 60% \( \approx \) \( X = 0.25 \)

\[
X' = 507
\]

c) 92% \( \approx \) \( X = 1.11 \)

\[
X' = 507
\]

51. Significance Level

\[
1 - P(X \geq 0.93) = 1 - 0.8238 = 0.1762
\]

53. Significance Level

\[
1 - P(X \geq 2.33) = 1 - 0.9991 = 0.0009
\]

55. Significance Level

\[
P(X \leq -1.24) = 0.1055
\]

57. Critical Value is 2.33

59. Critical Value is -1.64

61. Critical Value is -2.58

63. \[
Z_t = \frac{9 - 19}{SD} = \frac{-10}{SD}
\]

\[
P(Z_t \leq Z \leq 0) = 0.4922
\]

\[
Z_2 = -2.42
\]

\[
SD = \frac{-1}{-2.42} = 0.413231405
\]

65. \[
Z_1 = \frac{-6 - (-3)}{3} = -3
\]

\[
Z_2 = \frac{0 - (-3)}{3} = 1
\]

67.

\[
P(Z \geq -1) = 0.40
\]

\[
P(Z < -1) = 1 - 0.40 = 0.59
\]

\[
Z = \frac{0.23}{2} = -1 - E
\]

\[
E = -1 - 2(0.23) = -1.46
\]

69. \[
Z_1 = \frac{2.8 - E}{SD} = \frac{10.3 - E}{SD}
\]

\[
P(Z \geq Z_1) = 1 - P(Z < Z_1)
\]

\[
Z_1 = 0
\]

\[
P(Z \leq Z_2) = 0.8844
\]

\[
Z_2 = 1.25
\]

Thus,

\[
\frac{2.8 - E}{SD} = 0
\]

\[
E = 2.8
\]

\[
\frac{10.3 - E}{SD} = 1.25
\]

\[
1.25SD + 2.8 = 10.3
\]

\[
SD = 6
\]
71. \[ Z_1 = \frac{-3.1 - E}{SD}, \quad Z_2 = \frac{1.4 - E}{SD} \]

\[
P(Z \geq Z_1) = 1 - P(Z < Z_1)
\]

\[
= 0.1539
\]

\[
Z_1 = 0.4
\]

\[
P(Z \leq Z_2) = 0.8461
\]

\[
Z_2 = 1
\]

Thus,

\[
\frac{-3.1 - E}{SD} = 0.4
\]

\[
\frac{0.4SD + E}{SD} = -3.1
\]

\[
\frac{1.4 - E}{SD} = 1
\]

\[
\frac{SD + E}{SD} = 1.4
\]

Solving the system

\[
E = -0.85
\]

\[
SD = 2.25
\]

73. The reason is the symmetric nature of the Normal(\( \mu, \sigma \)) distribution about \( \mu \)

75. a) \( \mu = 2000, \sigma = 40 \)

\[
P(1949.5 \leq Y \leq 2000.5) = 0.40160
\]

b) \( Z_1 = \frac{1950 - 2000}{40} = -1.25 \)

\[
Z_2 = \frac{2000 - 2000}{40} = 0
\]

\[
P(-1.25 \leq Z \leq 0) = 0.3044
\]

e) 0.00720

77. As \( n \) increases the difference between probability with the continuity correction and the probability without the continuity correction becomes smaller.

79. \( n = 2 \rightarrow 0.341529 \)

\( n = 4 \rightarrow 0.341355 \)

\( n = 6 \rightarrow 0.341347 \)

\( n = 8 \rightarrow 0.341345 \)