FEEDBACK STABILIZATION, HOMOGENEITY, AND NONLINEAR DYNAMICS ON SPHERES AND SO(3)

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ABSTRACT
This article further explores the employment of a generalized notion of homogeneity with respect to dilation groups in conjunction with other symmetries for the construction of asymptotically stabilizing feedback laws for highly nonlinear control systems. We further show how to explicitly compute functions and vector fields that are homogeneous w.r.t. generalized dilation groups; in particular of dilation groups whose associated orbit structure is a focus rather than a node.

Key words: Nonlinear control, feedback stabilization, homogeneity, dilation groups.

1. INTRODUCTION
This article is motivated by the problem of finding algorithms for the construction of asymptotically stabilizing feedback laws for nonlinear control systems of the form
\[ \dot{x} = f(x) + ug(x) \]  
where \( x \in \mathbb{R}^n \) and \( f \) and \( g \) are sufficiently smooth vector fields.

In the case that the vector fields are linear and constant, respectively, a rather complete theory is available. Recently, also many different approaches for nonlinear systems have been proposed. These reach from strategies that try to generalize linear concepts, such as minimum-phase and zero dynamics (see e.g. [3]), over more algebraic investigations (see e.g. [2]), to designs employing optimal control designs such as in the nonlinear regulator approach (compare e.g. [5]). Here we shall further follow the ideas first explored by Dayawansa (unpublished) and concurrently in [9].

The basic idea is to look for infinitesimal symmetries \( \nu \) (in the sense of Sophus Lie, compare [10]) of the closed-loop vector field \( F = f + ug \) that are such that \( x = 0 \) is an asymptotically stable equilibrium of \( \dot{x} = -\nu(x) \). Then one investigates the dynamics on the space of nontrivial trajectories of \( \nu \) (which is homeomorphic to a sphere) that is induced by \( F \), and aims at inferring the asymptotic stability of \( F \) from the dynamics on the cone over the limit-set of the induced dynamics.

In practical applications one typical tries to first find an infinitesimal symmetry of the drift vector field \( f \), and then chooses the feedback law \( u = u(x) \) such that this is also a symmetry of \( ug \) and thus of \( F \). Of course, more general cases with full nonlinear dependence on the control \( u \) can be handled this way, too;

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however, currently no algorithms for the construction of suitable \( \nu \) seem to be available.

In the most simple case, one has \( \nu(x) = \sum x_i \frac{\partial}{\partial x_i} \) in which case the symmetry of \( F \) is just homogeneity in the classical sense, with linearity of \( F \) the basic example.

Results about stability in the mth-approximation which in some sense generalize Lyapunov's first method are available for the basic \( \nu(x) = \sum \tau_i x_i \frac{\partial}{\partial x_i} \) in [4], and for general \( \nu \) in [6], compare also [11].

In this article, after a brief review of basic definitions and the main theorem, we shall further explore the geometry of general families of symmetry groups.

2. BASIC TERMINOLOGY.

We throughout assume that all vector fields under discussion are such that unique solutions exist locally for the associated differential equation. A vector field \( F \) that vanishes at \( x = 0 \) is called (locally) asymptotically stable if there exists some neighbourhood \( N_0 \) of \( x = 0 \) such that all solution curves of \( \dot{x} = F(x) \) starting in \( N_0 \) are defined for all positive \( t \) and converge to \( x = 0 \), and such that for every neighbourhood \( N \subseteq N_0 \) of \( x = 0 \) there exists a neighbourhood \( N' \) of \( x = 0 \) such that all solution curves starting inside \( N' \) stay inside \( N \) for all positive times.

For two \( C^1 \)-vector fields \( F \) and \( G \) the Lie bracket \( [F, G] = FG - GF \) denotes their Lie bracket, which in local coordinates may be defined in terms of the Jacobians as \( [F, G] = \text{Jac}(F) - \text{Jac}(G) \). An infinitesimal symmetry of a \( C^1 \)-vector field \( F \) is a \( C^1 \) vector field \( \nu \) such that \( \nu \cdot F = mF \) for some constant \( m \) (if \( m \neq 0 \)) then \( \nu \) is often referred to as generalized symmetry, or symmetry of order \( m \). In this case we say that \( F \) is homogeneous of order \( m \) w.r.t. \( \nu \) and write \( F \in \mathfrak{H}_m \). We throughout shall only consider the case that the vector field \( \nu \) is asymptotically stable.

