EXAM 1

MAT 494 A · Spring 2001

This take-home exam is due at 5:00 pm on Thursday, March 8, 2001, in my office, PSA 723. Your notes, my notes, your homework, my homework, Spivak’s book [Sp65], and Jones’ book [Jon00] are the only allowable resources. Consulting other books, other notes, or other people is not allowed. Please remember that, in accordance with the University Student Academic Integrity Policy, the highest standards of academic integrity are expected of all students at all times.

Problem 1 (25 points). A function \( \Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^\ell \) is bilinear if for each \( x_0 \in \mathbb{R}^n \) and \( y_0 \in \mathbb{R}^m \), the functions \( x \mapsto \Phi(x, y_0) \) and \( y \mapsto \Phi(x_0, y) \) are linear.

(a) Prove that every bilinear function \( \Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^\ell \) is differentiable on \( \mathbb{R}^n \times \mathbb{R}^m \), and find a formula for \( \Phi'(x, y) \). (Here we have \( \mathbb{R}^n \times \mathbb{R}^m \) with \( \mathbb{R}^{n+m} \) in the usual way.)

(b) Use the result of part (a) to prove the Product Rule: if \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^m \rightarrow \mathbb{R} \) are differentiable, then \( fg \) (pointwise product) is differentiable, with

\[
(fg)'(x)(h) = f'(x)(g(x)) + f(x)g'(x)(h)
\]

for each \( x, h \in \mathbb{R}^n \).

(See Problem 2-12 in [Sp65].)

Problem 2 (25 points). Let \( a_{ij} : \mathbb{R} \rightarrow \mathbb{R} \) and \( b_t : \mathbb{R} \rightarrow \mathbb{R} \) be continuously differentiable for each \( 1 \leq i, j \leq n \), and let \( A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) and \( b : \mathbb{R} \rightarrow \mathbb{R}^n \) be the functions defined by \( A(t) = (a_{ij}(t)) \) and \( b(t) = (b_t) \) for each \( t \in \mathbb{R} \). Use the Implicit Function Theorem to prove that if \( A(0) \) is invertible, then there exists an open neighborhood \( U \) of \( 0 \) in \( \mathbb{R} \) and a differentiable function \( x : U \rightarrow \mathbb{R}^n \) such that

\[
A(t)x(t) = b(t) \quad \text{for all } t \in U.
\]

Also find an expression for \( x'(t) \). (See Problems 2-15 and 2-40 in [Sp65].)

Problem 3 (25 points). Let \( A_n \) be Lebesgue measurable for each \( n \in \mathbb{N} \).

(a) Prove that if \( \lambda(A_n \cap A_k) = 0 \) for each \( n \neq k \), then

\[
\lambda\left( \bigcup_{n=1}^\infty A_n \right) = \sum_{n=1}^\infty \lambda(A_n).
\]

(b) Also prove, conversely, that if \( \lambda\left( \bigcup_{n=1}^\infty A_n \right) < \infty \) and (1) holds, then \( \lambda(A_n \cap A_k) = 0 \) for each \( n \neq k \).

(c) Finally, give an example to show that the assumption \( \lambda\left( \bigcup_{n=1}^\infty A_n \right) < \infty \) in (b) is necessary.

(See Problems 43 and 44 in [Jon00, Section 2.B].)

Problem 4 (25 points). Prove that there exist disjoint sets \( A \) and \( B \) in \( \mathbb{R}^n \) such that

\[
\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)
\]

and

\[
\lambda^*(A \cup B) > \lambda^*(A) + \lambda^*(B).
\]

Also prove, by contrast, that if \( A \) and \( B \) are subsets of \( \mathbb{R}^n \) and there exists a Lebesgue measurable set \( C \) such that

\[
A \subseteq C \quad \text{and} \quad B \subseteq C^c,
\]

then \( \lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \). (See Problems 1 and 2 in [Jon00, Section 4.B].)

References


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