EXAM 2

MAT 494 A • SPRING 2002

This take-home exam is due in class at the start of class, 10:40am on Tuesday, April 9, 2002. Your notes, my notes, your homework, my homework, and Jones’ book ([Jon00]) are the only allowable resources. Other books, other notes, and other people are not allowed. Please remember that, in accordance with the University Student Academic Integrity Policy, the highest standards of academic integrity are expected of all students at all times.

Problem 1 (25 points). Prove that every open subset $G$ of $\mathbb{R}^n$ is a countable union of nonoverlapping cubes. Hint: Tile $\mathbb{R}^n$ with unit cubes, and keep only those which lie inside $G$; now tile $\mathbb{R}^n$ with half-unit cubes, etc. (This is Problem 9 on page 35 of [Jon00].)

Problem 2 (25 points). For every set $A \subseteq \mathbb{R}^n$, prove that
\[
\lambda^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(I_k) \right\} \quad \text{each } I_k \text{ is a special rectangle and } A \subseteq \bigcup_{k=1}^{\infty} I_k \).
\]

Hint: For $\geq$, let $G$ be an open set containing $A$ and use Problem 1 above. (This is Problem 27 on page 56 of [Jon00].)

Problem 3 (25 points). Prove that for any Lebesgue measurable set $A \subseteq \mathbb{R}^n$ and any $c \in \mathbb{R}$ such that $0 < c < \lambda(A)$, there exists a Lebesgue measurable set $B \subseteq A$ such that $\lambda(B) = c$.

Problem 4 (25 points). Let $(f_n)$ be a sequence of Lebesgue measurable $\mathbb{R}$-valued functions on $\mathbb{R}^n$. Prove that the set $\{ x \mid \lim_n f_n(x) \text{ exists } \}$ is Lebesgue measurable.

References


S. Kaliszewski, Department of Mathematics and Statistics, Arizona State University

Date: April 2, 2002.