ANALYSIS OF A CONSERVATION LAW MODELING
A HIGHLY RE-ENTRANT MANUFACTURING SYSTEM

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Abstract. This article studies a hyperbolic conservation law that models a highly re-entrant manufacturing system as encountered in semi-conductor production. Characteristic features are the nonlocal character of the velocity and that the influx and outflux constitute the control and output signal, respectively. We prove the existence and uniqueness of solutions for $L^1$-data, and study their regularity properties. We also prove the existence of optimal controls that minimizes in the $L^2$-sense the mismatch between the actual and a desired output signal. Finally, the time-optimal control for a step between equilibrium states is identified and proven to be optimal.

1. Introduction and prior work. This article studies optimal control problems governed by the scalar hyperbolic conservation law

$$\partial_t \rho(t,x) + \partial_x \left( \lambda(W(t)) \rho(t,x) \right) = 0 \quad \text{where} \quad W(t) = \int_0^1 \rho(t,x) \, dx,$$

(1)
on a rectangular domain $[0,T] \times [0,1]$ or the semi-infinite strip $[0,\infty) \times [0,1]$. We assume that $\lambda(\cdot) \in C^1([0, +\infty); (0, +\infty))$ in the whole paper.

This work is motivated by problems arising in the control of semiconductor manufacturing systems which are characterized by their highly re-entrant character, see below for more details. In the manufacturing system the natural control input is the influx, which suggests the boundary conditions

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Various different choices of the space of admissible controls are of both practical and mathematical interest, each leading to distinct mathematical problems. Motivated by this application from manufacturing systems, natural control objectives are to minimize the error signal that is the difference between a given demand forecast \( y_d \) and the actual out-flux \( y(t) = \lambda(W(t))\rho(t,1) \). An alternative to this problem modeling a perishable demand, is the similar problem that permits backlogs. In that case, the objective is to minimize in a suitable sense the size of the different error signal

\[
\beta(t) = \int_0^t y_d(s) \, ds - \int_0^t \lambda(W(s))\rho(s,1) \, ds, \tag{3}
\]

while keeping the state \( \rho(\cdot, x) \) bounded. This article only considers the problem of perishable demand and the minimization in the \( L^2 \)-sense.

Partial differential equations models for such manufacturing systems are motivated by the very high volume (number of parts manufactured per unit time) and the very large number of consecutive production steps which typically number in the many hundreds. They are popular due to their superior analytic properties and the availability of efficient numerical tools for simulation. For more detailed discussions see e.g. [?, ?, ?, ?, ?, ?, ?]. In many aspects these models are very similar to those of traffic flows, compare e.g. [?].

The study of hyperbolic conservation laws, and especially of control systems governed by such laws, have a rich history. A modern introduction to the subject is the text [?]. From a mathematical perspective, the choice of spaces in which to consider the conservations laws (and their data) provides for distinct levels of challenges. Fundamental are question of wellposedness, regularity properties of solutions, controllability, existence, uniqueness and regularity of optimal controls. Existence of solutions, regularity and well-posedness of nonlinear conservation laws have been widely studied under diverse sets of hypotheses, commonly in the context of vector values systems of conservation laws, see e.g. [?, ?, ?]. Further results on uniqueness may be found in [?], while [?] introduced an a distinct notion of differentiability of the solution of hyperbolic systems. For the controllability of linear hyperbolic systems, see, in particular, the important survey [?]. The attainable sets of nonlinear conservation laws are studied in [?, ?, ?, ?, ?, ?], while [?] provides a comprehensive survey of controllability that also includes nonlinear conservation laws.

This article is, in particular, motivated by the recent work [?] which, among others, considered the optimal control problem of minimizing \( \|y - y_d\|_{L^2(0,T)} \) (the \( L^2 \) norm of the difference between a demand forecast and the actual outflux). That work derived necessary conditions and used these to numerically compute optimal controls corresponding to piecewise constant desired outputs \( y_d \).

The organization of the following sections is as follows: First we rigorously prove the existence of weak solutions of the Cauchy problem for the conservation law (1) for the case when the initial data and boundary condition (2) lie in \( L^1(0,1) \) and \( L^1(0,T) \), respectively. Next we establish the existence and uniqueness of solutions for the optimal control problem of minimizing the \( L^2 \)-norm of the difference between any desired \( L^2 \)-demand forecast \( y_d \) and actual outflux \( y(t) = \lambda(W(t))\rho(t,1) \). Finally, in the classical special case where
\[ \lambda(W) = \frac{1}{1 + W}, \quad (4) \]

we prove that the natural candidate control for transferring the system from one equilibrium state to another one is indeed time-optimal.

