Nilpotent Lie algebras of vectorfields

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The paper is organized as follows: In part 1 we introduce some notation and briefly review dilations. In part 2 we state the main theorem and some noteworthy sideresults and lemmata. In part 3 we prove the lemmata preparing the proof of the theorem in part 4.

1. Notation and definitions

Let $X^1, \ldots, X^k$ be real analytic vectorfields on a real analytic $n$-dimensional manifold $M$. $L = L(X^1, \ldots, X^k)$ is the Lie algebra generated by the vectorfields $X^1, \ldots, X^k$, where $[V, W] = VW - WV$ is the Lie product of the vectorfields $V$ and $W$. We use $(\text{ad} V, W) = [V, W]$ and $(\text{ad}^{k+1} V, W) = [V, (\text{ad}^{k} V, W)]$. The central descending series of $L$ is defined by $L^{(1)} = L$ and inductively $L^{(k+1)} = [L, L^{(k)}]$. (Caution: Sometimes in the literature one uses $L^{(0)}$ for $L^{(1)}$.) The Lie algebra $L$ is nilpotent if for some smallest integer $q \geq 1$, $L^{(q)} = \{0\}$.

Vectorfields generating a nilpotent Lie algebra are very useful for practical purposes as they, for example, allow one to compute solution of ordinary differential equations explicitly by decomposing the equation into a finite number of possibly much less difficult equations. Specifically in control systems of the form

\begin{equation}
\dot{x} = X^0(x) + u X^1(x), \quad x(0) = 0,
\end{equation}

with an integrable bounded control function $u$, one has a natural decomposition of the right side. When $L(X^0, X^1)$ is nilpotent, then it is possible to write the solution $x(\cdot, u)$ of (1) as a composition of a finite number of solutions to possibly less complicated ordinary differential equations.

It is relatively simple to give examples of vectorfields generating a nilpotent Lie algebra if one uses dilations. We will show that in the proper local coordinates, every finite collection of vectorfields generating a nilpotent Lie algebra is associated to a dilation.

*) This work was supported by NSF Grant DMS 8500911 and is part of the doctoral dissertation of the author at the University of Colorado, Boulder.
We next give a brief review of *dilations* and *filtrations/gradations* on the Lie algebra of vectorfields with polynomial coefficients, where we try to follow as far as possible the notation as introduced in [1].

For a fixed choice of coordinates \( x = (x_1, \ldots, x_n) \) on \( \mathbb{R}^n \) and a nondecreasing sequence of positive integers \( 1 = r_1 \leq r_2 \leq \cdots \leq r_n \) define a one-parameter family of *dilations* \( \delta: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) by \( \delta_t(x) = (t^{r_1}x_1, \ldots, t^{r_n}x_n) \).

A polynomial \( p = p(x) \) is homogeneous of degree \( m \) with respect to the dilation \( \delta_t \) if \( p(\delta_t(x)) = t^m p(x) \). Let \( H_m \) be the set of all polynomials homogeneous of degree \( m \) w.r.t. \( \delta_t \), and set \( H_m = \{0\} \) if \( m < 0 \).

**Example.** If \( \delta_t(x_1, x_2, x_3) = (tx_1, t^2x_2, t^6x_3) \) is a dilation on \( \mathbb{R}^3 \), then
\[
p(x) = 7x_3 - x_2x_1^4 + 3x_1^6
\]
is homogeneous of degree 6 w.r.t. \( \delta_t \), i.e. \( p \in H_6 \).

A vectorfield \( X(x) = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} \) with polynomial coefficients \( a_i(x) \) is homogeneous of degree \( -k \in \mathbb{Z} \) if \( XH_m \subseteq H_{m-k} \) for all \( m \geq 0 \); this is equivalent to \( a_i(x) \in H_{r_i-k} \) for \( i = 1, \ldots, n \). Let \( n_{-k} \) be the set of all *vectorfields homogeneous of degree* \( -k \), and set \( n_{-k} = \{0\} \) if \( k > r_n \).

**Example.** With the same dilation as above
\[
X(x) = 3 \frac{\partial}{\partial x_1} + 7x_1 \frac{\partial}{\partial x_2} + (3x_1^5 + x_1x_2^3) \frac{\partial}{\partial x_3} \in n_{-1},
\]
i.e. \( X \) is homogeneous of degree \( -1 \) w.r.t. \( \delta_t \).

One readily verifies \( H_m \cdot H_n \subseteq H_{m+n} \) if \( m, n \geq 0 \), and \( [n_k; n_l] \subseteq n_{k+l} \) for all \( k, l \in \mathbb{Z} \): If \( X \in n_{-k}, Y \in n_{-l} \), then
\[
[X; Y] H_m = YXH_m - XYH_m \subseteq YH_{m-k} + XH_{m-l} \subseteq H_{m-k+l}.
\]
The spaces \( H_m \) give a *gradation* on the algebra \( P \) of polynomials in \( (x_1, \ldots, x_n) \): \( P = \bigoplus H_m \). Similarly the spaces \( n_k \) give a gradation on the Lie-algebra \( n \) of vectorfields with polynomial coefficients, \( n = \bigoplus n_k \).

Defining \( P_m = \sum_{k=0}^{m} H_k \) and \( N_{-k} = \sum_{l=-r_n}^{-k} n_l \), clearly
\[
P_{m+1} \supseteq P_m, \quad m \geq 0, \quad \text{and} \quad N_{-k+1} \supseteq N_{-k}, \quad -k \geq -r_n,
\]
and one obtains the *filtrations*.
\[
P = \bigcup_{k=0}^{\infty} P_k \quad \text{and} \quad n = \bigcup_{l=-r_n}^{\infty} n_l, \quad N_{-1} = \bigcup_{l=-r_n}^{-1} n_l.
\]
In the following we will call a polynomial \( p(x) \in P_m \) “of degree \( m \)” and a vectorfield \( X \in \mathcal{N}_{-k} \) “of degree \(-k\)”.

If \( X^1, \ldots, X^k \subseteq \mathcal{N}_{-1} \) are vectorfields with polynomial coefficients, then clearly \( L(X^1, \ldots, X^k) = L \subseteq \mathcal{N}_{-1} \) and \( L^0 \subseteq \mathcal{N}_{-1} \) for \( i \geq 1 \). But \( \mathcal{N}_{-k} = \{0\} \) for \( k > r_n \), and thus \( X^1, \ldots, X^k \) generate a nilpotent Lie algebra.

Usually we will use greek letters \( \alpha = (\alpha_1, \ldots, \alpha_k), \nu = (\nu_1, \ldots, \nu_n) \) to denote multi-indices with \( \alpha_i, \nu_i \in \mathbb{Z}_0^+ \). As usual

\[
|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \\
\alpha! = \alpha_1! \cdots \alpha_k!, \\
x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \\
D^\alpha = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}},
\]

and

\[ r \cdot \alpha = r_1 \alpha_1 + \cdots + r_k \alpha_k, \]

where we allow \( 1 \leq k \leq n \), even though \( r = (r_1, \ldots, r_n) \) is a fixed \( n \)-vector throughout this work.