Note that if \( L_\nu F = 0 \), then by antisymmetry also \( L_\nu \nu = 0 \), and hence the symmetry may be interpreted in either direction. However, homogeneity of order \( m \neq 0 \) is always in one direction only.

A \( C^1 \) function \( \varphi \) is homogeneous of order \( m \) if \( L_\nu \varphi = m \varphi \), and in this case we write \( \varphi \in H_m \). One easily verifies that \( H_k H_m \subseteq H_{k+m} \), \( L_\nu H_m \subseteq H_{m-h} \), and \( \mathfrak{H}_m \subseteq \mathfrak{H}_{m+h} \), which are the underlying algebraic properties that give rise to gradations and filtrations on the algebra of homogeneous functions, and on the Lie algebra of (smooth) homogeneous vector fields.

In order to state the fundamental theorem that underlies all applications of homogeneity to stabilization we need the following further terminology.

Definition (compare [8]). A smooth one-parameter family of dilations on \( \mathbb{R}^n \) is a one parameter group \( \Delta \) acting on \( \mathbb{R}^n \) that satisfies:

(i.) \( (s, x) \rightarrow s \Delta \) is the flow of a globally asymptotically stable differential equation \( \dot{x} = -\nu(x) \) which has unique solutions that are globally defined.

(ii.) There exists a smooth Lyapunov function \( V : \mathbb{R}^n \rightarrow [0, \infty) \), and a level set \( Q_\epsilon = \{ x : V(x) = \epsilon \} \) that is diffeomorphic to the standard sphere \( S^{n-1} \).

The second requirement is necessary in view of the occurrence of level surfaces that are homogeneous spheres, but that are not diffeomorphic to the standard sphere [12]. If \( F \) is \( \Delta \) homogeneous of order \( m \) (i.e. \( L_\nu F = mF \)), then \( \tilde{F} = h^{-m} F \in \mathfrak{H}_m \) for any smooth function \( h \in H_k \) (e.g. a homogeneous norm) that only vanishes at \( x = 0 \), and \( \tilde{F} \) induces a dynamical system \( \dot{y} = \tilde{F}_\nu(y) \Rightarrow (\nu, (\tilde{F}_\nu)) \) on the quotient space \( Q = (\mathbb{R}^n \setminus \{0\})/\Delta \), which is diffeomorphic to the level set \( Q_\epsilon \) for any \( \epsilon > 0 \).

Here \( \pi \) is the canonical projection of \( \mathbb{R}^n \setminus \{0\} \) onto \( Q \). Define \( \Omega \subseteq Q \) as the closure;
of the union of all ω-limit sets of the dynamical system defined by \( F \), and let \( F_0 \) be the restriction of \( F \) to \( \overline{\Omega} = \pi^{-1}\Omega \cup \{0\} \subseteq \mathbb{R}^n \) which is the cone over \( \Omega \).

**Theorem** [9, 8]. Let \( F \) be a vectorfield on \( \mathbb{R}^n \) that is \( \Delta \)-homogeneous (where \( \Delta \) is a smooth family of dilations), such that initial value problems to \( \dot{x} = F(x) \) have locally unique solutions. Then the differential equation \( \dot{x} = F(x) \) is asymptotically stable about \( x = 0 \) if and only if the differential equation \( \dot{z} = F_0(z) \) on the cone \( \overline{\Omega} \) is asymptotically stable about \( x = 0 \).

### 3. SOME EXAMPLES AND BASIC GEOMETRY.

For practical computations it usually is most convenient to first find (if possible) \( n \) \( C^1 \)-functions \( \varphi_i \in H_1 \) such that \( d\varphi_i \) are linearly independent (near \( x = 0 \)), and to also find (if possible) \( n \) \( C^1 \)-vectorfields \( Z' \in H_2 \) that are linearly independent near \( x = 0 \) if these do not commute, one may use these to actually find homogeneous functions since \( [Z', Z'] \subseteq H_2 \) implies that there exist functions \( h_k \in H_1 \) such that \( [Z', Z'] = \sum h_k Z' \).

In the standard case of \( \nu = \sum_{j=1}^n r_j(x_j \frac{\partial}{\partial x_j}) \) with \( r_j \geq 1 \) odd integers the functions \( \phi_j = \frac{r_j}{x_j} \in H_1 \) are as above, and the corresponding coordinate vectorfields satisfy \( \frac{\partial}{\partial x_j} \in \mathbb{N}_{-r_j} \).