While preparing the final version of this article, the authors received a copy of the related manuscript [?] which is also motivated in part by [?, ?, ?] and which addresses wellposedness for systems of hyperbolic conservation laws with a nonlocal speed on all of \( \mathbb{R}^n \). It also includes a study of the solutions with respect to the initial datum and a necessary condition for the optimality of integral functionals. There are substantial differences between [?] and our paper, especially the treatment of the boundary conditions and the method of proof.

2. Existence, uniqueness, and regularity of solutions in \( L^1 \).

2.1. Technical preliminaries and notation. For any \( \lambda \in C^1([0, +\infty); (0, +\infty)) \) define the functions \( \bar{\lambda}, \overline{\lambda} \in C^0([0, \infty); (0, \infty)) \) and \( d \in C^0([0, \infty); [0, \infty)) \) with respect to \( \lambda \) as

\[ \bar{\lambda}(M) := \inf_{0 \leq W \leq M} \lambda(W), \quad \overline{\lambda}(M) := \sup_{0 \leq W \leq M} \lambda(W), \quad d(M) := \sup_{0 \leq W \leq M} |\lambda'(W)|. \quad (5) \]

For convenience we extend \( \lambda \) to all of \( \mathbb{R} \) in such a way that this extension, still denoted \( \lambda \), is in \( C^1(\mathbb{R}; (0, +\infty)) \).

2.2. Weak solutions of the Cauchy problem. First we recall, from [?, Section 2.1], the usual definition of a weak solution to the Cauchy problem (1) and (2).

**Definition 2.1.** Let \( T > 0 \), \( \rho_0 \in L^1(0, 1) \) and \( u \in L^1(0, T) \) be given. A weak solution of the Cauchy problem (1) and (2) is a function \( \rho \in C^0([0, T]; L^1(0, 1)) \) such that, for every \( t \in [0, T] \) and every \( \varphi \in C^1([0, \tau] \times [0, 1]) \) such that

\[ \varphi(\tau, x) = 0, \forall x \in [0, 1] \quad \text{and} \quad \varphi(t, 1) = 0, \forall t \in [0, \tau], \quad (6) \]

one has

\[ \int_0^\tau \int_0^1 \rho(t, x)(\varphi_t(t, x) + \lambda(W(t))\varphi_x(t, x))dxdt \]

\[ + \int_0^\tau u(t)\varphi(t, 0)dt + \int_0^1 \rho_0(x)\varphi(0, x)dx = 0. \quad (7) \]

One has the following lemma, which will be useful to prove a uniqueness result for the Cauchy problem (1) and (2).

**Lemma 2.2.** If \( \rho \in C^0([0, T]; L^1(0, 1)) \) is a weak solution to the Cauchy problem (1) and (2), then for every \( \tau \in [0, T] \) and every \( \varphi \in C^1([0, \tau] \times [0, 1]) \) such that

\[ \varphi(t, 1) = 0, \forall t \in [0, \tau], \quad (8) \]

one has

\[ \int_0^\tau \int_0^1 \rho(t, x)(\varphi_t(t, x) + \lambda(W(t))\varphi_x(t, x))dxdt + \int_0^\tau u(t)\varphi(t, 0)dt \]

\[ - \int_0^1 \rho(\tau, x)\varphi(\tau, x)dx + \int_0^1 \rho_0(x)\varphi(0, x)dx = 0. \quad (9) \]
Proof. The case \( \tau = 0 \) is trivial. For every \( \tau \in (0, T] \) and \( \varepsilon \in (0, \tau) \), let \( \eta_{\varepsilon} \in C^1([0, \tau]) \) such that

\[
\eta_{\varepsilon}(\tau) = 0 \quad \text{and} \quad \eta_{\varepsilon}(t) = 1, \; \forall t \in [0, \tau - \varepsilon] \quad \text{and} \quad \eta_{\varepsilon}'(t) \leq 0, \; \forall t \in [0, \tau].
\]  

(10)