For analytic vectorfields \( Y^1, \ldots, Y^k, Z \) and \( f: M \to \mathbb{R} \) analytic we define

\[
(ad^\alpha Y^\alpha; Z) = (ad^{\alpha_1} Y^1; \cdots; (ad^{\alpha_k} Y^k; Z)) \cdots
\]

and \( Y^f = (Y^1)^{\nu_1} (Y^2)^{\nu_2} \cdots (Y^k)^{\nu_k} f \) denoting an \(|\alpha|\)-th order partial derivative of \( f \).

2. Statement of the theorem and side results

As already mentioned in the introduction, vectorfields \( X^1, \ldots, X^k \) which are of degree (at most) \(-1\) w.r.t. some dilation \( \delta_i \), generate a nilpotent Lie algebra \( L = L(X^1, \ldots, X^k) \).

We show that (in the proper local coordinates) the converse is also true.

**Theorem.** Let \( X^1, \ldots, X^k \) be real analytic vectorfields on the real analytic \( n \)-dimensional manifold \( M^n \) which generate the nilpotent Lie algebra \( L = L(X^1, \ldots, X^k) \). If \( p \in M \) is such that \( \dim L(p) = n \), then there are local coordinates \( (x_1, \ldots, x_n) \) in a neighborhood of \( p \) and a dilation \( \delta_i(x) = (t^{\nu_in} x_1, \ldots, t^{\nu_in} x_n) \) such that relative to these coordinates \( X^1, \ldots, X^k \) have polynomial coefficients and are of degree \(-1\) w.r.t. the dilation \( \delta_i \).

This immediately leads to: If \( X^i(x) = \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j}, i = 1, \ldots, n \); then \( a_{ij}(x) \) is a polynomial of degree \( r_j - 1 \) w.r.t. \( \delta_i \) and depends on \( x_1, \ldots, x_{j-1} \) only. In particular \( a_{ii} = \text{const.} \) for \( i = 1, \ldots, k \).
An immediate corollary to the theorem is the known result:

**Corollary.** If $G$ is a nilpotent Lie group locally acting transitively on a real analytic manifold $M^n$, i.e. $\dim g(p) = n$, then there are local coordinates on $M$ such that the action of $G$ relative to these coordinates is polynomial.

At this point we state two other interesting facts as lemmata. Later they are needed in the proof of the theorem.

It is well-known that $\dim L^{(j)}$ is strictly decreasing for $j = 1, 2, \ldots, q$ if $L$ is nilpotent. However $\dim L^{(j)}(p)$ need not be strictly decreasing — at least, it cannot increase since $L^{(j)}(p) \supseteq L^{(k)}(p)$ for $j \leq k$. But we need the following lemma which also holds under milder hypotheses than requiring analyticity.

**Lemma 1.** If $L = L(X^1, \ldots, X^k)$ is a nilpotent Lie algebra of vectorfields on $M^n$ with $\dim L(p) = n$, then $\dim L^{(2)}(p) < n$.

Recall that a distribution $\Delta$ is called regular at $p$ if there is a neighborhood $U$ of $p$ such that for all $i \geq 0$ the $i$th derived distribution $\Delta^{(i)}$ is of constant dimension throughout $U$. (The $i$th derived distribution is the span of all Lie products of at most $i$ vectorfields in $\Delta$.) We need a similar property, but note that the $L^{(i)}$ are decreasing, whereas the $\Delta^{(i)}$ are increasing.

**Lemma 2.** If $L$ is a nilpotent Lie algebra of analytic vectorfields on $M^n$ and $p \in M$ is such that $\dim L(p) = n$, then there is a neighborhood $U$ of $p$ such that for all $i = 1, 2, \ldots, q \dim L^{(i)}(q)$ is constant for $q \in U$.

Finally a large portion of the proof of the theorem is based on

**Lemma 3.** If $Z^i = \frac{\partial}{\partial x_i} + \sum_{j=i+1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}, i = 1, 2, \ldots, n$ are vectorfields of degree $-r_i$ w.r.t. the dilation $\delta_i$ (i.e. $a_{ij}(x)$ are polynomials of degree $r_j - r_i$ w.r.t. $\delta_i$) and $\psi: \mathbb{R}^n \to \mathbb{R}$ is analytic and satisfies $(\mathcal{L}^s \psi)(0) = 0$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $r \cdot \alpha > s$, then $\psi \in H_s$, i.e. $\psi$ is a polynomial of degree $s$ w.r.t. $\delta_i$.

If $a_{ij} \equiv 0$, $i = 1, \ldots, n$, $j = i+1, \ldots, n$, one has $Z^i = D^i = -\frac{\partial}{\partial x_i}$ and the lemma is immediate. But since the $Z^i$ may be noncommuting, the proof of lemma 3 requires some work.

**Remark.** In general it is not possible to achieve that the vectorfields $X^i$ in the theorem are homogeneous of degree $-1$ (or $-d_j$), or equivalently that the polynomials $a_{ij}(x)$ are homogeneous w.r.t. some dilation, since this would immediately lead to a gradation of the Lie algebra $L(X^1, \ldots, X^k)$, which in turn would allow one to construct dilating automorphisms of $L = L(X^{-1}, \ldots, X^k)$. However, there are examples of nilpotent Lie algebras in the literature, which do not admit any dilating automorphisms [2].

3. Proofs of the lemmata and technical claims

We start by proving lemma 1. This enables us to construct local coordinates and a dilation on $M^n$, which in turn we will use to prove lemma 2. Finally, we will verify several technical claims and prove lemma 3.
**Proof of lemma 1.** Since $L$ is generated by a finite number of vectorfields and is nilpotent, $L$ and each ideal $L^{(i)}$ is finite dimensional.

Choose a basis $\{Z^1, \ldots, Z^s\}$ for $L$ such that $\{Z^1(p), \ldots, Z^s(p)\}$ are linearly independent and $\{Z^{n+1}, \ldots, Z^s\}$ vanish at $p$. We will show, if $\dim L^{(2)}(p) = n$, then $\dim L^{(i)}(p) = n$ for all $i \geq 1$, thus contradicting nilpotency of $L$.

