(1) (10 points) No partial credit for this problem. Let $\mathbf{a} = \langle 1, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 3, -1 \rangle$ and $\mathbf{c} = 2\mathbf{i} + 4\mathbf{j}$. Find the following

(a) $3\mathbf{a} - \mathbf{b} = \langle 3, -6, 9 \rangle - \langle 2, 3, -1 \rangle = \langle 1, -9, 10 \rangle$

(b) $|\mathbf{a} + \mathbf{c}| = |\langle 3, 2, 3 \rangle| = \sqrt{9 + 4 + 9} = \sqrt{22}$

(c) $\mathbf{a} \cdot \mathbf{b} = 2 - 6 - 3 = -7$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & 3 & -1 \end{vmatrix} = \langle -7, 7, 7 \rangle$

(e) A unit vector in the direction of $\mathbf{a}$

$|\mathbf{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$ so $\mathbf{u} = \left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$

(2) (15 pts) Let $A = (0, 2, -1)$, $B = (-1, 0, 1)$ and $C = (1, -1, 0)$

(a) Find the equation of the plane that contains these three points.

$\overrightarrow{AB} = \langle -1, -2, 2 \rangle$, $\overrightarrow{AC} = \langle 1, -3, 1 \rangle$ so

$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 2 \\ 1 & -3 & 1 \end{vmatrix} = \langle 4, 3, 5 \rangle$

the equation of the plane is $4x + 3(y - 2) + 5(z + 1) = 0$

(b) Find the area of the triangle ABC.

area of the triangle $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{16 + 9 + 25} \frac{1}{2} \sqrt{50}$

(c) Find the angle at the vertex C in radians.

$\overrightarrow{CB} = \langle 2, -1, 1 \rangle$ and $\overrightarrow{CA} = \langle 1, -3, 1 \rangle$ so

$\cos \theta = \frac{\langle 2, -1, 1 \rangle \cdot \langle 1, -3, 1 \rangle}{\sqrt{4 + 1 + 1} \cdot \sqrt{1 + 9 + 1}} = \frac{6}{\sqrt{66}}$ and $\theta = 1.05$ rad
(3) (15pts)

(a) Is the line through the points (-4,1,3) and (-6,3,5) parallel to the plane $2x - 3y + 5z = 1$?

Explain your answer!

the direction vector of the line $y = (-2, 2, 2)$ and the normal vector of the plane is $n = (2, -3, 5)$. Since their dot product is $-4 - 6 + 10 = 0$ they are orthogonal which means that the line and the plane are parallel.

(b) Find the distance between the plane and the line if they are parallel or find the intersection of the plane and the line if they are not parallel.

the distance between the plane and the line using the point (-4,1,3) is given by

$$\frac{|2(-4) - 3(1) + 5(3) - 1|}{\sqrt{4 + 9 + 25}} = \frac{3}{\sqrt{38}}$$

(4) (10 pts) Find the distance between the plane $2x - y + z = 5$ and the point (3,-2,5)

the distance between the plane and the point is given by

$$\frac{|2(3) - (-2) + (5) - 5|}{\sqrt{4 + 1 + 1}} = \frac{8}{\sqrt{6}}$$

(5) (5 pts) Find $k$ such that the point $P = (5, 3, k)$ is on the line $x = 5, y = 4 - t$ and $z = 2t$.

since $4 - t = 3$ we know that $t = 1$. Therefore $k = 2t = 2$

(6) (12 pts)

(a) does not match
(b) 3
(c) 6
(d) 2
(e) 4
(f) does not match
(g) 1
(h) 5
(7) (10 pts) A 100-meter dash is run on a track in the direction of the vector \( \mathbf{a} = \langle 2, 6 \rangle \). The wind velocity is \( \mathbf{v} = \langle 5, 1 \rangle \). The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/h. Will the race results be disqualified due to illegal wind? Find the speed of the wind in the direction of the track.

The speed of the wind in the direction of the track is given by the scalar projection of the wind vector in the direction of the track.

\[
\text{comp}_{\mathbf{a}} \mathbf{v} = \frac{(5)(2) + (1)(6)}{\sqrt{4 + 36}} = \frac{16}{\sqrt{40}} = 2.53 < 5
\]

so the results will be legal.

(8) (15 pts) A particle passes the point \( P = (5, 4, -2) \) at a time \( t = 3 \) is moving with velocity \( \mathbf{v}(t) = \langle e^{2t}, \cos(t), 3t \rangle \).

(a) Find the acceleration at \( t = 3 \).

\[
\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2e^{2t}, -\sin(t), 3 \rangle
\]

\[
\mathbf{a}(3) = \langle 2e^{6}, -\sin(3), 3 \rangle
\]

(b) Find the vector function for the particle’s position.

\[
\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \langle \frac{1}{2}e^{2t}, \sin(t), \frac{3}{2}t^2 \rangle + \mathbf{c}
\]

using the initial condition given we get

\[
\mathbf{r}(3) = \langle 5, 4, -2 \rangle = \langle \frac{1}{2}e^{6}, \sin(3), \frac{3}{2}3^2 \rangle + \mathbf{c}
\]

\[
\mathbf{c} = \langle \frac{1}{2}e^{6} - 5, \sin(3) - 4, \frac{31}{2} \rangle
\]

therefore,

\[
\mathbf{r}(t) = \langle \frac{1}{2}e^{2t} + \frac{1}{2}e^{6} - 5, \sin(t) + \sin(3) - 4, \frac{3}{2}t^2 + \frac{31}{2} \rangle
\]

(c) Find the particle’s speed at \( t = 0 \).

\[
\text{speed} = |\mathbf{v}(0)| = |(1, 1, 0)| = \sqrt{2}
\]

(9) (10 pts) Find the parametric equation of the line tangent to the curve \( \mathbf{r}(t) = \langle t^2, t^4, t^5 \rangle \) at the point \( P = (1, 1, 1) \)

\[
\mathbf{r}'(t) = \langle 2t, 4t^3, 5t^4 \rangle
\]

\[
\mathbf{r}'(1) = \langle 2, 4, 5 \rangle \text{ which gives the direction vector of the line.}
\]

thus,

\[
x(t) = 1 + 2t \quad y(t) = 1 + 4t \quad z(t) = 1 + 5t
\]