It is easy to prove that, for every \( h \in C^0([0, \tau]) \),

\[
\lim_{\varepsilon \to 0} \int_{\tau-\varepsilon}^{\tau} \eta_{\varepsilon}'(t) h(t) dt = -h(\tau).
\]  

(11)

Then, for every \( \varphi \in C^1([0, \tau] \times [0, 1]) \) satisfying (8), let \( \varphi_{\varepsilon}(t, x) := \eta_{\varepsilon}(t) \varphi(t, x) \). This obviously verifies

\[
\varphi_{\varepsilon}(\tau, x) = 0, \; \forall x \in [0, 1] \quad \text{and} \quad \varphi_{\varepsilon}(t, 1) = 0, \; \forall t \in [0, \tau].
\]  

(12)

Since \( \rho \in C^0([0, T]; L^1(0, 1)) \) is a weak solution to the Cauchy problem (1) and (2), we have

\[
\int_0^\tau \int_0^1 \rho(t, x)((\varphi_{\varepsilon})_t(t, x) + \lambda(W(t))(\varphi_{\varepsilon})_x(t, x)) dx dt
+ \int_0^\tau u(t)(\varphi_{\varepsilon})(t, 0) dt + \int_0^1 \rho_0(x)(\varphi_{\varepsilon})(0, x) dx = 0.
\]  

(13)

Using the definition of \( \varphi_{\varepsilon} \), (10) and (13), one has

\[
\int_0^\tau \int_0^1 \rho(t, x)(\varphi_{\varepsilon}(t, x) + \lambda(W(t))\varphi_{\varepsilon}(t, x)) dx dt
+ \int_0^\tau u(t)\varphi(t, 0) dt + \int_0^1 \rho_0(x)\varphi(0, x) dx
= \int_{\tau-\varepsilon}^{\tau-\varepsilon} \int_0^1 (1 - \eta_{\varepsilon}(t))\rho(t, x)(\varphi_{\varepsilon}(t, x) + \lambda(W(t))\varphi_{\varepsilon}(t, x)) dx dt
+ \int_{\tau-\varepsilon}^{\tau-\varepsilon} \int_0^\tau \eta_{\varepsilon}'(t)\rho(t, x)\varphi(t, x) dx dt - \int_{\tau-\varepsilon}^{\tau-\varepsilon} \int_0^1 \eta_{\varepsilon}'(t)\rho(t, x)\varphi(t, x) dx dt.
\]  

(14)

Observing that \( \rho \in C^0([0, T]; L^1(0, 1)) \), \( \lambda \in C^1(\mathbb{R}; [0, \infty)) \) and \( \varphi \in C^1([0, \tau] \times [0, 1]) \), we point out that the functions \( W(\cdot) = \int_0^1 \rho(\cdot, x) dx \), \( \int_0^1 \rho(\cdot, x) \varphi(\cdot, x) dx \) and \( \lambda(W(\cdot)) \) are all in \( C^0([0, T]) \).

We can estimate the first two terms on the right hand side of (14) as

\[
\left| \int_{\tau-\varepsilon}^{\tau-\varepsilon} \int_0^1 (1 - \eta_{\varepsilon}(t))\rho(t, x)(\varphi_{\varepsilon}(t, x) + \lambda(W(t))\varphi_{\varepsilon}(t, x)) dx dt \right| \leq K\varepsilon,
\]  

(15)

and

\[
\left| \int_{\tau-\varepsilon}^{\tau-\varepsilon} (1 - \eta_{\varepsilon}(t))u(t)\varphi(t, 0) dt \right| \leq K \int_{\tau-\varepsilon}^{\tau-\varepsilon} u(t) dt,
\]  

(16)

where \( K \) is a constant independent of \( \varepsilon \). While for the last term on the right hand side of (14), we get from (11) that

\[
\int_{\tau-\varepsilon}^{\tau-\varepsilon} \int_0^1 \eta_{\varepsilon}'(t)\rho(t, x)\varphi(t, x) dx dt = \int_{\tau-\varepsilon}^{\tau-\varepsilon} \eta_{\varepsilon}'(t) \left( \int_0^1 \rho(t, x)\varphi(t, x) dx \right) dt
\]  

\[
\rightarrow - \int_0^1 \rho(\tau, x)\varphi(\tau, x) dx \quad \text{as} \quad \varepsilon \to 0.
\]  

(17)

In view of (15)-(17), letting \( \varepsilon \to 0 \) in (14) one gets (9). \( \square \)
Theorem 2.3. If $\rho_0 \in L^1(0,1)$ and $u \in L^1(0,T)$ are nonnegative almost everywhere, then the Cauchy problem (1) and (2) admits a unique weak solution $\rho \in C^0([0,T];L^1(0,1))$, which is also nonnegative almost everywhere in $Q = [0,T] \times [0,1]$.