Choose vectorfields $\{W^{2,1}, \ldots, W^{2,n}\} \in L^{(2)}(p)$ such that

$$W^{2,i}(p) = Z^i(p), \quad i = 1, 2, \ldots, n.$$ 

Thus there are $a_{ijk}, b_{ij} \in \mathbb{R}$ such that

$$W^{2,i} = Z^i - \sum_{j=n+1}^s b_{ij} Z^j$$

and

$$W^{2,i} = \sum_{1 \leq j < k \leq s} a_{ijk} [Z^j; Z^k], \quad i = 1, 2, \ldots, n.$$ 

Combining these two relations gives

$$Z^i = \sum_{1 \leq j < k \leq s} a_{ijk} [Z^j; Z^k] + \sum_{j=n+1}^s b_{ij} Z^j, \quad i = 1, 2, \ldots, n. \quad (2)$$

Now substitute for each $Z^v, 1 \leq v \leq n$ on the right side of (2) the right side of the $v$th equation of (2). The result is $n$ equations of the form

$$Z^i = W^{3,i} \mod \mathcal{H}_p, \quad i = 1, 2, \ldots, n, \quad (3)$$

where $W^{3,i} \in L^{(3)}$ and $\mathcal{H}_p = \text{span} \{Z^{n+1}, \ldots, Z^s\}$ is the isotropy subalgebra of $L$ at $p$.

Repeated substitution of (2) for all $Z^v, v = 1, 2, \ldots, n$, on the right side of (3) leads to equations of the form

$$Z^i = W^{h,i} \mod \mathcal{H}_p, \quad i = 1, 2, \ldots, n, \quad h \geq 2 \quad (4)$$

with $W^{h,i} \in L^{(h)}$ and $\mathcal{H}_p$ as above. Evaluating (4) at $p$ shows $\dim L^{(h)} = n$, all $h \geq 2$, thus finishing the proof of lemma 1.

We proceed by defining a family of dilations and local coordinates in a neighborhood of $p$. For $i = 1, \ldots, n$ define

$$r_i = \max \{j \in \mathbb{Z}^+: \dim L^{(j)}(p) = (n + 1) - i\},$$

$$s_i = \min (\{n + 1\} \cup \{j \in \mathbb{Z}^+: r_j > r_i\}). \quad (5)$$

From $\dim L^{(1)}(p) = n$ and lemma 1, $r_1 = 1$; and $r_n = q - 1$, where $q \in \mathbb{Z}^+$ is such that $L^{(q)} = \{0\}$ and $L^{(q-1)} \neq \{0\}$. 

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Choose vectorfields \( Y^1, \ldots, Y^n \in L = L(X^1, \ldots, X^k) \) such that
\[
Y^i \in L^{(r_i)},
\]
i = 1, 2, \ldots, n and \( Y^1(p), \ldots, Y^n(p) \) are linearly independent. Then \( Y^1(q), \ldots, Y^n(q) \) are linearly independent for \( q \) sufficiently close to \( p \) and hence there is a neighborhood \( U = U^1 \) of \( 0 \in \mathbb{R}^n \) such that the inverse of the map \( \gamma: U \rightarrow M^n \),
\[
\gamma(x) = (\exp x_n Y^n) \circ \cdots \circ (\exp x_1 Y^1)(p)
\]
defines local coordinates on the neighborhood \( \gamma(U) \) of \( p \in M^n \).

From now on we will work entirely in this neighborhood \( \gamma(U) \) of \( p \), and identify the point \( q \in \gamma(U) \) with its coordinates \( \gamma^{-1}(q) \in \mathbb{R}^n \), and similarly identify \( U \) with \( \gamma(U) \) if this will cause no confusion.

Furthermore we will have to restrict our work several more times (but a finite number of times) to a neighborhood \( U = U^1 \) of \( p \). To avoid unnecessary confusing notation we will call all these neighborhoods \( U \), but understand that \( U \) stands for the intersection \( U = \bigcap_i U^i \), on which all our statements hold.

Finally define a one-parameter family of dilations \( \delta: [0, 1] \times U \rightarrow U \) by
\[
\delta_t(x) = (t^{r_1}x_1, \ldots, t^{r_n}x_n)
\]
and let \( r = (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n \). (If necessary, shrink \( U \) such that \( \delta_t(U) \subseteq U \) for all \( t \in [0, 1] \).)

**Proof of lemma 2.** With \( Y^1, \ldots, Y^n \), \( r \), and local coordinates defined as above, there are uniquely determined analytic functions \( \alpha_{f,g}^h: U \rightarrow \mathbb{R}^n \); \( f, g, h = 1, 2, \ldots, n \), such that
\[
[Y^f, Y^g] = \sum_{h=1}^n \alpha_{f,h}^g(x) Y^h(x); \quad f, g, h = 1, \ldots, n.
\]
For \( g = 1, \ldots, n \), let \( A^g \) be the \( n \times n \)-matrix \( A^g = (\alpha_{f,h}^g)_{f=1,2,\ldots,n} \) and for \( g = 1, \ldots, n \), \( i = 1, \ldots, n \) let \( A_i^g \) be the \( i \times i \)-matrix \( A_i^g = (\alpha_{f,h}^g)_{f=1,2,\ldots,i} \). By skew-symmetry of the Lie product \([\cdot, \cdot]\), we have
\[
\alpha_{f,h}^g + \alpha_{g,f}^h = 0, \quad f, g, h = 1, \ldots, n.
\]
For a point \( x = (x_1, \ldots, x_n) \) in \( U \), i.e. sufficiently close to zero, define \( n \) smooth curves \( \xi^i \), \( i = 1, \ldots, n \), by
\[
\xi^i(0) = 0, \quad \xi^{i+1}(0) = \xi^i(x_1), \quad \xi^i(s) = (\exp s Y^i)(\xi^i(0)), \quad i = 1, 2, \ldots, n;
\]
in particular \( \xi^n(x_n) = x \). (Since we will work with one fixed \( x \) at a time, we suppress the dependence \( \xi(\cdot) = \xi_x(\cdot) \).)
We need a very simple form of Gronwall’s lemma: If \( f: \mathbb{R} \to \mathbb{R} \) is smooth, \( f(0) = 0 \) and \( |f(t)| \leq K \int_0^t |f(s)| \, ds \) for all \( t \geq 0 \), some constant \( K \geq 0 \), then \( f(\cdot) \equiv 0 \).

We will show by induction on \( \lambda \), \( \lambda \) decreasing, that if

\[
L^{(\lambda)}(0) = \text{span} \{ Y^\mu + 1(0), \ldots, Y^n(0) \},
\]

then \( L^{(\lambda)}(x) \subseteq \text{span} \{ Y^{\mu + 1}(x), \ldots, Y^n(x) \} \) for \( x \in U \), the other inclusion being trivial.

Starting with \( \lambda = \rho \), we have \( L^{(0)} = 0 \).

Next suppose \( L^{(\lambda + 1)}(x) = \text{span} \{ Y^{\mu + 1}(x), \ldots, Y^n(x) \} \) for all \( x \in U \) and

\[
L^{(\lambda)}(0) = \text{span} \{ Y^{\mu + 1}(0), \ldots, Y^n(0) \}.
\]

We claim

\[
(10) \quad x_f^g = 0 = x_f^g \quad \text{for} \quad f = \mu + 1, \mu + 2, \ldots, n, \quad h = 1, 2, \ldots, \nu, \quad \text{all} \quad g
\]
as a consequence of the induction hypothesis. (We have \( Y_f^{(\lambda)} \in L^{(\lambda)} \) for \( f = \mu + 1, \mu + 2, \ldots, n \), and thus for any \( g = 1, 2, \ldots, n \), \( [Y_f^{(\lambda)}; Y^g] \in L^{(\lambda + 1)} \), which by induction hypothesis is a \( C^\infty \)-module of \( \{ Y^{\mu + 1}, \ldots, Y^n \} \), thus giving (10).)