As a first nonlinear example consider the vectorfield \( \nu = 2x_1 \frac{\partial}{\partial x_1} + 3(x_2 + cx_2) \frac{\partial}{\partial x_2} \) with \( c = \pm 1 \). Note that in the case of \( c = 1 \), the vectorfield \( \nu \) exhibits finite escape time, whereas in the case \( c = -1 \) the vectorfield \( -\nu \) is only locally asymptotically stable. Thus continuing with local calculations only, we easily find that \( x_1 \in H_2 \) and both \( x_1 \frac{\partial}{\partial x_1} \in H_2 \) \( (x_2 + cx_2) \frac{\partial}{\partial x_2} \in H_2 \) whereas it already requires some work to find that e.g. \( \frac{x_2}{\sqrt{x_1 + c x_2}} \in H_2 \) and hence \( \frac{x_1}{\sqrt{x_1 + c x_2}} \in H_2 \). From these one then basically can build up all homogeneous functions and vectorfields. Moreover, one may actually use these to construct a nonlinear coordinate change that transforms \( \nu \) into a linear vectorfield.

Generally, one may find homogeneous functions by solving the first order partial differential equation \( L \varphi = m \varphi \), which when written out becomes (if \( \nu = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j} \))

\[
a_1(x) \frac{\partial \varphi}{\partial x_1}(x) + \ldots + a_n(x) \frac{\partial \varphi}{\partial x_n}(x) = m \varphi(x)
\]

The possibility of finding such independent homogeneous functions is closely related to the local phase portrait of \( \nu \) near the equilibrium point. For two dimensional systems one pictorially distinguishes between focal and (proper or improper) nodes. Also, if the linear part of \( \nu \) does not have any zero eigenvalues, then one analytically defines node/focus in terms of the imaginary parts of the eigenvalues.

For general nonlinear vectorfield we prefer the following characterization: We say that \( \nu \) has a node of order \( k \) (at \( x = 0 \)) if \( k \) is the maximal number of functions \( \varphi_k \) that are \( C^1 \) near \( x = 0 \) and are such that \( \varphi_k(L \varphi_k) \leq 0 \) near \( x = 0 \) and such that \( \{d\varphi_k(0)\} \) are linearly independent. In the case of smooth vectorfields the importance of the order of the node is illustrated in that if \( k < n \) then apparently we can not find \( n \) independent vectorfields \( Z' \) that are simultaneously homogeneous of order 1 and nonvanishing at \( x = 0 \).

### 4. GEOMETRY OF GENERAL DILATION GROUPS.

For many calculations it is advantageous to work with a frame of homogeneous vectorfields (on \( \mathbb{R}^n \setminus \{0\} \)) that contains as one of its members the symmetry gen-

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erator $\nu$. The most simple example is that of polar coordinates in the two dimensional case (e.g. if $\nu(x, y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$). More generally, we may construct some such complementary vectorfields as follows: Suppose we have two homogeneous vectorfields $Z^1 \in \mathfrak{m}_n$ and $Z^2 \in \mathfrak{m}_k$ that are linearly independent everywhere, and suppose that $h \in H_{m+k}$ is a positive definite function that is homogeneous of order $m + k > 0$.

Claim. The vectorfield $X = (L_{Z^1} h) Z^1 - (L_{Z^2} h) Z^2$ satisfies $X \in \mathfrak{m}_0$ and $L_X h = 0$. Moreover, $\{X, \nu\}$ are linearly independent (commuting) vectorfields on $\mathbb{R}^n \setminus \{0\}$.

Proof. Regarding the homogeneity, we use the gradations mentioned in the previous section to find that e.g. $X = (L_{Z^1} h) \in H_m$ and hence $X = (L_{Z^1} h) Z^1 \in \mathfrak{m}_0$, i.e. $[\nu, X] = 0$. Applying $X$ to the function $h$, we only need to rewrite $Z^1 h = L_{Z^1} h$ to see that indeed $L_X h = 0$ and hence $h$ is constant along integral curves of $X$, or in other words, $X$ is tangent to any one of the level surfaces $\{x : h(x) = c\}$ for any constant $c > 0$. To see that $\{X, \nu\}$ are linearly independent for $z = 0$ we only note that $L_{P} h = (m + k) h \neq 0$, whereas $L_{X} h = 0$.

