Notes for 6.4 – Math 211

Let \( z = f(x, y) \) be a continuous, finite-valued function for all points \((x, y)\) within a closed and bounded region \( R \) in the \( xy \)-plane. The Extreme Value Theorem states that the function \( z = f(x, y) \) is then guaranteed to have both an absolute maximum point and an absolute minimum point when restricted to the region \( R \). The terms closed and bounded have distinct mathematical meanings, but for our purposes it suffices to state that “closed” means that the boundaries of the region are included within the region \( R \), and “bounded” means that the region is finite in area and does not extend to infinity in any direction.

The typical problem will be given with a series of constraints, which are usually inequalities. For example, the set of constraints

\[
x \geq 0, \ y \geq 0, \ x + 2y \leq 12
\]

form a closed and bounded region in the \( xy \)-plane. The set of constraints

\[
x \geq 0, \ y \geq 0, \ 2x + y \geq 6, \ x + y \geq 4
\]

form an unbounded region. You should sketch and shade the solution sets to both to gain a visual understanding.

In each problem, you will

- Sketch and shade the region formed by the constraints (called the region of feasibility).
- Determine all vertex points where two constraint curves intersect (if any).
- Analyze each vertex point, each edge (boundary) and the inside of the region for potential extrema points.

**Example:** Let \( f(x, y) = x^2 + y^2 + xy - 5x - 4y + 1 \) be constrained by the region \( R \) defined by the inequalities \( 0 \leq x \leq 4, \ 0 \leq y \leq 4 \). Find the minimum and maximum values of \( f(x, y) \) over \( R \).

**Solution:** Sketch the region and you will see that it is a square, with vertex points \((0,0), (0,4), (4,0)\) and \((4,4)\). By direct evaluation we have the following functional values:

\[
f(0,0) = 1, \ \ f(4,0) = -3, \ \ f(4,4) = 13, \ \ f(0,4) = 1
\]

Now we look at each edge systematically. For the rightmost edge, which is defined as \( x = 4 \) for \( 0 \leq y \leq 4 \), we evaluate

\[
f(4,y) = 16 + y^2 + 4y - 20 - 4y + 1 = y^2 - 3
\]

The derivative of \( f(4, y) = y^2 - 3 \) is \( f_y = 2y \). Set equal to zero, we get that \( y = 0 \). We make sure that this value of \( y \) falls within the stated bounds for \( y \). In this case it’s actually the corner point we’ve already found, so we are done with this edge.

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The top edge is \( y = 4 \) for \( 0 \leq x \leq 4 \). Replace \( y \) with 4, and we get, after simplification:

\[
f(x,4) = x^2 - x + 1
\]

The derivative is \( f_x = 2x - 1 \), which gives a critical value of \( x = \frac{1}{2} \). This falls within the required bounds for \( x \), hence it is acceptable. (If it fell outside the bounds, we’d simply ignore it.) We now have another critical value, the point \( \left( \frac{1}{2}, 4, \frac{3}{2} \right) \).

The left edge is \( x = 0 \) with \( 0 \leq y \leq 4 \). We get \( f(0,y) = y^2 - 4y + 1 \). Its critical value is \( y = 2 \), which is acceptable. Hence we have a critical point of \( (0,2,-3) \).

The bottom edge is \( y = 0 \) for \( 0 \leq x \leq 4 \). We get \( f(x,0) = x^2 - 5x + 1 \) with a critical value of \( x = \frac{5}{2} \) and a resulting critical point of \( \left( \frac{5}{2}, 0, -\frac{9}{4} \right) \).

Lastly, we need to check the inside of the region for critical points. We do this by finding the first partials of \( f(x,y) \), setting them to zero and solving the system:

\[
\begin{align*}
 f_x &= 2x + y - 5 \\
 f_y &= 2y + x - 4
\end{align*}
\]

\[
\begin{align*}
 2x + y - 5 &= 0 \\
 x + 2y - 4 &= 0
\end{align*}
\]

\[x = 2, y = 1\]

These values are both within the constraint region so we use them. The corresponding point is \( (2,1,-6) \).

We now have a total of eight critical points. If we plot them on a graph we get the following:

Just by reading off the graph we see the maximum value of the function is 13 at the upper right corner while the minimum value is -6 at the point \( (2,1,-6) \).
The graph of $f(x, y) = x^2 + y^2 + xy - 5x - 4y + 1$ restricted to $0 \leq x \leq 4$, $0 \leq y \leq 4$ and with the critical points included.

Your challenge: Find the minimum and maximum value on the same function for the following constraints: $x \geq 0$, $y \geq 0$, $2x + y \leq 12$, $x \leq 3$. 

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