Let \( W \in L^{(\lambda)} \) w.l.o.g. be such that \( W(0) = 0 \). (If \( W(0) \neq 0 \), then \( W(0) = \sum_{j = \mu + 1}^n w_j Y^j(0) \), and we consider \( W' = W - \sum_{j = \mu + 1}^n w_j Y^j \).

There are analytic functions \( \beta_j: U \to \mathbb{R} \) such that \( W(x) = \sum_{j = 1}^n \beta_j(x) Y^j(x) \). We have \( \beta_j(0) \) and will show \( \beta_j \equiv 0, j = 1, 2, \ldots, \mu \). Compute for \( i = 1, \ldots, n \)

\[
(11) \quad [W; Y^i] = \sum_{j = 1}^n (Y^i \beta_j) Y^j + \beta_j [Y^j; Y^i]
\]

\[
= \sum_{j = 1}^n (Y^i \beta_j) Y^j + \sum_{j = 1}^n \sum_{k = 1}^n x_k^j \beta_j Y^k
\]

\[
= \sum_{j = 1}^n \left\{ (Y^i \beta_j) + \sum_{k = 1}^n x_k^j \beta_k \right\} Y^j, \quad i = 1, \ldots, n.
\]

Since \( [W; Y^i] \in L^{(\lambda + 1)} \) for \( i = 1, 2, \ldots, n \), there are analytic functions \( d_{ij}: U \to \mathbb{R} \) such that \( [W; Y^i] = \sum_{j = \nu + 1}^n d_{ij} Y^j, i = 1, \ldots, n \) (by the induction hypothesis).

Combining this with the last equation gives

\[
(12) \quad 0 = \sum_{j = 1}^n \left\{ (Y^i \beta_j) + \sum_{k = 1}^n x_k^i \beta_k \right\} Y^j + \sum_{j = \nu + 1}^n \left\{ (Y^i \beta_j) - d_{ij} + \sum_{k = 1}^n x_k^i \beta_k \right\} Y^j.
\]
Using that \( \{ Y^1, \ldots, Y^m \} \) are linearly independent (over \( C^\infty(U) \)), and that \( \alpha^i_k = 0 \) if \( k \geq \mu + 1 \) and \( j \leq \nu \), all \( i \), (note \( \nu \geq \mu \)), one obtains

\[
(Y^i \beta_j) = - \sum_{k=1}^\mu \alpha^i_k \beta_k, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \mu, \ldots, \nu.
\]

If we write this in vector form, \( b = (\beta_1, \ldots, \beta_\mu) \) and \( b^\mu = (\beta_1, \ldots, \beta_\mu) \), then we obtain

\[
(13) \quad Y^i b^\mu = b^\mu A^i_\mu, \quad i = 1, 2, \ldots, n.
\]

By induction on \( i \) we show \( b^\mu(\xi^i(0)) = 0 \), \( i = 1, 2, \ldots, n + 1 \).

\[
(14) \quad b^\mu(\xi^i(t)) = b^\mu(\xi^i(0)) + \int_0^t (Y^i b^\mu)(\xi^i(s)) \, ds
\]

Introducing norms, \( \| b^\mu \| = \sum_{l=1}^\mu |\beta_l| \), etc.), there is a constant \( K < \infty \) such that \( \| A(\cdot) \| < K \) on \( U \), and

\[
|b^\mu(\xi^i(t))| \leq \int_0^t K |b^\mu(\xi^i(s))| \, ds, \quad t \in [0, x_i]
\]

and thus \( b^\mu(\xi^i(t)) \equiv 0 \), \( 0 < t < x_i \) and finally \( b^\mu(\xi^i(x_i)) = b^\mu(\xi^{i+1}(0)) = 0 \), hence \( \beta_j(x) = 0 \) for \( j = 1, 2, \ldots, \mu \) and \( W(x) = \sum_{j=\mu+1}^n \beta_j(x) Y^j(x) \), finishing the proof of lemma 2.

We continue by introducing the distributions \( \Lambda^i \) on \( U \), \( i = 1, \ldots, n \),

\[
\Lambda^i(x) = \text{span} \{ Y^{n+1-i}(x), \ldots, Y^n(x) \}.
\]

We claim:

\[
\text{Claim 0.} \quad L^{(i)}(x) = \Lambda^{n-i+1}(x), \text{ where } l \text{ is such that } r_{l-1} < j \leq r_l, \ j = 1, 2, \ldots, q-1.
\]

\[
\text{Proof.} \quad \text{We only have to consider } r_{l-1} < r_l. \text{ By choice of the vectorfields } Y^i, \ L^{(i)}(0) = \text{span} \{ Y^1(0), \ldots, Y^n(0) \} = \Lambda^{n-i+1}(0). \text{ From } j \leq r_l, \ L^{(j)} \supseteq L^{(r_l)}. \text{ From } r_{l-1} < j, \ \text{dim} \ L^{(j)}(0) = \text{dim} \ L^{(r_l)}(0), \text{ and by lemma 2, the dimensions of } L^{(j)} \text{ evaluated at a point in } U \text{ are constant, thus}
\]

\[
L^{(j)}(x) = \Lambda^{n-i+1}(x) \quad \text{for all } x \in U.
\]
Claim 1. Relative to the coordinates \((x_1, \ldots, x_n)\), the vectorfields \(Y^i\) are of the form:

\[
Y^i(x) = \frac{\partial}{\partial x_i} + \sum_{j=x_{i+1}}^{n} c_{ij}(x) \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n
\]

with analytic functions \(c_{ij}\) vanishing at \(x = 0\). (For notational convenience let \(c_{ij} \equiv 0\) for \(i = 1, \ldots, n; j = i + 1, \ldots, s_{i+1} - 1.\))

**Proof.** (By induction on \(i\), \(i\) decreasing.) \(Y^n = \frac{\partial}{\partial x_n}\) is immediate.

Next suppose claim 1 is true for \(l = i + 1, i + 2, \ldots, n\). Compute

\[
\frac{\partial}{\partial x_i} = (\exp x_n Y^n)_e \circ \cdots \circ (\exp x_{i+1} Y^{i+1})_e Y^i((\exp x_i Y^i) \circ \cdots \circ (\exp x_1 Y^1)(0))
\]

where \(v = (v_{i+1}, \ldots, v_n)\). Each iterated bracket in the sum contains a factor \([Y^l; Y^j]\) for some \(l \geq i + 1\) and therefore lies in \(L^{(l+1)}\) which is a \(C^\omega\)-module of \(\{Y^{s_{i+1}}, \ldots, Y^n\}\).