Proof. We first prove the existence of weak solution for small time: there exists a small $\delta \in (0,T]$ such that the Cauchy problem (1) and (2) has a weak solution $\rho \in C^0([0,\delta];L^1(0,1))$. The idea is to find first the characteristic curve $\xi = \xi(t)$ passing through $(0,0)$, then construct a solution to the Cauchy problem.

Let

$$\Omega_{\delta,M} := \{ \xi \in C^0([0,\delta]) : \xi(0) = 0, \quad \tilde{\lambda}(M) \leq \frac{\xi(s) - \xi(t)}{s - t} \leq \overline{\lambda}(M), \forall s,t \in [0,\delta], \; s > t \},$$

(18)

where $\tilde{\lambda}, \overline{\lambda}$ are defined by (5) and

$$M := \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,1)}.$$  

(19)

We point out here that the case $d(M) = 0$ (by (5), $\lambda$ is a constant in $[0,M]$) is trivial. We only prove Theorem 2.3 for the case $d(M) > 0$.

We define a map $F : \Omega_{\delta,M} \rightarrow C^0([0,\delta])$, $\xi \mapsto F(\xi)$, as

$$F(\xi)(t) := \int_0^t \lambda(t) \int_{1-\xi(s)}^1 u(\sigma)d\sigma + \int_0^1 \rho_0(x)dx ds, \forall \xi \in \Omega_{\delta,M}, \forall t \in [0,\delta].$$  

(20)

It is obvious that $F$ maps into $\Omega_{\delta,M}$ itself if

$$0 < \delta < T \text{ and } \delta < \frac{1}{\overline{\lambda}(M)}.$$  

(21)

Now we prove that, if $\delta$ is small enough, $F$ is a contraction mapping on $\Omega_{\delta,M}$ with respect to the $C^0$ norm defined by

$$\|\xi\|_{C^0([0,\delta])} := \sup_{0 \leq t \leq \delta} |\xi(t)|.$$  

Let $\xi_1, \xi_2 \in \Omega_{\delta,M}$. We define $\overline{\xi}_1 \in C^0([0,\delta])$ and $\overline{\xi}_2 \in C^0([0,\delta])$ by $\overline{\xi}_1(t) := \max\{\xi_1(t), \xi_2(t)\}$ and $\overline{\xi}_2(t) := \min\{\xi_1(t), \xi_2(t)\}$. By (5) and changing the order of the integrations (see Figure ??), we have

$$|F(\xi_2)(t) - F(\xi_1)(t)| \leq d(M) \int_0^t \left| \int_{1-\xi_1(s)}^{1-\xi_2(s)} \rho_0(x)dx \right| ds$$

$$= d(M) \int_{1-\xi_1(t)}^{1-\xi_2(t)} \rho_0(x)(t - \overline{\xi}_1^{-1}(1 - x))dx$$

$$+ d(M) \int_{1-\xi_2(t)}^{1} \rho_0(x)(\overline{\xi}_2^{-1}(1 - x) - \overline{\xi}_1^{-1}(1 - x))dx$$

$$\leq d(M) \int_{1-\xi_1(t)}^{1} \rho_0(x)dx \cdot (\overline{\xi}_2^{-1}(\overline{\xi}_2(t)) - \overline{\xi}_1^{-1}(\overline{\xi}_2(t)))$$

$$+ d(M) \int_{1-\xi_2(t)}^{1} \rho_0(x)(\overline{\xi}_2^{-1}(1 - x) - \overline{\xi}_1^{-1}(1 - x))dx$$

$$\leq d(M) \int_{1-\xi_1(t)}^{1} \rho_0(x)dx \cdot \sup_{0 \leq y \leq \overline{\xi}_2(t)} (\overline{\xi}_2^{-1}(y) - \overline{\xi}_1^{-1}(y)).$$  

(22)