1. (9) Complete the following definitions.

(a) A subset $S$ of $\mathbb{R}^n$ is open if . . .

(b) If $D \subseteq \mathbb{R}^n$, $p_0$ is an accumulation point of $D$, $f : D \to \mathbb{R}^m$ is a function, and $L \in \mathbb{R}^m$, we say $\lim_{p \to p_0, p \in D} f(p) = L$ if . . .

(c) Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is uniformly continuous on $D$ if . . .
2. (9) Complete the following to give meaningful true statements (not the definitions).

(a) A set $S \subseteq \mathbb{R}^n$ is compact if and only if . . .

(b) If $D \subseteq \mathbb{R}^n$ is connected and $f : D \rightarrow \mathbb{R}^1$ is continuous, then . . .

(c) Suppose $p_n = (a_n, b_n, \ldots, z_n) \in \mathbb{R}^{26}$ for each $n \in \mathbb{N}$. The sequence $(p_n)$ converges if and only if . . .
3. (12) In each part, give an example or briefly explain why there is no such example.

(a) A set $S \subseteq \mathbb{R}^2$ which is closed but not compact.

(b) A bounded sequence in $\mathbb{R}^2$ which has no convergent subsequence.

(c) Two distinct sequences $(p_n)$ and $(q_n)$ in $\mathbb{R}^2$ which do not converge, but such that $|p_n - q_n| \to 0$. 
4. (10) Suppose $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ is uniformly continuous, and $(p_k) \subseteq S$ is a Cauchy sequence. Prove that $(f(p_k))$ is a Cauchy sequence. (Note that $(p_k)$ need not converge in $S$.)
5. (15) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, and for each $c \in \mathbb{R}$, let
\[ A_c = \{ p \in \mathbb{R}^2 \mid |f(p)| \geq c \}. \]

(a) Prove that $|f|$ is continuous.

(b) Prove that each $A_c$ is closed.

(c) Prove that if each $A_c$ is compact, then $f(n, n) \to (0, 0)$. 
Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} \frac{x^2 y + xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that $f$ is continuous at $(0, 0)$. 
MAT 372 A
EXAM 1
February 23, 2006

Instructions. You have until 10:30am to complete the 6 problems on this exam. Be sure to watch the clock and pace yourself accordingly. No notes, books, or calculators are allowed. Be sure to understand each problem carefully before starting work on it. Unless otherwise indicated, you may use any results from class, from the text (through Section 3.3), and from any homework you have turned in. When you use such results, be sure to make some acknowledgement. For example, “since every interval contains a rational number”, or “by the Squeeze Theorem...”

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