Note that since \(L\) is nilpotent, the sum (16) is finite, and using the induction hypothesis, claim 1 is immediate.

Also immediate from equation (16) is

**Claim 2.** Each \(c_{ij}(x)\) in (15) is a sum of terms of the form \(x_1 c_{ij}^l(x)\) with \(i < l \leq n\) and \(c_{ij}^l\) analytic (when identifying \(c_{ij}(x)\) with its Taylor series about \(x = 0\)).

Observe that by claim 1 \(\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}\) is an abelian basis for \(\Delta^{n-l+1}, l = 1, 2, \ldots, n\). (Each \(\Delta^l\) is involutive.)

**Claim 3.** Each of the functions \(c_{ij}(x)\) depends on \(x_1, \ldots, x_i\) only, where \(l\) is such that \(s_1 \leq j\) (i.e. in particular \(l < j\)), \(i = 1, 2, \ldots, n\).

**Proof.** (By induction on \(l\), \(l\) decreasing)

Since \(Y^n = \frac{\partial}{\partial x_n} \in L^{(n-1)}\), \([Y^n; Y^i] = 0, i = 1, \ldots, n\), and therefore

\[
0 \equiv \begin{bmatrix} Y^i; \frac{\partial}{\partial x_n} \end{bmatrix} = \sum_{n=s_{i+1}}^{n} \frac{\partial c_{ij}}{\partial x_n} \frac{\partial}{\partial x_j},
\]

and hence \(\frac{\partial c_{ij}}{\partial x_n} = 0; i = 1, 2, \ldots, n; j = s_{i+1}, \ldots, n\).
Next suppose \( \frac{\partial c_{ij}}{\partial x_h} \equiv 0 \) for all \( h > l \) if \( s_h > j \). To show \( \frac{\partial c_{ij}}{\partial x_i} \equiv 0 \) if \( s_i > j \), \( i = 1, \ldots, n \), \( j = s_{i+1}, \ldots, n \), we only have to investigate \( l < j \) and from \([Y^i; Y^j] \in L^{(n+1)} \subseteq A^{n-s_l+1}\):

\[
[Y^i; Y^j] = \sum_{j=s_{i+1}}^{n-1} \frac{\partial c_{ij}}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{l=s_l}^{s_{i+1}} \sum_{j=s_{l+1}}^{n-1} c_{lh} \frac{\partial c_{ij}}{\partial x_h} \frac{\partial}{\partial x_j} \mod A^{n-s_l+1}.
\]

By the induction hypothesis, the double sum vanishes, giving

\[
\frac{\partial c_{ij}}{\partial x_i} \equiv 0, \quad i = 1, \ldots, n, \quad j \leq s_l - 1
\]

and thus \( c_{ij} = c_{ij}(x_1, \ldots, x_n) \) with \( s_h \leq j \).

**Proof of lemma 3.** We will show that \((Z^a \psi)(0) = 0\) for all \( \alpha \cdot r > s \) implies \((D^a \psi)(0) = 0\) for all \( \alpha \cdot r > s \), the lemma then being immediate from the Taylor series expansion of \( \psi \) about 0. Let \( M_s = \{ \alpha = (\alpha_1, \ldots, \alpha_n): \alpha \cdot r > s \} \) and define an order relation on all multi-indices \( \alpha \in (Z_0^+)^n \) by

\[
\alpha > \beta \iff \begin{cases} 
(|\alpha| > |\beta|) \\
\text{or} \\
(|\alpha| = |\beta| \text{ and there is an } i \in \{1, \ldots, n\} \text{ such that } \alpha_i > \beta_i \text{ and } \alpha_j = \beta_j \text{ for all } j < i)
\end{cases}
\]

(We do not work with lexicographical ordering, instead we introduced "\(|\alpha| > |\beta| \) or " in order to keep the sections \( S_\alpha: \beta \in M_s: \beta < \alpha \) finite.)

Let \( m_0 = \min \{ m \in Z^+: m \cdot r_n > s \} \) and if \( n_0 \) is the smallest integer such that \( r_{n_0} = r_n \), define \( \alpha^0 \in M_s \) by \( \alpha_{n_0}^0 = m_0 \) and \( \alpha_j^0 = 0 \) for all \( j \neq n_0 \). Then \( \alpha^0 \) is the smallest element in \( M_s \).

We prove the lemma by induction on \( \alpha \), starting with \( \alpha^0 \). Since \( r_{n_0} = r_n \), \( Z^{n_0} = \frac{\partial}{\partial x_{n_0}} \) and thus \( 0 = (Z^{a^0} \psi)(0) \) immediately gives \((D^{a^0} \psi)(0) = 0\).

Next suppose \((D^b \psi)(0) = 0\) for all \( \beta \in M_s \) with \( \beta < \alpha \) has already been shown and consider \( Z^a \psi \). To do further analysis, write each \( Z^i \) occurring in \( Z^a \psi \) in the form

\[
Z^i = \frac{\partial}{\partial x_1} + \sum_{j=1+1}^{n} a_{ij} \frac{\partial}{\partial x_j}
\]

and use the linearity of the differential operators to write \((Z^a \psi)\) as a sum of terms of the form

\[
(17) \quad a_{g_1 f_1} \frac{\partial}{\partial x_{f_1}} \left( a_{g_2 f_2} \frac{\partial}{\partial x_{f_2}} \left( \cdots \frac{\partial}{\partial x_{f_{|\alpha|}-1}} \left( a_{g_{|\alpha|} f_{|\alpha|}} \frac{\partial}{\partial x_{f_{|\alpha|}}} \psi \right) \cdots \right) \right)
\]

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with \( \alpha_i \equiv 1, i = 1, \ldots, n \), for notational ease), and apply Leibniz' rule repeatedly until no derivative acts on any product anymore. The result is

\[
(Z^\alpha \psi) = \sum_{\mu} p_{\mu}(x) (D^\mu \psi)
\]

where the \( p_{\mu}(x) \) are sums of products of partial derivatives of the coefficients \( a_{ij}(x) \).

We claim \( p_{\mu} \equiv 1, \mu \) ranges over all multi-indices \( \mu \leq \alpha \) and \( p_{\mu} \equiv 0 \) if \( \mu \notin \mathcal{M} \), i.e., \( \mu < \alpha^0 \).

To verify this claim observe that the order \( |\mu| \) of the derivative \( (D^\mu \psi) \) cannot be larger than the total number \( |\alpha| \) of \( Z^i \) applied to \( f_i \), hence \( |\mu| \leq |\alpha| \). Next observe that only if \( g_i = f_i \) for \( i = 1, 2, \ldots, |\alpha| \) in (17), one may get \( \mu = \alpha \) in (18), hence \( p_{\mu} = 1 \). Now fix one of those \( \mu \) in (18) with \( |\mu| = |\alpha| \), but \( \mu \neq \alpha \). Let \( k = \min \{j: \mu_j = \alpha_j\} \). Then \( \mu_1 = \alpha_1 \) implies \( g_i = f_i = 1 \) for \( i = 1, 2, \ldots, \alpha_1 \), and inductively \( \mu_j = \alpha_j \) for \( j < k \) implies \( g_i = f_i \) for \( i = 1, 2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} \). But now \( \alpha_k + \mu_k \) is only possible if \( \mu_k < \alpha_k \) and thus \( \mu < \alpha \).