As a simple example consider $\nu(x, y) = x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}$ and the function $h(x, y) = \frac{1}{2} x^e - 2y + \frac{3}{4} x^3 y^3$, which is positive definite for $e > 1$. One easily sees that $h \in H_2$, and that the coordinate vectorfields are homogeneous, i.e. we may take $Z^1 = \frac{\partial}{\partial x} \in \mathfrak{m}_2$ and $Z^2 = \frac{\partial}{\partial y} \in \mathfrak{m}_3$. In this case the vectorfield $X = X_4$ as above becomes $X_4 = (ey^{1/3} - x) \frac{\partial}{\partial x} + (y - x^3) \frac{\partial}{\partial y}$, which played a fundamental role in the asymptotic feedback stabilization of the system $\dot{x} = u$, $\dot{y} = y - x^3$ using a Hölder continuous feedback law, compare [7].

Given such a frame of homogeneous vectorfields $\{\nu, X^1, \ldots, X^{n-1}\}$, one may express any homogeneous vectorfield $F \in \mathfrak{m}_m$ as a homogeneous linear combination $F = F_0 \nu + \sum_{i=1}^{n-1} F_i X^i$ with $F_i \in H_m$. If $m \neq 0$ we as usual write $\tilde{F} = h^{-m} F$ using any smooth function $h \in H_1$ (e.g. use a homogeneous norm). Since vectorfields $X \in \mathfrak{m}_0$ project to vectorfields $X_\pi = \pi_\nu(X)$ on the quotient space $\Omega$, we easily find that the reduced dynamics $F_\pi$ is given by $F_\pi = \sum_{i=1}^{n-1} F_i \pi X^i$ (since $\nu_{\pi} \equiv 0$).

For elementary topological reasons one only can find such global frames in dimensions $n = 2, 4, 8$ (the spheres $S^1$, $S^3$ and $S^7$ are parallelizable). In the case of $n = 2$ such a frame basically looks like a nonlinear version of the coordinate vectorfields of polar coordinates. More interesting is the case of $n = 4$, which may lead to some very interesting dynamics on the manifold $SO(3)$ (provided the fields are all also odd, i.e. $X(\pm x) = -X(x)$).

5. LINEAR DILATION GROUPS IN TWO DIMENSIONS.

Here we analyze in detail the possible features of linear dilation groups in two dimensions. Let $A$ be a real $2 \times 2$ matrix, and let $\nu(x, y) = (x \ y)^T A^T \left( \frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \right)$ be the infinitesimal generator of the dilation group $\Delta$. The requirement that the vectorfield $-\nu$ is asymptotically stable requires that $A$ is positive definite. We have to distinguish three cases depending on whether $A$ has a pair of real eigenvalues, a pair of complex conjugate eigenvalues, or a repeated real eigenvalue with nontrivial Jordan block.

The first case is the one which has been popularized e.g. with investigations of hypoelliptic differential operators, and in connection with controllability preserving nilpotent approximations of smooth control system. We may assume that (after possibly a linear change of coordinates) $A$ is diagonal, and without loss of generality we assume that its first eigenvalue is $\alpha_{11} = 1$ (this normalization
corresponds to a reparametrization of the group $\Delta$). In this case the coordinate vectorfields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are homogeneous of orders $-1$ and $-r = -a_{21}$, respectively, i.e. $\frac{\partial}{\partial x} \in \mathbb{R}_{-1}$ and $\frac{\partial}{\partial y} \in \mathbb{R}_{-r}$.

Regarding homogeneous functions we find that a monomial $p(x, y) = x^\alpha y^\beta$ is of homogeneous degree $(\alpha + \beta r)$, written $x^\alpha y^\beta \in H_{\alpha + \beta r}$. The homogeneous (pseudo) norm $\| (x, y) \|_\Delta = (x^r + y^r)^{1/r}$ is a function in $H_1$, and in case of $r$ being an integer, it is analytic away from the origin. The homogeneous unit sphere, which is the level surface of $\| \cdot \|_\Delta$ corresponding to $c = 1$, and which is diffeomorphic to $Q$, is locally nicely coordinatized by e.g. $\Theta = (x/y^{1/r}) \in H_0$.

