This take-home final exam is due at 11:50 am on Friday, May 3, 2002, in my office, PSA 723. Your notes, my notes, your homework, my homework, Jones’ book [1], and Spivak’s book [2] are the only allowable resources. Consulting other books, other notes, or other people is not allowed. Please remember that, in accordance with the University Student Academic Integrity Policy, the highest standards of academic integrity are expected of all students at all times.

Problem 1 (30 points). Prove that there exists an open interval $I$ in $\mathbb{R}$ containing $-1$ and unique functions $u$ and $v$ from $I$ into $\mathbb{R}$ such that

$$xe^{u(x)} + u(x)e^{v(x)} = xe^{v(x)} + v(x)e^{u(x)} = 0$$

for all $x \in I$ and $u(-1) = v(-1) = 1$.

Problem 2 (30 points). Prove that there is no Lebesgue measurable set $A \subseteq \mathbb{R}$ such that

$$\mu(A \setminus [a, b]) = b - a$$

for all $a < b$ in $\mathbb{R}$.

Problem 3 (30 points). Suppose $f: [0, 1] \to \mathbb{R}$ is a function, and for every $\epsilon > 0$ there exists a Lebesgue measurable set $A \subseteq [0, 1]$ such that:

(i) the restriction of $f$ to $A$ is continuous, and

(ii) $\lambda([0, 1] \setminus A) < \epsilon$.

Prove that $f$ is Lebesgue measurable.

Problem 4 (30 points). Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^{-1/2}$ if $0 < x < 1$, and $f(x) = 0$ otherwise. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$, and for each $x \in \mathbb{R}$ let

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{q_n} f(x - q_n).$$

Prove that $g \in L^1(\mathbb{R})$, and evaluate $\int_{\mathbb{R}} g \, d\lambda$.

Problem 5 (30 points). Suppose $f \in L^1(\mathbb{R}^2)$. Carefully justify the formula

$$\int_{\mathbb{R}^2} f \, d\lambda = \int_{[0,2\pi]} \int_{[0,\infty)} f(r \cos \theta, r \sin \theta) r \, d\lambda(r) \, d\lambda(\theta).$$

References


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