Finally we investigate those \( \mu \) with \( \mu < \alpha^0 \), i.e. \( \mu \notin \mathcal{M} \). We consider one fixed summand resulting from applying Leibniz' rule repeatedly to one fixed term of form (17). Let \( I \subseteq \{1, 2, \ldots, |\alpha|\} \) be the set of all indices \( i \), such that by Leibniz' rule \( \frac{\partial}{\partial x_{f_i}} \) acts on \( \psi \) only, and similarly \( J = \{1, 2, \ldots, |\alpha|\} \setminus I \), the set of all indices \( i \) such that \( \frac{\partial}{\partial x_{f_i}} \) acts on some \( \alpha_{gh,f_n} \), \( i < h \leq |\alpha| \).

Using \( \alpha \in \mathcal{M}, \mu \notin \mathcal{M} \) or equivalently \( \alpha \cdot r > s \geq \mu \cdot r \) gives

\[
\alpha \cdot r = \sum_{i=1}^{[\alpha]} r_{gi} > s \geq \mu \cdot r = \sum_{i \in I} r_{fi},
\]

and thus

\[
\sum_{i \in J} r_{fi} = \sum_{i=1}^{[\alpha]} r_{fi} - \sum_{i \in I} r_{fi} > \sum_{i=1}^{[\alpha]} (r_{fi} - r_{gi}),
\]

which means the total degree \( \sum_{i \in J} r_{fi} \) of the derivatives acting on the coefficients \( \alpha_{gh,f_n} \) is higher than the total degree \( \sum_{i=1}^{[\alpha]} (r_{fi} - r_{gi}) \) of the product \( \sum_{h=1}^{[\alpha]} \alpha_{gh,f_n} \), and thus we may conclude \( p_{\mu} = 0 \) if \( \mu \cdot r \leq s < \alpha^0 \cdot r \leq \alpha \cdot r \).

**Caution:** Here we have \( \mu \cdot r < \alpha^0 \cdot r \leq \alpha \cdot r \). This needs not imply \( \mu < \alpha \) or \( \mu < \alpha^0 \) as illustrated in the following example.

If \( r = (1, 2, 6), s = 3, \) then the only multi-indices not in \( \mathcal{M} \) are \((0, 0, 0), (0, 1, 0), (1, 0, 0)\) and \((2, 0, 0)\). \( \alpha^0 = (0, 0, 1) \) and for all \( \mu \in \mathcal{M} \setminus \{(0, 0, 0)\} \) \( \mu \cdot r \leq s \leq \mu \cdot \alpha^0 \), but \( \mu > \alpha^0 \). (We need the ordering \( \alpha < \beta \) on the multi-indices only for this induction; whereas the degrees \( \alpha \cdot r \) w.r.t. \( \delta \), have geometric meaning.)
To conclude the proof of lemma 3, use \( p_\alpha \equiv 1 \), \( p_\mu \equiv 0 \) for \( \mu \notin \mathcal{M}_s \) and that the sum only ranges over \( \mu \) with \( \mu < \alpha \), to obtain \( 0 = (Z^s \psi)(0) = (D^s \psi)(0) + \sum_{\mu \in \mathcal{M}_s} p_\mu (D^s \psi)(0) \). By the induction hypothesis the sum is zero and thus \( (D^s \psi)(0) = 0 \).

We continue with

**Claim 4.** Each of the coefficients \( c_{ij}(x) \) in (15) is a polynomial in \( (x_1, \ldots, x_n) \) of degree \( r_j - r_i \) w.r.t. \( \delta_i \).

**Proof.** We will show \( (D^s c_{ij})(0) = 0 \) for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha \cdot r > r_j - r_i \). The claim then is immediate from the Taylor series expansion of \( c_{ij} \) about \( x = 0 \).

Almost the only fact we can use apart from the previous Claims and lemmata is that

\[
(ad^* Y^s; Y^s) = 0 \mod \Lambda^n - s_j + 1, \quad \text{if} \quad r \cdot \alpha + r_i > r_j.
\]

The problem, though, is that \( (ad^* Y^s; Y^s) \) consists of differences of products of derivatives of many coefficients \( c_{ij} \). But by a "tricky" use of the "triangular" structure of the matrix \( C = (c_{ij}) \) using the claims 1 through 3 we will be able to single out one specific derivative \( Y^s \left( \frac{\partial c_{ij}}{\partial x_k} \right) \) at a time and from \( Y^s \left( \frac{\partial c_{ij}}{\partial x_k} \right)(0) = 0 \) if \( \alpha \cdot r + r_k > r_i - r_j \) by virtue of lemma 3, we may conclude that \( \frac{\partial c_{ij}}{\partial x_k} \) is of degree \( r_i - r_j - r_k \) and finally that \( c_{ij} \) is of degree \( r_i - r_j \).

We will work through all \( \frac{\partial c_{ij}}{\partial x_k} \) by three nested inductions.

We identify \( c_{ij}(x) \) with its Taylor series about \( x = 0 \). The first induction is on \( j \), \( j \) increasing. The smallest \( j \) occurring is \( j = s_1 + 1 = s_2 \).

Letting \( j = s_2 \), then for \( i = 1, 2, \ldots, j - 2 \), \( c_{ij}(x) \) depends on \( (x_1, \ldots, x_{j-1}) \) only (by claim 3) and each nonvanishing summand of \( c_{ij}(x) \) must obtain a factor \( x_k \) with \( i < k \leq j - 1 \) (by claim 2). Therefore it suffices to show that \( \frac{\partial c_{ij}}{\partial x_k} \) (x) is a polynomial of degree \( r_j - r_i - r_k \) for all \( k \) such that \( i < k \leq j - 1 \).

The second induction is on \( k \), \( k \) decreasing. We start with \( k = j - 1 = s_2 - 1 \), and since \( s_{k+1} = s_j > j \) \( Y^{j-1} = \frac{\partial}{\partial x_{j-1}} \mod \Lambda^{n-j} \), and we compute

\[
[Y^s; Y^{j-1}] = \left( \frac{\partial c_{ij}}{\partial x_{j-1}} \right) \frac{\partial}{\partial x_j} \mod \Lambda^{n-j}
\]

\[
(ad^* Y^s; [Y^s; Y^{j-1}]) = (-)^{s_i} D^s \left( \frac{\partial c_{ij}}{\partial x_{j-1}} \right) \frac{\partial}{\partial x_j} \mod \Lambda^{n-j}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_{j-1}) \). (Note: there are not nonconstant \( c_{gh} \) with \( h < j \)).
For all $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$ with $\alpha \cdot r + r_i + r_{j-1} > r_j$, this expression vanishes modulo $A^{n-l}$, and thus $\frac{\partial c_{ij}}{\partial x_{j-1}}$ is a polynomial of degree $r_j - r_i - r_{j-1}$.