Next let us consider the case where $A$ has a pair of complex eigenvalues. Again after the standard normalization (reparametrization of time) and possibly a linear coordinate change we may assume that $\nu = (x - \omega y)\frac{\partial}{\partial x} + (\omega x + y)\frac{\partial}{\partial y}$ for some $\omega \in \mathbb{R}$. In this case it is convenient to work with complex valued functions on $\mathbb{R}^2$, and even allow for complex degrees of homogeneity! In particular, the eigenvalues of $A$ are $\lambda_{1,2} = (1 \pm \omega i)$, and working with the associated eigenvectors, we introduce the functions $u = (x + iy)$ and $v = (x - iy)$. One immediately verifies that $L_\nu u = (1 + i\omega)u$ and $L_\nu v = (1 - i\omega)v$. Thus we may as well write $u \in H_{1+i\omega}$ and $v \in H_{1-i\omega}$. This then allows for the straightforward calculation of independent homogeneous real-valued functions: In particular, the multiplication rules in the (complex) graded algebra of homogeneous functions are still valid, and we find for example

$$uv \in H_{1+i\omega} \cdot H_{1-i\omega} \subseteq H_{(1+i\omega) + (1-i\omega)} = H_2$$

and consequently $\sqrt{uv} = (x^2 + y^2)^{1/2} \in H_1$. Similarly, we have that the degree of homogeneity of $u/v$ is $(1 + i\omega) - (1 - i\omega) = 2i\omega$, and thus expect $(u/v)^{1/2i\omega}$ to be an independent function in $H_1$. Of course this should be related to the angular variable (of the polar coordinates), and indeed we find after some elementary complex arithmetic

$$\left( \frac{u}{v} \right)^{1/2i\omega} = \exp \left( \frac{1}{2i\omega} \right) \ln \frac{x + iy}{x - iy} \exp \left( \frac{1}{i\omega} \tan^{-1}(\omega y/x) \right)$$

In particular, $e^\omega \in H_\omega$, where we write $(\omega, \vartheta) = (\sqrt{x^2 + y^2}, \tan^{-1}(x/y))$ for the polar coordinates. Summarizing, we have $e^\omega \partial^\omega \in H_1$, and a suitable coordinate on the quotient space $Q$ is for example $e^\omega \partial^\omega$. In this case the corresponding vectorfields $\nu_0 = \partial^\omega \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \vartheta} = \frac{1}{i}(\nu - \nu_0)$ are clearly independent everywhere on $\mathbb{R}^2 \setminus \{0\}$, and both are in $H_0$.

Finally, we discuss the geometry of the dilation group in the case where $A$ has a repeated eigenvalue with nontrivial Jordan block. We normalize time such that $\nu = (x + \lambda y)\frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. First we look for independent homogeneous functions. An immediate solution is $\varphi_1 = y \in H_1$. A second independent solution of the equation (subscripts denote partial derivatives)

$$(x + \lambda y)\varphi_x + y\varphi_y = \varphi$$

is found as $\varphi_2 = x - \lambda y \ln |y| \in H_1$, and as a local homogeneous coordinate on the quotient space $Q$ we may take e.g. $\Theta = \left( \frac{\partial}{\partial x} - \lambda \ln |y| \right) \in H_0$. In this case the vectorfields $\nu$ and $y \frac{\partial}{\partial y}$ form a frame of homogeneous vectorfields, but only outside the the subspace $\{(x, y) : y = 0\}$. Given a general system $\dot{x} = f(x) + \rho(x)$
such that $f$ is homogeneous of order $0$, we try a class of feedback laws so that also $ug \in \mathbb{R}^n$, and then investigate the limit sets of the induced dynamics, i.e. $\Theta = (f + ug)_+$ ($\Theta$), and then proceed as in [8].

Concluding, we hope that these descriptions of basic geometric features of dilations groups and corresponding examples will stimulate further research, and maybe even be found useful for some practical stabilization problems. Further, we look ahead to fully nonlinear symmetry vectorfields $\nu$ on $\mathbb{R}^4$, whose local orbit structure may be even more complicated than those linear vectorfields which have repeated pairs of complex eigenvalues with nontrivial Jordan block. We even have to consider chaotic angular dynamics, and may use symmetry vectorfields with similar chaotic orbit structure to obtain useful information. Moreover, in this case of $n = 4$, the reduced space is $SO(3)$ which invites some nice animation of the reduced dynamics on graphics computers.

References


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