To continue the second induction, suppose $\frac{\partial c_{ij}}{\partial x_i}$ is a polynomial of degree $r_j - r_i - r_l$ for all $l$ such that $k < l < j - 1$, and we consider $\frac{\partial c_{ij}}{\partial x_k}$. By claim 2, we may restrict our considerations to $i < k$ and compute:

$$[Y^i, Y^k] = \left(\frac{\partial c_{ij}}{\partial x_k} - \frac{\partial c_{kj}}{\partial x_i}\right) \frac{\partial}{\partial x_j} \mod A^{n-j}$$

and for $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$ such that $\alpha \cdot r + r_k + r_i > r_j$:

$$0 = (\text{ad}^\alpha Y^z; [Y^i, Y^k]) \mod A^{n-j}$$

$$= (-)^{\alpha_i}\left(\left(Y^z \frac{\partial c_{ij}}{\partial x_k} - Y^z \frac{\partial c_{kj}}{\partial x_i}\right) \frac{\partial}{\partial x_j} \mod A^{n-j}\right)$$

(since there are no nonconstant $c_{gh}$ with $h < j$).

If $\alpha_i \neq 0$ for some $l > k$, then interchange the order of differentiation in the last term, i.e. choose $\beta$ so that $D^z \frac{\partial}{\partial x_i} = D^z \frac{\partial}{\partial x_l}$ and get $D^z \frac{\partial c_{kl}}{\partial x_i}(0) = D^z \frac{\partial c_{kl}}{\partial x_k}(0) = 0$ by induction (2) hypothesis since

$$\beta \cdot r + r_i = \alpha \cdot r + r_i > r_j - r_k.$$ 

If $\alpha_i = 0$ for all $l > k$, then each not yet vanishing summand in $D^z \left(\frac{\partial c_{kl}}{\partial x_i}\right)$ still contains a factor $x_i$ for some $l > k$ and therefore vanishes at $x = 0$. In both cases we obtain $D^z \frac{\partial c_{ij}}{\partial x_k} = 0$, thus finishing the second induction and the start of the first induction.

Before continuing induction (1), we introduce some notation.

Let $j \in \{s_2, \ldots, n\}$ fixed and define for $1 \leq f \leq g \leq j$, $s \in \mathbb{Z}^+$,

$$\mathcal{G}^f_g(-s) = \left\{ \sum_{l=f}^{g} p_l(x) \frac{\partial}{\partial x_l} : p_l \in H_{r_l-s} \right\},$$
and for $1 \leq f \leq g \leq j - 1$

$$\mathcal{F}^g(-s) = \left\{ \frac{\partial \phi(x)}{\partial x_j} : \phi(x) = \phi(x_1, \ldots, x_{j-1}) \text{ is analytic, each summand} \right\}$$

of $\phi$ contains a factor $x_h$ for some $h \in \{ f + 1, \ldots, j - 1 \}$

and $\frac{\partial \phi}{\partial x_h}(x)$ is a polynomial of degree $r_j - r_h - s$ if

$h \in \{ g + 1, \ldots, j - 1 \}$

(We assume $\phi$ is expanded in its Taylor series about $x = 0$.)

Then one easily verifies

$$\left[ \mathcal{F}^g(-s) ; \mathcal{F}^g(-s') \right] \subseteq \mathcal{F}^{\max(g, g')}(-s - s')$$

as a consequence of $\frac{\partial p_l}{\partial x_h} = 0$ if $h \geq l$ and

$$\left[ \mathcal{G}^g_{s+1}(-s) ; \mathcal{G}^g_{s+1}(-s') \right] \subseteq \mathcal{G}^g_{s+1}(-s - s'),$$

$$\left[ \mathcal{F}^g(-s) ; \mathcal{F}^g(-s') \right] \subseteq \mathcal{F}^g(-s - s')$$

$$\mathcal{F}^g(-s)(0) = 0 \text{ for all } s > 0; f, g = 1, 2, \ldots, j - 2,$$

$$\mathcal{G}^g(-s) = 0 \text{ if } s > r_j, 1 \leq f \leq g \leq j,$$

$$\mathcal{F}^g(-s) \subseteq \mathcal{F}^g(-s) \text{ if } f \leq h.$$

To continue induction (1) suppose $c_{ij}(x)$ is a polynomial of degree $r_i - i$ for all $l < j$ and consider $c_{ij}(x)$ with $i$ such that $s_i + 1 \leq j$. By claim 3, $c_{ij}(x)$ does not depend on $x_h$ with $h \geq j$ for all $l \leq j$ and therefore we may and will do all the following calculations modulo $\Delta^n - j$.

By induction (2') on $k$, $k$ decreasing, we show $\frac{\partial c_{ij}}{\partial x_k}$ is a polynomial of degree $r_j - r_i - r_k$ for $i < k \leq j$. To save unnecessary work start with $k = j$ and by claim 3, $\frac{\partial c_{ij}}{\partial x_j} = 0$. Next suppose $\frac{\partial c_{ij}}{\partial x_l}$ is a polynomial of degree $r_j - r_i - r_l$ for all $l > k$ (and $i < l \leq j$).

By claim 2 we only have to consider $k > i$.

By induction (3) on $\alpha$, the length $|\alpha|$ of $\alpha$ increasing we show

Claim 5. For each $\alpha = (\alpha_1, \ldots, \alpha_j - 1)$

$$(-)^{|\alpha|} (\text{ad}^a Y^z, [Y^I, Y^k]) - \left( Y^z \frac{\partial c_{ij}}{\partial x_k} \right) \frac{\partial}{\partial x_j} \in \mathcal{G}_{s+1}(-s) + \mathcal{F}_k(-s)$$

where $s = r \cdot \alpha + r_k + r_i$. 

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Observe that \( Y^k = \left( \frac{\partial}{\partial x_k} + \sum_{i=1}^{j-1} c_{kl} \frac{\partial}{\partial x_i} \right) + c_{kj} \frac{\partial}{\partial x_j} \), where by induction (1) hypothesis the expression in parenthesis is in \( \mathcal{F}_k^{-1}(-r_k) \) and by claim 2 and induction (2') hypothesis

\[
c_{kj} \frac{\partial}{\partial x_j} \in \mathcal{F}_k(-r_k) + \mathcal{F}_k^k(-r_k)
\]

and similarly

\[
Y^i - c_{ij} \frac{\partial}{\partial x_j} \in \mathcal{F}_j^{-1}(-r_i),
\]

by induction (2') hypothesis

\[
\left( \sum_{i=1}^{j-1} c_{kl} \frac{\partial}{\partial x_i} \right) c_{ij} \frac{\partial}{\partial x_j} \in \mathcal{F}_k(-r_i - r_k)
\]

and thus

\[
[Y^i; Y^k] - c_{ij} \frac{\partial}{\partial x_j} \in \mathcal{F}_{i+j+1}(-r_i - r_k) + \mathcal{F}_k^k(-r_k)
\]

using the relations (19). (We get \( \mathcal{F}_{i+j+1} \) instead of \( \mathcal{F}_i \) because \( c_{kk} = 1 = \text{const.} \).) This starts induction (3) with \(|\alpha| = 0\).

Next suppose claim 5 is true for all \( \beta \) with \(|\beta| < a_0 \) and fix \( \alpha = (\alpha_1, \ldots, \alpha_{j-1}) \) with \(|\alpha| = a_0\), letting \( l = \min \{ h : \alpha_h = 0 \} \), and define the multi-index \( \beta = \beta(\alpha) \) by \( \beta_i = \alpha_i - 1 \) and \( \beta_h = \alpha_h \) for all \( h > l \). Then by induction (3) hypothesis

\[
(-)^{|\beta|} (\text{ad}^a Y^\beta; [Y^i; Y^k]) - Y^\beta \frac{\partial c_{ij}}{\partial x_k} \frac{\partial}{\partial x_j} \in \mathcal{F}_{i+j+1}(-s) + \mathcal{F}_k^k(-s)
\]

with \( s = r \cdot \beta + r_k + r_i \).

Furthermore \( (\text{ad}^a Y^\beta; \cdot) = [Y^i; (\text{ad}^a Y^\beta; \cdot)] \), that is why we chose \( l \) to be the smallest integer such that \( \alpha_l = 0 \).

Finally \( Y^i \in \mathcal{F}_i(-r_i) + \mathcal{F}_k^k(-r_k) \) and using the relations (19) we obtain:

\[
(-)^{|\alpha|} (\text{ad}^a Y^\alpha; [Y^i; Y^k]) - \left( Y^\alpha \frac{\partial c_{ij}}{\partial x_k} \right) \frac{\partial}{\partial x_j} = \left[ Y^i; (-)^{|\beta|} (\text{ad}^a Y^\beta; [Y^i; Y^k]) - \left( Y^\beta \frac{\partial c_{ij}}{\partial x_k} \right) \frac{\partial}{\partial x_j} \right] \\
\in \left[ \mathcal{F}_i(-r_i) + \mathcal{F}_k^k(-r_k); \mathcal{F}_{i+j+1}(-s) + \mathcal{F}_k^k(-s) \right] \\
\subseteq \mathcal{F}_{i+j+1}(-s-r_i) + \mathcal{F}_k^k(-s-r_k)
\]

with \( s = r \cdot \beta + r_i + r_k \), thus finishing induction (3) and the proof of claim 5.
To finish both induction (1) and induction (2') consider $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$ such that $\sigma = \alpha \cdot r + r_i + r_k > r_j$. Then $\mathfrak{F}_{k+1} (-\sigma) = 0$,

$$(\mathfrak{F}_k (-\sigma))(0) = 0, \text{ and } (\text{ad}^\alpha Y^*; [Y^i; Y^k])(0) = 0 \mod A^{n+1-j}.$$ 

Therefore $Y^\alpha \frac{\partial c_{ij}}{\partial x_k}(0)=0$. By lemma 3, $\frac{\partial c_{ij}}{\partial x_k}$ is a polynomial of degree $r_j - r_i - r_k$ for $i < k \leq j - 1$ and thus $c_{ij}$ is a polynomial of degree $r_j - r_i$.

4. Proof of the theorem

We show that relative to the coordinates $(x_1, \ldots, x_n)$ and the dilation $\delta_i(x)$ defined in (6) and (7), the fields $X^i$ are of the form

$$X^i(x) = \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq k,$$

with $a_{ij}(x)$ being polynomials of degree $r_j - 1$. Since the proof is exactly the same for each $i$, we will suppress the index $i$, and just write $X$ for $X^i$.

Since $Y^1(0), \ldots, Y^n(0)$ (as defined in section 3) are linearly independent there are analytic functions $b_j, j = 1, \ldots, n$ such that

$$(20) \quad X(x) = \sum_{j=1}^n b_j(x) Y^j(x).$$

We show by induction on $j$, $j$ increasing, that $b_j$ is a polynomial in $x$ of degree $r_i - 1$.

Start with $j = 1$ and compute for $1 \leq i \leq n$

$$[X; Y^i] = (Y^i b_1) Y^1 + \sum_{i=1}^n ((Y^i b_1) Y^i - b_1 [Y^i; Y^1]) = (Y^i b_1) Y^1 \mod A^{n-2},$$

giving $Y^i b_1 \equiv 0$ for $i = 1, 2, \ldots, n$ and thus $b_1(x) = \text{const}$.

Next suppose $b_i(x)$ is a polynomial of degree $r_i - 1$ for all $l < j$.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha \cdot r > r_j - 1$ compute:

$$0 = (\text{ad}^\alpha Y^*; X) \mod A^{n+1-s_j} = \sum_{l=1}^{j-1} \left( \text{ad}^\alpha Y^*; b_l \frac{\partial}{\partial x_l} \right) + \left( \text{ad}^\alpha Y^*; b_j \frac{\partial}{\partial x_j} \right) \mod A^{n+1-s_j}.$$ 

By the inductions hypotheses, the sum vanishes identically modulo $A^{n+1-s_j}$ and since

$$\frac{\partial c_{k,l}}{\partial x_h} = 0 \text{ for } h \geq l,$$

$$(\text{ad}^\alpha Y^*; b_j \frac{\partial}{\partial x_j}) = (-)^{|\alpha|} (Y^* b_j) \frac{\partial}{\partial x_j} \mod A^{n+1-s_j}.$$
Now \((Y^*b_j) \equiv 0\) for all \(\alpha \cdot r > r_j - 1\), and by lemma 3, \(b_j\) is a polynomial of degree \(r_j - 1\), thus finishing the induction.

Finally to express \(X^i\) in terms of the abelian basis \(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\), combine (15) and (20) and use claim 4:

\[
X^i(x) = \sum_{j=1}^{n} b_{ij}(x) Y^j(x) = \sum_{l=1}^{n} \sum_{j=1}^{n} b_{ij}(x) c_{jl}(x) \frac{\partial}{\partial x_l}
\]

\[
= \sum_{l=1}^{n} a_{il}(x) \frac{\partial}{\partial x_l}
\]

with \(a_{il}(x) = \sum_{j=1}^{n} b_{ij}(x) c_{jl}(x)\)

and \(a_{il}\) is of degree \((r_j - 1) + (r_l - r_j) = r_l - 1\) and thus \(X^i(x)\) is of degree \(-1\) w.r.t. the dilation \(\delta_i(x)\), \(x = 1, \ldots, k\).

**Bibliography**


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Eingegangen 9. April 1987